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On the representation type of tensor product algebras

by

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Abstract. The representation type of tensor product algebras of finite-dimensional algebras is considered. The characterization of algebras A, B such that $A \otimes B$ is of tame representation type is given in terms of the Gabriel quivers of the algebras A, B.

Introduction. In this paper by an algebra we mean a finite-dimensional algebra over a fixed algebraically closed field K. All algebras are assumed to be basic indecomposable with respect to the direct product. Our aim is to determine the representation type of the tensor product algebra $B \otimes_K C$ of two algebras B and C in terms of the quivers with relations describing the algebras B and C.

One of the motivations for our study is to introduce a unified approach to the investigation of the representation type of several important classes of algebras including:

(i) The group algebras $B[G]$ of a finite group G with coefficients in an algebra B (studied in [MS, S1]).

(ii) The lower triangular $n \times n$ matrix algebras

$$T_n(B) = \begin{bmatrix} B & 0 & \dots & 0 \\ B & B & \dots & 0 \\ \vdots & \vdots & \ddots & \\ B & B & \dots & B \end{bmatrix} \tag{0.1}$$

with $n \geq 2$ and with coefficients in an algebra B (studied in [AR, Br2, L1, L2, LS, S2]).

(iii) The factor algebras $T_{n,r}(B) := T_n(B)/J_n^r(B)$ of $T_n(B)$, n, $r \geq 2$, studied in [HM], where $J_n(B)$ the ideal of strictly lower triangular $n \times n$ matrices.

(iv) The path algebra BQ of a bound quiver Q with coefficients in an algebra B.

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Indeed, we note that there are algebra isomorphisms

$$\begin{aligned} &B[G] \cong B \otimes_K K[G], \quad T_n(B) \cong B \otimes_K T_n(K)\,, \\ &BQ \cong B \otimes_K KQ, \quad T_{n,r}(B) \cong B \otimes_K T_{n,r}(K)\,, \end{aligned} \tag{0.2}$$

in cases (i)–(iv) respectively. The algebras $T_n(K)$, $T_{n,r}(K)$ are very simple of finite representation type, whereas the representation type of $K[G]$ is well understood for a long time [BD].

Another motivation is the study of the category of B-representations of a quiver over an algebra B [Br1] (see 1.6). On the other hand, some of our results on the representation type of $B \otimes_K C$ are inspired by the facts established earlier for the classes (i)–(iii). In particular, we essentially use the results of Skowroński on tame group algebras [S1] and on tame triangular matrix algebras over Nakayama algebras [S2].

In Section 1 we make some preliminary observations including the assertion that the tensor product $B\otimes_K C\otimes_K D$ of indecomposable non-semisimple algebras B, C, D is of tame type if and only if $B \cong C \cong D \cong T_2(K)$. Moreover, we show that if $B\otimes C$ is of tame representation type, then the algebras B and C are standard [R] of finite representation type.

This approach is also used in Section 2 to show that $B \otimes_K T_n(K)$ is of wild type for $n \geq 6$, and to describe all simply connected algebras B such that $B \otimes T_n(K)$ is of tame type for some $n \geq 3$. The case $n = 2$ is studied in [LSk].

One of our main results is Theorem 3.2 containing a full classification of pairs of weakly sincere simply connected algebras B and C which are not of type $T_n(K)$ and have the property that $B \otimes_K C$ is of tame representation type. All such pairs are listed in a table. In the final part of the paper the representation type of $B \otimes_K C$ when at least one of the algebras B and C is not simply connected is briefly discussed.

The reader is referred to [G1], [D] and [S1] for the definitions of the finite (resp. tame, wild) representation type and of the polynomial growth of an algebra.

1. Preliminaries and notations. In this section we collect basic definitions and elementary properties of bound quivers and their tensor products.

We recall from [G1] that a *bound quiver* is a pair $\widehat{Q} = (Q, I)$, where $Q = (Q_0, Q_1)$ is an oriented graph with a set of vertices Q_0 and a set of arrows Q_1, and I is an admissible two-sided ideal (called a relation ideal) in the path algebra KQ of Q, that is, I is generated by a set of K-combinations of paths of length ≥ 2 in Q and $\exists_n\, Q_1^n \subseteq I$. We call $\widehat{Q}$ *trivial* if $|Q_0| = 1$, Q_1 is empty and I is zero. If $I = 0$ we write Q instead of (Q, I). If (Q, I) is a bound quiver, then the factor algebra

$$K(Q, I) := KQ/I$$

is called the *bound quiver algebra* of (Q,I). It is known [G1] that any basic K-algebra is isomorphic to a bound quiver algebra.

In order to describe the bound quiver of the algebra $B\otimes_K C$ we define the *tensor product* $(Q,I)\otimes(Q',I')$ of two bound quivers (Q,I), (Q',I') to be the bound quiver $(Q\otimes Q',\ I\,\square\, I')$, where

$$(Q\otimes Q')_0 = Q_0\times Q'_0,\qquad (Q\otimes Q')_1 = (Q_0\times Q'_1)\cup(Q_1\times Q'_0)$$

and $I\,\square\, I'$ is the ideal in $K(Q\otimes Q')$ generated by $(Q_0\times I')\cup(I\times Q'_0)$ and by elements of the form

$$(\alpha_{rp},t)\circ(p,\beta_{ts})-(r,\beta_{ts})\circ(\alpha_{rp},s) \tag{1.1}$$

where α_{rp} and β_{ts} run through all arrows $\alpha_{rp}:p\to r$ in Q_1 and $\beta_{ts}:s\to t$ in Q'_1.

If $\widehat{Q}$ and $\widehat{Q}'$ are non-trivial, then $I\,\square\, I'\neq 0$ (even in case $I=0$, $I'=0$), because $I\,\square\, I'$ contains the ideal generated by the set of elements (1.1). It is clear that $I\,\square\, I'$ is an admissible ideal in $K(Q\otimes Q')$ and we denote the bound quiver $(Q\otimes Q',\ I\,\square\, I')$ by $\widehat{Q}\otimes\widehat{Q}'$ (even for $I=0$, $I'=0$). The following simple lemma can be easily verified.

