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The theory of dual groups

by

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Abstract. We study the $L_{\infty,\omega}$-theory of sequences of dual groups and give a complete classification of the $L_{\infty,\omega}$-elementary classes by finding simple invariants for them. We show that nonstandard models exist.

1. Introduction. In this paper we begin a study of dual groups and duality from a logical point of view. Recall that if A is an Abelian group, its *dual* is the group $A^* = \mathrm{Hom}(A, \mathbb{Z})$, where $\mathbb{Z}$ denotes the group of integers. There is a canonical map σ_A from a group A to its double dual $A^{**} = (A^*)^*$ taking an element a to evaluation at a. The group A is said to be *reflexive* if σ_A is an isomorphism.

In his lecture series [7], Reid was interested in the structure of dual groups, and asked whether every dual group is reflexive. In recent years there have been several different constructions showing otherwise ([3], [5], [4]), and some progress has been made on other questions about their structure ([4], [8]). Here we continue this investigation by analyzing the theory of dual groups.

In fact, we will not concentrate literally on the theory of dual groups, but rather on the theory of structures derived naturally from a given dual group. For example:

Definition 1.1. A *standard short sequence* is a 4-sorted structure

$$\mathfrak{A} = (A_i\ (i < 3);\ \langle -, - \rangle;\ \sigma;\ \mathbb{Z})$$

for the language $(\boldsymbol{\langle -, - \rangle}, \boldsymbol{\sigma})$, where A_0 is a dual group, $A_{i+1} = A_i^*$, σ is the natural map from A to A^{**}, for $(a, f) \in A_i \times A_{i+1}$, $\langle a, f \rangle = f(a)$ (for $i = 0, 1$), and $\mathbb{Z}$ is a copy of the Abelian group of integers. The sequence $\mathfrak{A}$ will be called the *short sequence associated with* A_0.

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It is easy to check that for any dual group A, the map σ_A is an embedding (such A are said to be *torsionless*), so that A may be seen as a subgroup of A^{**}, which is itself a subgroup of $\mathbb{Z}^{A^*}$. It follows that A is $\aleph_1$-free, that is, every countable subgroup is free. (In fact, the map σ_A is even a pure embedding, so that A is separable.) Thus A is $L_{\infty,\omega}$-equivalent to a free group (see [1]). So in the language $L_{\infty,\omega}$, one can only distinguish between dual groups of differing rank. (Here and below, the rank of a group will take values in the set $\{0,1,2,\ldots,\infty\}$.) Hence we cannot, for example, distinguish between a nonreflexive and a reflexive dual group (both of infinite rank), an important structural division among dual groups. In the structure defined in Definition 1.1 one can do this, however; we will consider these structures in Section 2 and show exactly what may be said in the language $L_{\infty,\omega}$.

The important gain made by considering these extended structures in place of dual groups is that we are able to speak about how a dual group sits inside its double dual; this suggests extending the idea by considering the following structures:

Definition 1.2. A *standard* (*long*) *sequence* is an $\omega+1$-sorted structure

$$\mathfrak{A} = (A_n\ (n\in\omega);\ \langle -,-\rangle;\ \sigma_n\ (n\in\omega);\ \mathbb{Z})$$

for the language $(\langle \boldsymbol{-}, \boldsymbol{-}\rangle, \boldsymbol{\sigma}_n\ (n\in\omega))$, in which A_0 is a dual group, $A_{i+1} = A_i^*$, σ_n is the natural map from A_n to A_{n+2}, and for $(a,f)\in A_i\times A_{i+1}$, $\langle a,f\rangle = f(a)$ (for $i\in\omega$). We shall say that the sequence $\mathfrak{A}$ is *associated with*, or *generated by*, A_0.

Sometimes we will subscript the maps of a long sequence when it is necessary to distinguish between the maps of two different sequences.

In Section 3 we will give invariants for the $L_{\infty,\omega}$-theories of long sequences, which will describe exactly what the theory can say. But we also show that the nontrivial theories are undecidable (Section 4). Our work in Section 3 will enable us to show that there are nonstandard, even countable, models of the $L_{\infty,\omega}$-theory of long sequences, structures related to the "models" of [5].