LEMMA 1.2. *If $\widehat{Q}$, $\widehat{Q}'$, $\widehat{Q}''$ are bound quivers then there exist bound quiver isomorphisms*:

(a) $\widehat{Q}\otimes\widehat{Q}'\cong\widehat{Q}'\otimes\widehat{Q}$,
(b) $\widehat{Q}\otimes(\widehat{Q}'\otimes\widehat{Q}'')\cong(\widehat{Q}\otimes\widehat{Q}')\otimes\widehat{Q}''$,
(c) $\widehat{Q}\otimes(\widehat{Q}'\cup\widehat{Q}'')\cong(\widehat{Q}\otimes\widehat{Q}')\cup(\widehat{Q}\otimes\widehat{Q}'')$,
(d) $\widehat{Q}\otimes\widehat{Q}'\cong\widehat{Q}$ *if and only if* $\widehat{Q}'$ *is trivial*,
(e) $(\widehat{Q}\otimes\widehat{Q}')^{\mathrm{op}}\cong\widehat{Q}^{\mathrm{op}}\otimes(\widehat{Q}')^{\mathrm{op}}$.

Moreover,

(f) *the bound quiver $\widehat{Q}\otimes\widehat{Q}'$ is connected if and only if the quivers $\widehat{Q}$ and $\widehat{Q}'$ are connected.* ■

Now we are able to prove a useful technical result.

LEMMA 1.3. *For any bound quivers $\widehat{Q}$ and $\widehat{Q}'$ there is a K-algebra isomorphism*

$$(K\widehat{Q})\otimes_K(K\widehat{Q}')\cong K(\widehat{Q}\otimes\widehat{Q}')\,.$$

Proof. Let us first construct the isomorphism for the case $I=0$, $I'=0$. Let I_c denote the ideal in $K(Q\otimes Q')$ generated by the set of elements (1.1). We define an algebra epimorphism

$$\underline{f}:K(Q\otimes Q')\to(KQ)\otimes_K(KQ')$$

by setting

$$\begin{aligned}
\underline{f}(p,q) &= p\otimes q \quad \&\text{for } p\in Q_0,\ q\in Q'_0\,,\\
\underline{f}(p,\beta) &= p\otimes\beta \qquad \text{for } p\in Q_0,\ \beta\in Q'_1\,,\\
\underline{f}(\alpha,q) &= \alpha\otimes q \qquad \text{for } \alpha\in Q'_0,\ q\in Q'_0\,.
\end{aligned}$$

The epimorphism $\underline{f}$ induces an epimorphism $f : K(\widehat{Q}\otimes\widehat{Q}')\to (KQ)\otimes_K (KQ')$, because $\operatorname{Ker}\underline{f}$ contains the ideal I_c. Now, we define an epimorphism

$$\underline{h} : K(Q)\times K(Q')\to K(Q\otimes Q')/I_c\cong K(\widehat{Q}\otimes\widehat{Q}')$$

by setting

$$\begin{aligned}
\underline{h}(p,q) &= p\otimes q + I_c \quad \text{for } p\in Q_0,\ q\in Q'_0\,,\\
\underline{h}(p,\beta) &= p\otimes\beta + I_c \quad \text{for } p\in Q_0,\beta\in Q'_1\,,\\
\underline{h}(\alpha,q) &= \alpha\otimes q + I_c \quad \text{for } \alpha\in Q'_0,\ q\in Q'_0\,.
\end{aligned}$$

One can check that $\underline{h}$ induces an epimorphism $h : (KQ)\otimes_K (KQ')\to K(Q\otimes Q')/I_c$. Observe that h is the inverse to f.

Now we note that $h(I\otimes KQ'+KQ'\otimes I')\subseteq I\,\Box\, I'/I_c$ and $f(I\,\Box\, I'/I_c)\subseteq I\otimes KQ'+KQ\otimes I'$, because $\underline{f}(I\,\Box\, I')\subseteq I\otimes KQ'+KQ\otimes I'$. Hence $f(I\,\Box\, I'/I_c) = I\otimes KQ'+KQ\otimes I'$. Let $\pi : (KQ)\otimes_K (KQ')\to (K\widehat{Q})\otimes_K (K\widehat{Q}')$ denote the tensor product of the natural epimorphisms $KQ\to K\widehat{Q}$, $KQ'\to K\widehat{Q}'$. One can see that $\operatorname{Ker}\pi = I\otimes KQ'+KQ\otimes I'$. Hence, there exists an isomorphism $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd}
K(Q\otimes Q')/I_c \arrow[r, "f"] \arrow[d, "\pi'"] & (KQ)\otimes_K (KQ') \arrow[d, "\pi"] \\
K(\widehat{Q}\otimes\widehat{Q}') \arrow[r, "\tilde{f}"] & (K\widehat{Q})\otimes_K (K\widehat{Q}'),
\end{tikzcd}$$

where π' is a natural epimorphism. ∎

Given a bound quiver $\widehat{Q}=(Q,I)$ and $n\ge 2$ we define the triangular bound quiver (see [LS] for the case $n=2$) to be the tensor product quiver

$$T_n(\widehat{Q})=\widehat{Q}\otimes A_n\,,$$

where A_n is the quiver

(1.4) $$A_n :\quad 1\to 2\to\ldots\to n,\qquad n\ge 1\,.$$

Corollary 1.5. *There are algebra isomorphisms*

$$T_n(K\widehat{Q})\cong KT_n(\widehat{Q})\cong KA_n\otimes_K K\widehat{Q}\,.$$

Proof. In view of the obvious K-algebra isomorphism $KA_n\cong T_n(K)$, Lemma 1.3 together with (0.2) yields $T_n(K\widehat{Q})\cong K\widehat{Q}\otimes_K T_n(K)\cong K\widehat{Q}\otimes_K KA_n\cong K(\widehat{Q}\otimes A_n)=KT_n(\widehat{Q})$. ∎

Remark 1.6. The category of B-representations of a bound quiver $\widehat{Q}$ over an algebra B (see [Br1]) is equivalent to the category $\operatorname{mod}(B\widehat{Q})$, where $B\widehat{Q}$ is the bound quiver algebra of $\widehat{Q}$ with coefficients in B. In view of the isomorphism $B\widehat{Q} \cong B \otimes_K K\widehat{Q}$ the study of indecomposable B-representations of $\widehat{Q}$ reduces to the study of indecomposable $(B \otimes_K K\widehat{Q})$-modules. ■

We recall from [G4, DLS] that a surjective bound quiver map $f : (\widetilde{Q}, \widetilde{I}) \to (Q, I)$ is a *Galois covering with group* G if G is a group of automorphisms of $(\widetilde{Q}, \widetilde{I})$ acting freely on $\widetilde{Q}_1$ and such that $f(\alpha) = f(\beta)$ if and only if $G^*\alpha = G^*\beta$, that is, Q is a quiver of G-orbits of $\widetilde{Q}$ and the ideal I is induced by $\widetilde{I}$.

Lemma 1.7. *Suppose that* $f : (\widetilde{Q}, \widetilde{I}) \to (Q, I)$ *and* $f' : (\widetilde{Q}', \widetilde{I}') \to (Q', I')$ *are bound quiver Galois coverings with groups* G *and* G', *respectively.*

(a) *The induced maps*

$$f \otimes f' : (\widetilde{Q}, \widetilde{I}) \otimes (\widetilde{Q}', \widetilde{I}') \to (Q, I) \otimes (Q', I'),$$
$$f \otimes \mathrm{id} : (\widetilde{Q}, \widetilde{I}) \otimes (Q', I') \to (Q, I) \otimes (Q', I') \quad \textit{and}$$
$$T_n(f) : T_n(\widetilde{Q}, \widetilde{I}) \to T_n(Q, I)$$

are Galois coverings with groups $G \times G'$ (*for the first map*) *and* G, *respectively.*

(b) *The bound quiver* $T_n(\widehat{Q})$ *is simply connected* (*in the sense of* [AS, DLS, DS1]) *if and only if so is* $\widehat{Q}$. *The bound quiver* $\widehat{Q} \otimes \widehat{Q}'$ *is simply connected if and only if so are* $\widehat{Q}$ *and* $\widehat{Q}'$.

The proof is left to the reader.

A bound quiver $\widehat{Q} = (Q, I)$ is called of tame representation type (resp. finite, wild representation type, or of polynomial growth) if the algebra $K\widehat{Q} = KQ/I$ is of tame representation type (resp. finite, wild representation type, or of polynomial growth).

In the paper a subquiver means a convex subquiver.

We start by proving some necessary conditions for tameness of tensor product algebras.

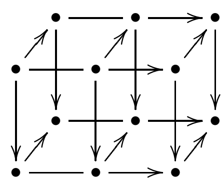
Proposition 1.8. *Suppose that* $\widehat{P} = (P, L)$, $\widehat{Q} = (Q, I)$ *and* $\widehat{R} = (R, M)$ *are non-trivial connected bound quivers.*

(a) *If the algebra* $K(\widehat{P} \otimes \widehat{Q})$ *is not of wild representation type*, *then the algebras* $K\widehat{P}$, $K\widehat{Q}$ *are standard* (*in the sense of* [G4]) *and of finite representation type.*

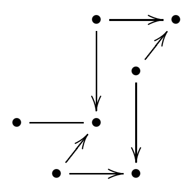
(b) *The algebra* $K(\widehat{P} \otimes \widehat{Q} \otimes \widehat{R})$ *is of tame representation type if and only if* $P = Q = R = A_2$ (see (1.4)).

Proof. (a) If $\widehat{P}\otimes\widehat{Q}$ is not of wild representation type and $(\widetilde{P},\widetilde{L})$ is the universal covering of $\widehat{P}$, then $(\widetilde{P},\widetilde{L})\otimes\widehat{Q}$ is a covering of $\widehat{P}\otimes\widehat{Q}$, and hence is not of wild representation type. Since $\widehat{P}$ is non-trivial, the quivers $\widehat{P}$ and $(\widetilde{P},\widetilde{L})$ each contain a subquiver A_2. Hence $(\widetilde{P},\widetilde{L})\otimes\widehat{Q}$ contains $A_2\otimes\widehat{Q}$ and there is an algebra isomorphism $K(\widehat{A}_2\otimes Q)\cong T_2(K(\widehat{Q}))$. According to [S2], $K\widehat{Q}$ must be a standard algebra of finite representation type (the same is true for $K\widehat{P}$).

(b) We recall from [R] that a standard quiver has no double arrow nor double loop. Assume that $\widehat{P}\otimes\widehat{Q}\otimes\widehat{R}$ is not of wild representation type and denote by $(\widetilde{P},\widetilde{L})$, $(\widetilde{Q},\widetilde{I})$, $(\widetilde{R},\widetilde{M})$ Galois coverings of $\widehat{P}$, $\widehat{Q}$, $\widehat{R}$ respectively. The quiver $\widetilde{P}\otimes\widetilde{Q}\otimes\widetilde{R}$ contains a subquiver of the form $A_2\otimes A_2\otimes A_2$, and if $P\neq A_2$, then $\widetilde{P}\neq A_2$. In this case $\widetilde{P}$ (or $\widetilde{P}^{\mathrm{op}}$) contains $\bullet - \bullet \longrightarrow \bullet$, and $\widetilde{P}\otimes\widetilde{Q}\otimes\widetilde{R}$ (or $(\widetilde{P}\otimes\widetilde{Q}\otimes\widetilde{R})^{\mathrm{op}}$) contains



Hence $(\widetilde{P},\widetilde{L})\otimes(\widetilde{Q},\widetilde{I})\otimes(\widetilde{R},\widetilde{M})$ contains



which is of wild type (see [DR]) and we get a contradiction. It follows that $\widehat{P}=\widehat{Q}=\widehat{R}=A_2$ and the "if part" is proved.

Conversely, if $\widehat{P}=\widehat{Q}=\widehat{R}=A_2$ then in view of Corollary 1.5 there is an isomorphism $K\widehat{P}\otimes K\widehat{Q}\otimes K\widehat{R}\cong T_2(T_2(T_2(K)))$ and according to [FGR, R] the last algebra is of tame representation type. ■

2. The representation type of algebras $B\otimes_K T_n(K)$ for $n\geq 3$. Our aim is to study algebras of tame type of the form $B\otimes_K T_n(K)\cong T_n(B)$, where $n\geq 3$. The case $n=2$ is considered in [LSk]. We use freely the elementary facts and notations pertaining to tame and wild algebras collected in [DS2]. In particular, we assume that algebras of finite representation type are of tame representation type.

Let us start with a general observation.

PROPOSITION 2.1. *Let B, C, D be indecomposable non-semisimple bound quiver algebras. Then*

(a) *The algebra $B \otimes_K C \otimes_K D$ is of tame representation type if and only if $B \cong C \cong D \cong T_2(K)$.*

(b) *If $n \geq 2$ and $B \otimes_K T_n(K)$ is of tame* (*resp. of finite*) *representation type*, *then B is standard of finite representation type and $n \leq 5$* (*resp.* $n \leq 4$).

Proof. Statement (a) is a consequence of Proposition 1.8.