For future reference, note that as we have already done in the definitions, we shall not explicitly include the language of Abelian groups in our structures. All universes of structures discussed should be regarded as Abelian groups. By "group" we will mean "Abelian group."

Suppose we have a pair of groups A_1 and A_2 along with a bilinear map $\langle -,-\rangle : A_1\times A_2\to\mathbb{Z}$. If H is a subgroup of A_1, then $H^\perp$ will denote the subgroup of A_2 consisting of elements which annihilate H,

$$H^\perp = \{f\in A_2 \mid \langle a,f\rangle = 0\ \forall a\in H\}\,.$$

(When there is a third group A_3 also acting on A_1 the location of $H^\perp$ will be made clear from context.)

The book [4] contains most of what is known about dual groups. Any concepts we leave undefined here may be found there.

2. Short sequences of dual groups. In this section we show what may be said in $L_{\infty,\omega}$ about short sequences of dual groups. The methods used here will be important when we consider long sequences in Section 3. Our main result is the following

THEOREM 2.1. *Suppose that A_0 and B_0 are dual groups. Then* $\mathrm{rank}(A_0) = \mathrm{rank}(B_0)$ *and* $\mathrm{rank}(A_2/\sigma_0[A_0]) = \mathrm{rank}(B_2/\sigma_0[B_0])$ *iff the associated short sequences $\mathfrak{A}$ and $\mathfrak{B}$ are $L_{\infty,\omega}$-equivalent.*

Proof. Towards a proof of "necessity", but also with an axiomatization of the theory of short sequences in mind, we isolate some facts which are true of any short standard sequence $\mathfrak{A}$. The reader may easily verify (see [1] if necessary) that all of the following statements may be formulated in $L_{\infty,\omega}$.

AXIOM (i). The groups A_i are $\aleph_1$-free, for $i = 0, 1, 2$. (In particular, this implies that if $a \in A_i$, there is an element $a' \in A_i$ so that $a = na'$ for some integer n and $\langle a' \rangle$ is a pure subgroup.)

AXIOM (ii). If $\langle a \rangle$ is pure and nonzero in A_i, then there exists $f \in A_{i+1}$ so that $\langle a, f \rangle = 1$ (for $i = 0, 1$).

AXIOM (iii). If $\langle f \rangle$ is pure and nonzero in A_{i+1} then there is $a \in A_i$ so that $\langle a, f \rangle = 1$ (for $i = 0, 1$).

AXIOM (iv). $\langle a - b, f \rangle = \langle a, f \rangle - \langle b, f \rangle$ for all $a, b \in A_i$ and $f \in A_{i+1}$.

AXIOM (v). $\langle a, f - g \rangle = \langle a, f \rangle - \langle a, g \rangle$ for all $a \in A_i$ and $f, g \in A_{i+1}$.

AXIOM (vi). $\langle a, f \rangle = \langle f, \sigma(a) \rangle$. (Notice that Axioms (ii)–(v) imply that σ is a pure embedding.)

AXIOM (vii). If H is a finite rank pure subgroup of $A_2/\sigma_0[A_0]$ then $(A_2/\sigma_0[A_0])/H$ is $\aleph_1$-free.

(Axiom (vii) holds for any standard sequence since $\sigma_0[A_0]$ splits A_2, by Lemma 3.3.)

Notice that the proof of "necessity" is trivial if $\mathrm{rank}(A_0)$ is finite, for then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Suppose then that $\mathrm{rank}(A_0) = \infty$.

We define a set P of isomorphisms between substructures of $\mathfrak{A}$ and $\mathfrak{B}$ and show that it has the back-and-forth property. This suffices to show that $\mathfrak{A}$ and $\mathfrak{B}$ are $L_{\infty,\omega}$-equivalent (see [1]).