(b) Assume that $B = K\widehat{Q}$, where $\widehat{Q}$ is not trivial. It follows that the universal bound quiver covering $(\widetilde{Q}, \widetilde{I})$ of $\widehat{Q}$ contains the quiver A_2 (1.4) and therefore $T_n((\widetilde{Q}, \widetilde{I}))$ contains $T_n(A_2) \cong T_2(A_n)$, which is of wild representation type for $n \geq 6$ [L1] and is not of finite representation type for $n \geq 5$ [LS]. Since $B \otimes_K T_n(K) \cong T_n(B) \cong KT_n(Q, I)$, it follows that $B \otimes_K T_n(K)$ is of wild representation type for $n \geq 6$ and is not of finite representation type for $n \geq 5$. ∎

We recall that B is called a *Nakayama algebra* if all submodules of any indecomposable projective right and left ideal of B form a uniserial chain.

Lemma 2.2. *A bound quiver algebra $B = K\widehat{Q}$ is a Nakayama algebra if and only if the quiver Q contains neither $A_3^* = \bullet \longleftarrow \bullet \longrightarrow \bullet$* (*with two extremal points*) *nor $(A_3^*)^{\mathrm{op}}$.*

The proof is left to the reader.

Corollary 2.3. *If $B = K\widehat{Q}$ is a Nakayama algebra*, *then either $Q = A_n$ for some n* (*if $\widehat{Q}$ is simply connected*) *or Q is an oriented cycle* (*if $\widehat{Q}$ is not simply connected*, *and then $I \neq 0$*). ∎

Theorem 2.4. (a) *The algebra $T_s(K) \otimes_K (KA_2)$ is of finite representation type if and only if $s \leq 4$.*

(b) *If $B = K\widehat{Q}$ is a Nakayama algebra with $J^2(B) = 0$, then $T_3(B)$ is of finite representation type.*

Proof. Statement (a) follows from the isomorphisms $T_s(KA_2) \cong T_2(KA_s)$, $T_s(K) \otimes_K (KA_2) \cong T_s(KA_2)$ and [AR, LS].

In case (b) the quiver $T_3(\widehat{Q})$ of $T_3(B)$ is of the form

$$\begin{tikzcd}
\cdots & \bullet \arrow[r, "\alpha_1"] \arrow[d] & \bullet \arrow[r, "\alpha_2"] \arrow[d] & \bullet \arrow[r, "\alpha_3"] \arrow[d] & \bullet \arrow[d] & \cdots \\
\cdots & \bullet \arrow[r, "\alpha_1'"] \arrow[d] & \bullet \arrow[r, "\alpha_2'"] \arrow[d] & \bullet \arrow[r, "\alpha_3'"] \arrow[d] & \bullet \arrow[d] & \cdots \\
\cdots & \bullet \arrow[r, "\alpha_1''"] & \bullet \arrow[r, "\alpha_2''"] & \bullet \arrow[r, "\alpha_3''"] & \bullet & \cdots
\end{tikzcd}$$

with $\alpha_{i+1}\alpha_i \in I$, $\alpha'_{i+1}\alpha'_i \in I$, $\alpha''_{i+1}\alpha''_i \in I$, $i \in Q_0$, where all squares commute. In view of Corollary 2.3, Q must be an oriented cycle or $Q = A_n$ for some n. In the case $Q = A_n$, $T_3(\widehat{Q})$ does not contain any subquiver $\widehat{R}$ such that the algebra $K(\widehat{R})$ is concealed (see [R] for the definition of a concealed algebra) and $T_3(K\widehat{Q})$ is of finite representation type. If Q is a cycle then $T_3(\widehat{Q})$ is a cylinder and has a Galois covering $T_3(\widetilde{Q})$ (where $\widetilde{Q}$ is a Galois covering of $\widehat{Q}$) such that $T_3(\widetilde{Q})$ does not contain infinite lines without relations and any of its finite subquivers is contained in a subquiver of the form $T_3(A_m)$ (for some natural m). Hence $T_3(K\widehat{Q})$ is of finite representation type [DS1]. ∎

Suppose that $\widehat{Z} = (Z, J)$ is a bound quiver. The figure

$$\bullet \xrightarrow{\gamma_1} \bullet \xrightarrow{\gamma_2} \bullet \cdots \bullet \xrightarrow{\gamma_{n-1}} \bullet \xrightarrow{\gamma_n} \bullet$$

where $\gamma_1, \ldots, \gamma_n$ are composable arrows from Z_1 means that $\gamma_n \circ \gamma_{n-1} \circ \ldots \circ \gamma_2 \circ \gamma_1 \in J$ and $\gamma_{n-1} \circ \ldots \circ \gamma_2 \circ \gamma_1 \notin J$, $\gamma_n \circ \gamma_{n-1} \circ \ldots \circ \gamma_2 \notin J$.

We say that the quiver $\widehat{Z}$ has the *property* $(*)$ if either Z is an oriented cycle or $Z = A_s$ (for some natural number s) and for any composable arrows α_2, α_3 in Z_1 such that $\alpha_3\alpha_2 \notin J$ we have $\alpha_2\alpha_1 \in J$ and $\alpha_4\alpha_3 \in J$ for any sequence

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \xrightarrow{\alpha_4} \bullet$$

of composable arrows in Z_1.

It is easy to see that $\widehat{Z}$ has the property $(*)$ if and only if $\widehat{Z}$ is a factor quiver of

$$\alpha \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \quad \cdots \quad \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \omega$$

(possibly $\alpha = \omega$), or equivalently, if and only if $\widehat{Z}$ is either a factor of some oriented cycle or a factor quiver of A_s (for some natural s) such that $Z_1^3 \subseteq J$ and $\widehat{Z}$ does not contain $\bullet \to \bullet \to \bullet \to \bullet$ (so J is generated by paths of length 2).

Theorem 2.5. *Suppose that $\widehat{Q} = (Q, I)$ is a connected and simply connected bound quiver. Then*

(a) *The algebra $K\widehat{Q} \otimes T_5(K) \cong T_5(K\widehat{Q})$ is of tame representation type if and only if $\widehat{Q} = A_2$.*

(b) $K\widehat{Q} \otimes T_4(K) \cong T_4(K\widehat{Q})$ *is of tame representation type if and only if* $\widehat{Q} = A_n$ (*for some natural number* $n \geq 3$) *with* $J^2(K\widehat{Q}) = 0$.

(c) $K\widehat{Q} \otimes T_3(K) \cong T_3(K\widehat{Q})$ *is of tame representation type if and only if one of the following conditions holds*:

(i) $Q = A_3^*$ *or* $Q^{\mathrm{op}} = A_3^*$ (see Lemma 2.2).

(ii) *There exists* $n \geq 1$ *such that* $\widehat{Q} = A_n$ *and* $\widehat{Q}$ *has the property* $(*)$.

P r o o f. (a) If $Q = (\bullet)$, then $T_s(KQ) \cong T_s(K)$ is of finite representation type. If $\widehat{Q}$ contains

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet$$

then $T_5(\widehat{Q})$ contains

$$\begin{tikzcd}
 & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] \arrow[d] & \bullet & \\
\bullet & &
\end{tikzcd}$$

(even in the case $\alpha_2\alpha_1 \in I$), which is of wild representation type as a quiver of a one-point coextension by a simple injective module of a concealed tame algebra of type $\widetilde{E}_7$ [R, p. 130]. The tameness of $T_5(A_2) \cong T_2(A_5)$ has been established in [L1].

(b) Suppose $\widehat{Q}$ contains either A_3^* or $(A_3^*)^{\mathrm{op}}$. Hence $T_4(\widehat{Q})$ or $T_4(\widehat{Q})^{\mathrm{op}}$ contains

$$\begin{tikzcd}
 & & \bullet \arrow[d] \\
 & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \\
 & \bullet &
\end{tikzcd}$$

which is of wild representation type as a one-point extension by a projective module of a concealed quiver of type $\widetilde{D}_5$. In a similar way one can show that if $\widehat{Q}$ contains A_3 then $T_4(\widehat{Q})$ is of wild representation type.

Assume now that for the quiver $\widehat{Q} = (A_n, I)$ we have $Q_1^2 \subseteq I$. Then $T_4(\widehat{Q})$ is the quiver

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

with four rows and n columns and with the following relations:

(i) The squares are commutative.

(ii) The composition of any pair of composable horizontal arrows is zero.

If $n = 3$, then $T_4(Q)$ contains exactly one concealed quiver

$$\begin{tikzcd}
 & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
 & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet &
\end{tikzcd}$$

(of type $\widetilde{D}_7$) and $T_4(\widehat{Q})$ is of tame representation type as a finite extension and a coextension of the above quiver [R, p. 130]. If $\operatorname{card}(Q_0) \geq 4$, then $T_4(\widehat{Q})$ contains a finite number of concealed quivers each of them isomorphic to the above one or to the quiver

$$\begin{tikzcd}
& & \bullet \arrow[r] \arrow[d] & \bullet \\
& \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
& \bullet \arrow[r] \arrow[d] & \bullet & \\
\bullet \arrow[r] & \bullet & &
\end{tikzcd}$$

(of type $\widetilde{E}_7$). The reader can show, applying the method of a one-point (co-) extension, that the algebra $T_4(K\widehat{Q})$ is a finite extension and a finite coextension of tubular algebra. Ringel has studied the representation type of tubular algebras obtained by extensions by simple regular modules [R, p. 228]. The above tubular algebra is of type (2, 3, 6) in Ringel notation and it is of tame representation type.

(c) Suppose that $\widehat{Q}$ contains A_3^* or $(A_3^*)^{\mathrm{op}}$. If $\widehat{Q}$ contains the quiver

$$\bullet \xleftarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet$$

with $\gamma\beta \in I$, then $T_3(\widehat{Q})$ or $T_3(\widehat{Q})^{\mathrm{op}}$ contains the quiver

$$\begin{tikzcd}
\bullet \arrow[d] & & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet &
\end{tikzcd}$$

(of type $\widetilde{\widetilde{E}}_7$). Hence $T_3(\widehat{Q})$ is of wild representation type. If Q (or Q^{op}) is equal to A_3^*, one can show that the algebra $T_3(K\widehat{Q})$ is of tame representation type as a tubular algebra with two concealed subalgebras with quivers of the form [R, p. 228]

$$\begin{tikzcd}
\bullet \arrow[d] & & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[d] \\
\bullet & & \bullet
\end{tikzcd}$$

Assume now that $\widehat{Q}$ contains neither A_3^* nor $(A_3^*)^{\mathrm{op}}$. If $\widehat{Q}$ contains

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet$$

with $\beta\alpha \notin I$, $\gamma\beta \notin I$, then $T_3(\widehat{Q})$ is of wild type, because it contains

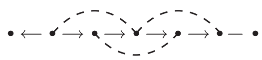
$$\begin{tikzcd}
& \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet & \\
\bullet \arrow[r] & \bullet & &
\end{tikzcd}$$

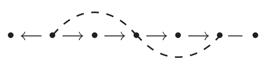
(of type $\widetilde{\widetilde{D}}_6$). If $\widehat{Q}$ is the quiver of a Nakayama algebra and satisfies the condition $(*)$, then one can show that $T_4(K\widehat{Q})$ is an iterated tubular algebra of type (2, 3, 6) and it is of tame representation type. ∎

3. Tameness of $B \otimes C$ in the general case. Throughout we assume that the algebras $B = K\widehat{P}$, $C = K\widehat{Q}$ are not hereditary Nakayama (i.e. neither B nor C is isomorphic to the algebra $T_n(K)$, see (1.4)), because the case when one of the quivers $\widehat{P}$, $\widehat{Q}$ is equal to A_n was studied in the preceding section.

We consider the following quivers:

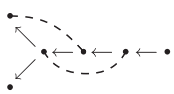
(1) 

(2) 

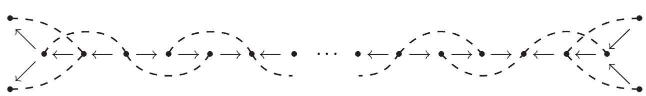
(3) 

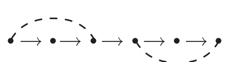
(4) 

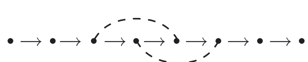
(5) 

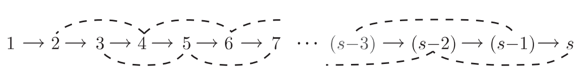
(6) 

(7) 

(8) 

(9) 

(10) 

(11) 

where $\bullet - \bullet$ means either $\bullet \longrightarrow \bullet$ or $\bullet \longleftarrow \bullet$ and a dotted curve over (under) an oriented path means that the composition of the arrows from that path lies in I.