The set P consists of isomorphisms $\phi = (\phi_0, \phi_1, \phi_2)$ between *substructures* $\mathfrak{F} = (F_0, F_1, F_2)$ of $\mathfrak{A}$ and $\mathfrak{G} = (G_0, G_1, G_2)$ of $\mathfrak{B}$ with the following properties:

(A) The groups F_i are finite-rank pure subgroups of A_i (for $i = 0, 1, 2$). (And likewise for the G_i.)
(B) The groups F_0 and F_1 have dual bases, that is, bases $\{a_1, \ldots, a_n\}$ and $\{f_1, \ldots, f_n\}$ so that $\langle a_i, f_j\rangle = 0$ if $i \neq j$ and 1 otherwise. (And likewise for the G_i.)
(C) $\sigma[F_0] = F_2 \cap \sigma[A_0]$, and likewise for $\mathfrak{G}$.
(D) $(F_2 + \sigma[A_0])/\sigma[A_0]$ is pure in $A_2/\sigma[A_0]$ and of the same rank as $G_2 + \sigma[B_0]/\sigma[B_0]$ (which is pure in $B_2/\sigma[B_0]$).

Note that (B) implies that $F_0 \oplus F_1^{\perp} = A_0$ and $F_1 \oplus F_0^{\perp} = A_1$ (and likewise for $\mathfrak{B}$).

Assume then that we have an isomorphism ϕ satisfying statements (A) through (D). We must show how to extend ϕ when an element is added to F_0, F_1 or F_2. (The cases for the G_i are symmetric.) The reader should be careful to note that our arguments use only properties implied by Axioms (i) through (vii).

Case I: We pick $a \in A_0 - F_0$.

Write $a = a_0 + a_1$ where $a_0 \in F_0$ and $a_1 \in F_1^{\perp}$, and $a_1 = na_2$ for some integer n, so that $\langle a_2\rangle$ is pure. Since $\sigma[B_0]/(\sigma[B_0] \cap G_2)$ is of infinite rank (by hypothesis), we may choose $b_2 \in G_1^{\perp}$ so that $\langle b_2\rangle$ is pure and $\sigma(b_2) \notin G_2$. Now that we have b_2, pick $f \in F_0^{\perp}$ so that $\langle f, a_2\rangle = 1$ and $g \in G_0^{\perp}$ so that $\langle g, b_2\rangle = 1$ (using Axiom (ii)). Now extend F_0 to $F_0 \oplus \langle a_2\rangle$, F_1 to $F_1 \oplus \langle f\rangle$ and F_2 to $F_2 + \langle\sigma(a_2)\rangle$, and likewise for the G_i. Extend ϕ to ϕ' by letting ϕ' take a_2 to b_2, f to g, and $\sigma(a_2)$ to $\sigma(b_2)$. It is clear that hypotheses (B), (C) and (D) are still met in the extended structures. Obviously requirement (A) still holds for $i = 1, 2$; we check that $F_2 + \langle\sigma(a_2)\rangle$ is pure in A_2. (The case for G_2 is similar.)

Suppose then that $h \in A_2$ and $nh = j + m\sigma(a_2)$ for some $j \in F_2$ and $m \in \mathbb{Z}$. By hypothesis (D) on $\mathfrak{F}$, we know that $h \in F_2 + \sigma[A_0]$, so that $h = k + \sigma(c)$ for some $k \in F_2$ and $c \in A_0$. Then

$$nk - j + n\sigma(c) - m\sigma(a_2) = 0$$

so that $n\sigma(c) - m\sigma(a_2)$ is in F_2, and therefore by hypothesis (C) on $\mathfrak{F}$, is in $\sigma(F_0)$. Thus $n\sigma(c) \in \sigma[F_0 + \langle a_2\rangle]$. Now $\sigma[F_0 + \langle a_2\rangle]$ is a pure subgroup of A_2 by hypothesis (A) on $\mathfrak{F}$, the choice of a_2, and the fact that σ is a pure embedding (Axiom (vi)). Thus $\sigma(c) \in \sigma[F_0 + \langle a_2\rangle]$, so that $h \in F_2 + \langle\sigma(a_2)\rangle$.

Since a is in the domain of ϕ', we are done. This concludes Case I.

For the next case we will need the following lemma, due to Chase [2] (or see Theorem XI.3.2 in [4]).

LEMMA 2.2. *Suppose that A_1 and A_2 are groups, and $\langle -, -\rangle : A_1 \times A_2 \to \mathbb{Z}$ is a bilinear map so that Axioms* (i)–(v) *are satisfied. If $a_1, \ldots, a_n$ are independent elements of A_1 which generate a pure subgroup, and $m_1, \ldots, m_n$*

are integers, then there is an element $f \in A_2$ *so that* $\langle f, a_i \rangle = m_i$ *for* $i = 1, \ldots, n$.