We say that the quiver $\widehat{Z}$ has the *property* $(**)$ if either Z is an oriented cycle or $Z = A_s$ (for some natural number s) and for any composable arrows α_3, α_4 in Z_1 such that $\alpha_4\alpha_3 \notin J$, we have $\alpha_2\alpha_1 \in J$, $\alpha_3\alpha_2 \in J$, $\alpha_5\alpha_4 \in J$, $\alpha_6\alpha_5 \in J$ for any sequence

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \xrightarrow{\alpha_4} \bullet \xrightarrow{\alpha_5} \bullet \xrightarrow{\alpha_6} \bullet$$

of composable arrows in $\widehat{Z}$.

Note that $\widehat{Z}$ has the property $(**)$ if and only if $\widehat{Z}$ is a factor quiver of

(possibly $\alpha = \omega$).

One can see that $(**)$ implies $(*)$.

Definition 3.1. The quiver $\widehat{P} \otimes \widehat{Q}$ is called *weakly sincere* if there exists an indecomposable representation of $\widehat{P} \otimes \widehat{Q}$ such that its support is not contained in a tensor product $\widehat{R} \otimes \widehat{T}$, where $\widehat{R} \subseteq \widehat{P}$, $\widehat{T} \subseteq \widehat{Q}$ are subquivers and $\widehat{R} \neq \widehat{P}$ or $\widehat{T} \neq \widehat{Q}$.

It follows that a non-weakly sincere bound quiver $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if $\widehat{R} \otimes \widehat{T}$ is of tame representation type for any subquiver $\widehat{R}$ of $\widehat{P}$ and any subquiver $\widehat{T}$ of $\widehat{Q}$ (see [D]). Therefore, the study of tame algebras $B \otimes_K C$ reduces to the study of weakly sincere tensor product bound quiver algebras.

We recall that in this paper finite representation type algebras are assumed to be tame.

Theorem 3.2. *Suppose $\widehat{P} = (P, L)$, $\widehat{Q} = (Q, I)$ be bound quivers such that the tensor product $\widehat{P} \otimes \widehat{Q}$ is weakly sincere. Assume that $B = K\widehat{P}$, $C = K\widehat{Q}$ are connected and simply connected bound quiver K-algebras non-isomorphic to the algebra $T_n(K)$ for any $n \geq 1$.*

(a) *If B and C is not a Nakayama algebra, then the following conditions are equivalent*:

(i) *$B \otimes_K C$ is of tame representation type*;

(ii) $\dim_K B = \dim_K C = 5$;

(iii) *$\widehat{P}$ and $\widehat{Q}$ is isomorphic either to A_3^* or to $(A_3^*)^{\mathrm{op}}$.*

(b) *Suppose that $C = K\widehat{Q}$ is a Nakayama algebra and B is not. Then $B \otimes_K C$ is of tame representation type if and only if one of the following conditions holds*:

(i) *$J^2(C) = 0$, $\dim_K C \geq 9$, $\dim_K B \geq 6$ and $\widehat{P}$ or $\widehat{P}^{\mathrm{op}}$ is the quiver* (1).

(ii) *$J^2(C) = 0$, $6 \leq \dim_K C \leq 8$, $\dim_K B \geq 6$ and $\widehat{P}$ is a factor quiver of the quiver* (2).

(iii) $J^2(C) = 0$, $\dim_K C \leq 5$, $\dim_K B \geq 6$ *and* $\widehat{P}$ *is a factor quiver of one of the quivers* $(2), \ldots, (8)$.

(iv) $\dim_K B \leq 5$ *and* $\widehat{Q}$ *has the property* $(**)$.

(c) *Suppose that* B *and* C *are Nakayama algebras.*

(c1) *If* $J^3(B) \neq 0$, *then the algebra* $B \otimes_K C$ *is of tame representation type if and only if* $J^2(C) = 0$, $\dim_K C = 5$ *and* $\widehat{P}$ *is a factor quiver of one of the quivers* (9), (10).

(c2) *If* $J^3(B) = 0$, $J^3(C) = 0$ *and* $\widehat{P}$ *contains* $\bullet \to \bullet \to \bullet \to \bullet$, *then* $B \otimes_K C$ *is of tame representation type if and only if one of the following conditions holds*:

(i) $\dim_K B \geq 10$, $J^2(C) = 0$ *and* $\dim_K C = 5$.

(ii) $\dim_K B \leq 9$ *and* $J^2(C) = 0$.

(c3) *If both* $\widehat{P}$ *and* $\widehat{Q}$ *have the property* $(*)$ *and* $\widehat{P}$ *does not have the property* $(**)$, *then* $B \otimes_K C$ *is of tame representation type if and only if* $J^2(C) = 0$.

(c4) *If* $J^2(B) \neq 0$, $J^2(C) \neq 0$, *and both* $\widehat{P}$ *and* $\widehat{Q}$ *have the property* $(**)$, *then* $B \otimes_K C$ *is of tame representation type if and only if one of the following conditions holds*:

(i) *Each of the quivers* $\widehat{P}$, $\widehat{Q}$ *is of the form* (11) (*for some natural numbers* s *and* s' *respectively*).

(ii) $\dim_K B = \dim_K C = 8$.

(Observe that the condition (c4; ii) for the Nakayama algebras means that $\widehat{P}$ or $\widehat{P}^{\mathrm{op}}$ is equal to the quiver (11) for $s = 4$ and $\widehat{Q}$ or $\widehat{Q}^{\mathrm{op}}$ is equal to the quiver (11) for $s = 4$.)

P r o o f. The division into types of algebras and the order of the formulation of the theorem agree with the table below. One can see that up to duality (see Lemma 1.2(e)) and up to the order of tensor products (Lemma 1.2(a)) all cases of pairs of bound quivers providing tameness of their tensor product quivers are mentioned in the table and all cases of pairs of bound quiver algebras providing tameness of their tensor products are mentioned in Theorem 3.2.

The letters (A), (B), ..., (F) in the table designate the following statements:

(A): $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if $\widehat{P}$ is of the form (1) (and in this case $\widehat{P} \otimes \widehat{Q}$ is of finite representation type).

(B): $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if $\widehat{P}$ is a factor quiver of the quiver (2).

(C): $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if $\widehat{P}$ is a factor quiver of one of the quivers (2), ..., (8).

D): $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if $\widehat{P}$ is a factor quiver of one of (9) or (10).

(E): $\widehat{P} \otimes \widehat{Q}$ is of tame representation type if and only if either each of $\widehat{P}$, $\widehat{Q}$ is a factor quiver of (11) or $\widehat{P} = \widehat{Q}^{\mathrm{op}} = \bullet \to \bullet \to \bullet \to \bullet$ or $\widehat{P}^{\mathrm{op}} = \widehat{Q} = \bullet \to \bullet \to \bullet \to \bullet$.

(F): If we divide this case into subcases: $\mathrm{card}(P_0) \geq 5$, $\mathrm{card}(P_0) = 4$, $\mathrm{card}(P_0) = 3$, then the representation type of each subcase is determined by statements (A), (B), (C) respectively (after interchanging $\widehat{P}$ and $\widehat{Q}$ wherever they occur).

Statement (F) follows from Lemma 1.2(e), (a) and from the fact that for any Nakayama algebra $K\widehat{Z}$, the dual algebra $(K\widehat{Z})^{\mathrm{op}}$ is Nakayama (see Corollary 2.3) and if $J^2(K\widehat{Z}) = 0$, then $\widehat{Z} = \widehat{Z}^{\mathrm{op}}$.

T A B L E. Type of $\widehat{P} \otimes \widehat{Q}$

		$\widehat{Q} \supsetneq A_3^*$	$\widehat{Q} = A_3^*$	N				
				$(*)$ holds for $\widehat{Q}$ and $(**)$ does not	$(**)$ holds for $\widehat{Q}$, $Q_1^2 \not\subseteq I$	$Q_1^2 \subseteq I$, $\mathrm{card}(Q_0) = s$		
						$s \geq 5$	$s = 4$	$s = 3$
$\widehat{P} \supsetneq A_3^*$ or $\widehat{P} \supsetneq (A_3^*)^{\mathrm{op}}$		WT	WT	WT	WT	(A)	(B)	(C)
$\widehat{P} = A_3^*$ or $\widehat{P} = (A_3^*)^{\mathrm{op}}$		WT	TT	WT	TT	TT	TT	TT
N	$P_1^3 \not\subseteq L$	WT	WT	WT	WT	WT	WT	(D)
	$\widehat{P} \supsetneq \bullet \to \bullet \to \bullet \to \bullet$ and $P_1^3 \subseteq L$	WT	WT	WT	WT	WT	WT	TT
	$\widehat{P} = \bullet \to \bullet \to \bullet \to \bullet$	WT	WT	WT	WT	TT	TT	TT
	$(*)$ holds for $\widehat{P}$, $(**)$ does not	WT	WT	WT	WT	TT	TT	TT
	$(**)$ holds for $\widehat{P}$, $P_1^2 \not\subseteq L$	WT	WT	WT	(E)	TT	TT	TT
	$P_1^2 \subseteq L$	(F)	TT	TT	TT	TT	TT	TT

“N” stands for “Quivers of Nakayama algebras”.

In the table, TT and WT means tame and wild representation type respectively.

The determination of the representation type of any tensor product quiver from the table is obtained by the use of the one point (co-) extension method [R].

(a) Assume that B, C are not Nakayama algebras. Then the quivers $\widehat{P}$, $\widehat{Q}$ contain A_3^* or $(A_3^*)^{\mathrm{op}}$. Suppose $\widehat{P}$ and $\widehat{Q}$ contains A_3^*, and $\widehat{P} \neq A_3^*$. If $\widehat{P}$ contains the quiver (1), then $\widehat{P} \otimes \widehat{Q}$ is of wild representation type, because it contains

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[r] & \bullet & \\
\bullet \arrow[u] \arrow[d] & & \bullet \arrow[u] \arrow[d] \arrow[r] & \bullet \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet &
\end{tikzcd}$$

which is a wild tree-quiver of type $\widetilde{\widetilde{A}}_8$.

If $\widehat{P}$ contains one of the quivers:

(3.3) $$\bullet \leftarrow \bullet \rightarrow \bullet - \bullet \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \bullet & \\ \bullet \arrow[r, dash] & \bullet \arrow[u] \arrow[r] & \bullet \end{tikzcd}$$

then similarly one can show that (even if $P_1^2 \subseteq L$) $\widehat{P} \otimes \widehat{Q}$ is also of wild representation type. Hence $P = A_3^*$. On the other hand, either $\widehat{Q} = A_3^*$ or $\widehat{Q} = (A_3^*)^{\mathrm{op}}$. This is equivalent (for non-Nakayama algebras) to $\dim_K B = \dim_K C = 5$.

Assume now that $\widehat{P} = A_3^* = \widehat{Q}$ (or $\widehat{Q}^{\mathrm{op}} = \widehat{P} = A_3^*$). One can prove by the one-point extension method [R] that $\widehat{P} \otimes \widehat{Q}$ is the quiver of a tubular algebra of type (2, 4, 4) and hence it is of tame representation type.

(b) First observe that a Nakayama bound quiver algebra $C = K\widehat{Q}$ with $J^2(C) = 0$ is of dimension s if and only if $s = 2n - 1$ for $n = \mathrm{card}(Q_0)$.

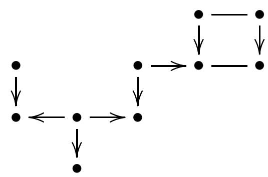
(i) Suppose $\widehat{P} \supseteq A_3^*$ and $\dim_K C \geq 9$ ($\Leftrightarrow$ $\mathrm{card}(Q_0) \geq 5$). In much the same way as above one can prove that the tameness of $B \otimes C$ implies $\widehat{P} = (1)$.

Assume now that $\widehat{P} = (1)$ and $Q_1^2 \subseteq I$. Then $\widehat{P} \otimes \widehat{Q}$ contains no quiver of a concealed algebra and it is of finite representation type.

(ii) Suppose $\widehat{P} \supseteq A_3^* \neq \widehat{P}$ and $\mathrm{card}(Q_0) = 4$. If $\widehat{P}$ contains one of the quivers (3.3), then $\widehat{P} \otimes \widehat{Q}$ contains a subquiver of wild representation type. For example, if $\widehat{P}$ contains the last quiver of (3.3), then $\widehat{P} \otimes \widehat{Q}$ contains a subquiver of the form

$$\begin{tikzcd}
 & & \bullet \arrow[d] & \\
 & & \bullet & \bullet \arrow[d] \\
 & \bullet \arrow[ur] \arrow[rr] \arrow[d] & & \bullet \\
\bullet \arrow[r] \arrow[d] & \bullet & & \\
\bullet & & &
\end{tikzcd}$$

which is a wild quiver of type $\widetilde{\widetilde{E}}_6$. Assume now that $\widehat{P}$ contains the quiver (1) and $\widehat{P} \neq (1)$. Suppose $\widehat{P}$ contains $\bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet - \bullet$. Then $\widehat{P} \otimes \widehat{Q}$ contains a subquiver of wild representation type of the form