Case II: We pick $f \in A_1 - F_1$.

Write $f = f_0 + f_1$ where $f_0 \in F_1$, $f_1 \in F_0^{\perp}$, and $f_1 = nf_2$ so that $\langle f_2 \rangle$ is pure. Next, pick $a \in F_1^{\perp}$ so that $\langle a, f_2 \rangle = 1$ (so that $\langle a \rangle$ is pure). Now we may engage Case I, finding a $b \in G_1^{\perp}$ corresponding to a. Extend the map ϕ_2 on F_2 to an isomorphism ϕ_2' from $F_2 + \langle \sigma(a) \rangle$ to $G_2 + \langle \sigma(b) \rangle$ in the obvious way (possible since $\sigma(a)$ is independent of F_2, by hypothesis (C)). Now we may apply Lemma 2.2 to find a $g_2 \in B_1$ so that $\langle g_2, \phi_2'(h) \rangle = \langle f_2, h \rangle$ for all $h \in F_2 + \langle \sigma(a) \rangle$. Then $g_2 \in G_0^{\perp}$ and $\langle b, g_2 \rangle = 1$, so that the extension $G_1 \oplus \langle g_2 \rangle$ is pure in B_2. We extend the other groups in the obvious way, and complete the extension of ϕ to an isomorphism between the extended structures, just as in Case I. As above, one can check that requirements (A) to (D) are still satisfied.

Case III: We pick $k \in A_2 - F_2$.

There are two possibilities:

Case III(a): If $k \in F_2 + \sigma_0[A_0]$, we may add the appropriate element to A_0, proceeding as in Case I.

Case III(b): $k \notin F_2 + \sigma_0[A_0]$. Notice that

$$\frac{A_2/\sigma[A_0]}{(F_2 + \sigma[A_0])/\sigma[A_0]}$$

is $\aleph_1$-free, by induction hypothesis (D) and Axiom (vii). So we may choose k_1 in A_2 so that $k + F_2 + \sigma[A_0] = n(k_1 + F_2 + \sigma[A_0])$ for some integer n, and $\langle k_1 + F_2 + \sigma[A_0] \rangle$ is pure in $A_2/(F_2 + \sigma[A_0])$. By hypothesis (D), and theorem hypothesis, we may choose $l_1 \in B_2$ so that $\langle l_1 + G_2 + \sigma[B_0] \rangle$ is pure and nonzero in $B_2/(G_2 + \sigma[B_0])$. Applying Lemma 2.2, we may choose $b \in G_0$ so that $\langle \phi_1(f), \sigma(b) + l_1 \rangle = \langle f, k_1 \rangle$, for all $f \in F_1$. Finally, we extend G_2 to $G_2 + \langle \sigma(b) + l_1 \rangle$, F_2 to $F_2 + \langle k_1 \rangle$, and ϕ to an isomorphism between the extended structures taking k_1 to $\sigma(b) + l_1$. Our choice of k_1 and l_1 ensure that requirement (D) is still met, and the other requirements obviously still hold.

Now we may engage Case III(a) to make the necessary extension to an isomorphism with k in its domain. This completes Case III and the proof of the "necessity" direction.

It is easy to see that two short sequences $\mathfrak{A}$ and $\mathfrak{B}$ which are $L_{\infty,\omega}$-equivalent must satisfy the rank conditions in the hypothesis. ∎

COROLLARY 2.3. *Axioms* (i)–(vii) *axiomatize the* $L_{\infty,\omega}$*-theory of short sequences of dual groups.*

Proof. Suppose $\mathfrak{B}$ is some structure $(B_i\ (i<3);\ \langle -,-\rangle;\ \sigma;\ \mathbb{Z})$ which satisfies Axioms (i)–(vii). Let $\mathfrak{A}$ be a short standard sequence for which $\operatorname{rank}(A_0) = \operatorname{rank}(B_0)$ and $\operatorname{rank}(A_2/\sigma[A_0]) = \operatorname{rank}(B_2/\sigma[B_0])$. (There are many constructions of such groups A_0, see for example [3], [4], [8].) Then we may carry out the argument in the proof of Theorem 2.1 to see that $\mathfrak{B}$ and $\mathfrak{A}$ are $L_{\infty,\omega}$-equivalent. It follows that $\mathfrak{B}$ models the $L_{\infty,\omega}$-theory of short sequences of dual groups. ∎