Hence $\widehat{P}$ contains $\bullet\leftarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet$. If $\widehat{P}$ is not equal to the latter quiver, then $\widehat{P}$ does not contain $\bullet\leftarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet-\bullet$ (for any orientation of the last arrow). Hence $\widehat{P}$ contains $\bullet\leftarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet$ and if it is not equal to the latter, then $\widehat{P}$ contains either (2) or $\widehat{U}$: $\bullet\leftarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\omega$. One can show that if $\widehat{P}$ contains (2), then $\widehat{P}=(2)$. It is easy to check that the quiver $\widehat{U}\otimes\widehat{Q}$ is not weakly sincere. If $\widehat{P}\neq\widehat{U}$, then by the above part of the proof we observe that (for $\widehat{P}\otimes\widehat{Q}$ of tame representation type) there is only one possibility to add some arrow to the quiver $\widehat{U}$, that is, to add an arrow reaching the point ω. Then $\widehat{P}\otimes\widehat{Q}$ also will not be weakly sincere. Hence $\widehat{P}=(2)$.

The proof of tameness of the tensor product $\widehat{P}\otimes\widehat{Q}$ for $\widehat{P}=(2)$ and the quiver $\widehat{Q}$ of the form $\bullet\rightarrow\bullet\rightarrow\bullet\rightarrow\bullet$ is left to the reader.

(iii) Now, if $J^2(C)=0$ and $\dim_K C\leq 5$ (and $C\not\cong T_2(K)$), then $\operatorname{card}(Q_0)=3$ and $\dim_K C=5$. By a discussion similar to the above (but with much more combinatorics) one can show that for $B\otimes_K C$ of tame representation type the quiver $\widehat{P}$ is of one of the forms (2), …, (8).

If $\widehat{P}$ is one of the quivers (2), …, (8) and $J^2(C)=0$ with $\operatorname{card}(Q_0)=3$ then one can show that the algebra is of tame representation type.

Similarly for the case (b; iv) with $\dim_K B\leq 5$ (i.e. $\widehat{P}=A_3^*$ or $\widehat{P}^{\mathrm{op}}=A_3^*$) one can prove that $B\otimes_K C$ is of tame representation type if and only if $\widehat{Q}$ has the property (∗∗).

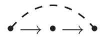
Similarly one can easily prove part (c) of the theorem.

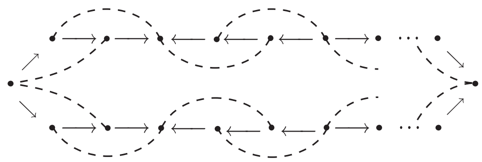
It might seem that we have to add a next case with both B, C having the property (∗∗) and $J^2(C)=0$, but that case is already solved in (c3), because (∗∗) implies (∗). This finishes the proof. ■

Now we present results for the general case with at least one of $\widehat{P}$, $\widehat{Q}$ not simply connected.

Proposition 3.4. *Suppose $\widehat{P}$, $\widehat{Q}$ are connected finite bound quivers such that $\widehat{P}\otimes\widehat{Q}$ is of tame type, and $\widetilde{P}$, $\widetilde{Q}$ are their simply connected coverings such that $\widetilde{P}\otimes\widetilde{Q}$ contains an infinite* (*free*) *line. Then one of the following conditions holds*:

(a) *The quivers $\widehat{P}$, $\widehat{Q}$ are Nakayama with $Q_1^2\subseteq I$ and the condition* (∗∗) *holds for $\widehat{P}$* (*or $P_1^2\subseteq L$ and* (∗∗) *holds for $\widehat{Q}$*);

(b) $\widehat{Q}$ *is the quiver*  (*with* $Q_1^2 \subseteq I$), *and* $\widehat{P}$ *is of the form*



(*with* $P_1^2 \subseteq L$).

P r o o f. By considering cases (of pairs) of quivers from the table of Theorem 3.2 as parts of some larger quivers (which are weakly sincere and of tame type) one can see that it is possible to construct an infinite line only for the above cases. ∎

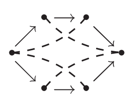
THEOREM 3.5. *Suppose that the finite connected bound quivers* $\widehat{P}$, $\widehat{Q}$ *are such that* $\widehat{P} \otimes \widehat{Q}$ *is of tame type*, *and* $\widetilde{P}$, $\widetilde{Q}$ *are their simply connected coverings such that* $\widetilde{P} \otimes \widetilde{Q}$ *contains an infinite line.*

(i) *If* $\widehat{P}$, $\widehat{Q}$ *are Nakayama quivers*, $\widehat{P}$ *satisfies* $(**)$ *and* $P_1^2 \nsubseteq L$, *then* $\widehat{P} \otimes \widehat{Q}$ *is not of polynomial growth* (see [S1]).

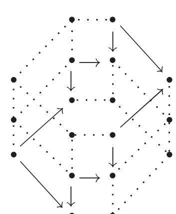
(ii) *If* $\widehat{P}$, $\widehat{Q}$ *are as in Proposition* 3.4(b), *then* $\widehat{P} \otimes \widehat{Q}$ *is of tame type.*

P r o o f. (i) The representation type of the bound quiver isomorphic to $\widehat{P} \otimes \widehat{Q}$ has been determined in [S1] (Prop. 3).

(ii) Since the covering $\widetilde{P} \otimes \widetilde{Q}$ of $\widehat{P} \otimes \widehat{Q}$ is also a covering of $\widehat{T} \otimes \widehat{Q}$, where $\widehat{T} = (T, M)$ is of the form



with $T_1^2 \subseteq M$, it is enough to show that the quiver $\widehat{T} \otimes \widehat{Q}$ is of tame type. The bound quiver $\widehat{T} \otimes \widehat{Q}$ is a finite extension and coextension of the quiver $\widetilde{A}_{10}$:



Hence it is of tame type [R]. ∎

To summarize, in the paper the representation type of tensor product algebras $B \otimes_K C$ is determined with the exception of the following cases:

(i) B or C is isomorphic to one of the algebras: K, $T_2(K)$,

(ii) $B = K\widehat{P}$, $C = K\widehat{Q}$, where $\widehat{P}$ and $\widehat{Q}$ are considered in statement (i) of Theorem 3.5.

The representation type of the tensor product algebra $B \otimes_K C$ for the algebras from (ii) is unknown to the author. The characterization of the bound quiver algebras A such that the algebra $A \otimes_K T_2(K)$ is of tame representation type contains long lists (to be published).

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