Since a dual group A_0 is a direct summand (under σ) of its double dual A_2 (Lemma 3.3), it may seem convenient to include in the structure of a short sequence a splitting $\varrho : A_2 \to A_0$, so that $\varrho\sigma = \mathrm{Id}_{A_0}$. Then in the proof of Theorem 2.1, Case III would seem easier to establish. Doing the analogous thing for long sequences will be especially tempting in Section 3. But for this strengthened notion of a short sequence, Theorem 2.1 does not hold:

EXAMPLE 1. Denote by G the torsionless but not separable group $\mathbb{Z}^{(\omega)} + 2\mathbb{Z}^{\omega}$, a subgroup of $\mathbb{Z}^{\omega}$. It is not hard to see that $G^* = \mathbb{Z}^{(\omega)}$. Let A be a torsionless group such that $A^{**}/\sigma[A] \cong G$ ([8]) and let B be a torsionless group such that $B^{**}/\sigma[B] \cong \mathbb{Z}^{(\omega)}$. Now consider the augmented short sequences

$$\mathfrak{A} = (A_i\ (i<3);\ \sigma;\ \varrho_{\mathfrak{A}};\ \mathbb{Z}) \quad \text{and} \quad \mathfrak{B} = (B_i\ (i<3);\ \sigma;\ \varrho_{\mathfrak{B}};\ \mathbb{Z})$$

in which $A_0 = A^*$, $B_0 = B^*$, and $\varrho_{\mathfrak{A}}$ and $\varrho_{\mathfrak{B}}$ are the corresponding restriction maps. (That is, for $a \in A$ and $f \in A_2$, $\varrho_{\mathfrak{A}}(f)(a) = f(\sigma_A(a))$, and likewise for B.)

Then we have $\operatorname{rank}(A_0) = \operatorname{rank}(B_0)$, and since $A_2/\sigma[A_0] \cong G^*$ (Lemma 3.3) (and similarly for $\mathfrak{B}$), $\operatorname{rank}(A_2/\sigma[A_0]) = \operatorname{rank}(B_2/\sigma[B_0])$. But $\mathfrak{A}$ and $\mathfrak{B}$ are not $L_{\infty,\omega}$-equivalent. For the fact that G and $\mathbb{Z}^{(\omega)}$ are torsionless ensures that the subgroup of A_1 (B_1) annihilated by the kernel of $\varrho_{\mathfrak{A}}$ ($\varrho_{\mathfrak{B}}$) is $\sigma_A[A]$ ($\sigma_B[B]$). Thus we can say in $L_{\infty,\omega}$ sentences about $\mathfrak{A}$ and $\mathfrak{B}$ that $A_1/\ker(\varrho_{\mathfrak{A}})^{\perp}$ ($\cong G$) is not separable, while $B_1/\ker(\varrho_{\mathfrak{B}})^{\perp}$ ($\cong \mathbb{Z}^{(\omega)}$) is separable. So $\mathfrak{A}$ and $\mathfrak{B}$ are not $L_{\infty,\omega}$-equivalent.

We will return to these observations in Section 4.

3. Sequences of dual groups. Here we consider the $L_{\infty,\omega}$-theories of the long sequences defined in Definition 1.2, and prove a result analogous to Theorem 2.1. We will need the following definition from [4].

DEFINITION 3.1. Given a dual group A, by induction on n define $D^0(A) = A$, and $D^{n+1}(A) = (D^n(A))^{**}/\sigma[D^n(A)]$. Then the *length rank* of A is the pair (n,m) of elements of $\omega \cup \{\infty\}$ where $n = \sup\{k \mid D^k(A) \neq 0\}$, $m = \infty$ if $n = \infty$ and $\operatorname{rank}(D^n(A))$ otherwise. (We set $\sup\emptyset = 0$.)

THEOREM 3.2. *The length rank of A_0 is equal to the length rank of B_0 iff the associated sequences $\mathfrak{A}$ and $\mathfrak{B}$ are $L_{\infty,\omega}$-equivalent.*

Towards a proof of this we make the following definitions. Given a (long) sequence $\mathfrak{A}$, we define the *quotient sequence* $\mathfrak{A}_1$ with universe $A_{1,n}$ $(n \in \omega)$, by $A_{1,0} = A_2/\sigma_0[A_0]$ and $A_{1,n} = (\sigma_{n-1}[A_{n-1}])^{\perp}$ (a subgroup of A_{n+2}) for $n \geq 1$. We denote the projection map from A_2 to $A_2/\sigma_0[A_0]$ by π and define the bracket map for $\mathfrak{A}_1$ as follows: Given $a \in A_{1,n}$ and $f \in A_{1,n+1}$, we let

$$\langle a, f\rangle_1 = \begin{cases} \langle a, f\rangle_{\mathfrak{A}} & \text{if } n \geq 1, \\ \langle a', f\rangle_{\mathfrak{A}} \text{ (where } \pi(a') = a) & \text{if } n = 0. \end{cases}$$

We claim that with this bracket map, each group $A_{1,n+1}$ may be seen as the dual of $A_{1,n}$. This follows easily from repeated applications of the following (purely categorical) result of [4]:

LEMMA 3.3. *Suppose that A is a dual group*, $A = B^*$. *Then A splits A^{**} via σ_A. That is*, *there is a homomorphism $\varrho : A^{**} \to A$ so that $\varrho\sigma = \mathrm{Id}_A$. Also*, $\ker(\varrho) = (\sigma_B[B])^{\perp}$, *and this kernel is itself a dual group.*

Proof. Let ϱ be the restriction map, that is, $\langle b, \varrho(f)\rangle = \langle \sigma_B(b), f\rangle$ for all $f \in A^{**}$ and $b \in B$. Then $\varrho\sigma = \mathrm{Id}_A$, clearly $\ker(\varrho) = (\sigma_B[B])^{\perp}$, and it is straightforward to show that $(\sigma_B[B])^{\perp} \cong (A^*/\sigma_B[B])^*$. ∎

Applying the lemma also establishes that $A_{1,0}$ is a dual group. Now since each group is the dual of the previous, we may define the σ maps appropriately, to obtain a standard sequence,

$$\mathfrak{A}' = (A_{1,n}\ (n \geq 0);\ \langle -, -\rangle_1;\ \sigma_{1,n}\ (n \geq 0);\ \mathbb{Z}),$$

isomorphic to the sequence associated with $A_2/\sigma_0[A_0]$.

We may extend the definition above, and define by induction the nth quotient model $\mathfrak{A}_n$ for $n \geq 0$ setting $\mathfrak{A}_0$ equal to $\mathfrak{A}$ and $\mathfrak{A}_{n+1}$ equal to the quotient sequence of $\mathfrak{A}_n$. For $n \geq 0$ we will denote the mth group in $\mathfrak{A}_n$ by $A_{n,m}$, the mth σ map by $\sigma_{n,m}$, and the bracket map by $\langle -, -\rangle_n$ (or possibly just $\langle -, -\rangle$ when no confusion is possible).

Let π_n be the projection map linking $\mathfrak{A}_n$ with $\mathfrak{A}_{n+1}$.

With quotient models in hand, we further define the *derived sequence* of a sequence $\mathfrak{A}$. This is an $\omega + 1$-sorted structure

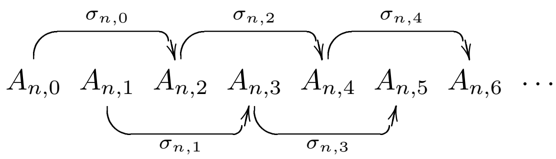
$$\mathcal{A} = (A_{n,i}\ (n \in \omega, i < 3);\ \langle -, -\rangle, \sigma_n, \pi_n, \mathbb{Z})$$

for the language $(\boldsymbol{\langle -, -\rangle};\ \boldsymbol{\sigma}_n\ (n \geq 0);\ \boldsymbol{\pi}_n\ (n \geq 0))$ for which

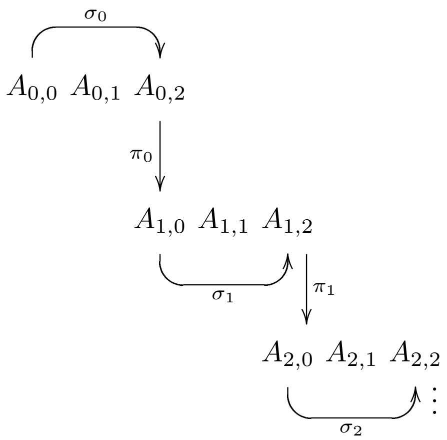
(i) $\sigma_n : A_{n,0} \to A_{n,2}$ has the same action as $\sigma_{n,0}$ in the nth quotient model $\mathfrak{A}_n$.

(ii) $\langle -, -\rangle : A_{n,i} \times A_{n,i+1} \to \mathbb{Z}$ $(i = 0, 1)$ has the same action as $\langle -, -\rangle_n$ restricted to $A_{n,i} \times A_{n,i+1}$ in $\mathfrak{A}_n$.

(iii) $\pi_n : A_{n,2} \to A_{n+1,0}$ has the action described above.



The nth quotient model $\mathfrak{A}_n$ (bracket maps not shown)



The derived sequence $\mathcal{A}$

In the sequel, we will reserve Fraktur letters for structures with the type of long sequences, and calligraphic letters for structures with the type of derived sequences. By the length rank of a structure

$$\mathcal{A} = (A_{n,i}\ (n \in \omega, i < 3);\ \langle -, - \rangle, \sigma_n, \pi_n, \mathbb{Z})$$

we mean the pair (n, m) where n is the greatest integer such that $A_{n,0} \neq \{0\}$ (or ∞ if no such integer exists), and $m = \infty$ if $n = \infty$ or $\operatorname{rank}(A_{n,0})$ otherwise. Thus the derived sequence $\mathcal{A}$ of the sequence $\mathfrak{A}$ generated by a group A_0 of length rank (n, m) is of length rank (n, m).

LEMMA 3.4. *If the derived sequences $\mathcal{A}$ and $\mathcal{B}$ of two models $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Proof. We show how to recover a model $\mathfrak{A}$ from its derived sequence $\mathcal{A}$. For later purposes we will isolate a portion of this proof, and set it in a more general context:

CONSTRUCTION TECHNIQUE. We present a technique for transforming a structure of the type of a derived sequence,

$$\mathcal{A} = (A_{n,i}\ (n \in \omega, i < 3);\ \langle -, - \rangle_{\mathcal{A}}, \sigma_{\mathcal{A},n}, \pi_n, \mathbb{Z}),$$

into a structure with the type of a long sequence,

$$\mathfrak{A} = (A_n\ (n \in \omega);\ \langle -, - \rangle_{\mathfrak{A}};\ \sigma_{\mathfrak{A},n}\ (n \in \omega);\ \mathbb{Z}).$$

The definition of the A_i proceeds by induction on i. First, we let $A_i = A_{0,i}$ $(i = 0, 1, 2)$. Now suppose $i \geq 3$, and we have given the technique for constructing the groups A_j for $j < i$. Let $A_i = A_{i-2} \oplus A'_{i-2}$ where A'_{i-2} is the result of applying the construction technique to the truncated structure

$$\mathcal{A}' = (A_{n,i}\ (n \geq 1, i < 3);\ \langle -, - \rangle_{\mathcal{A}}, \sigma_{\mathcal{A},n}, \pi_n, \mathbb{Z})\,.$$

Define the map $\tau_i : A_i \to A'_{i-2}$ to be π_0 if $i = 2$ and the natural projection if $i > 2$. Thus for example in the following diagram the top row consists of the groups A_i for $0 \leq i \leq 5$ and the bottom row consists of the groups A'_i for $0 \leq i \leq 3$.

$$\begin{array}{cccccc}
A_{0,0} & A_{0,1} & A_{0,2} & \begin{array}{c} A_{0,1} \\ \oplus \\ A_{1,1} \end{array} & \begin{array}{c} A_{0,2} \\ \oplus \\ A_{1,2} \end{array} & \begin{pmatrix} A_{0,1} & & A_{1,1} \\ \oplus & \oplus & \oplus \\ A_{1,1} & & A_{2,1} \end{pmatrix} \\
 & & \downarrow \tau_2 & \downarrow \tau_3 & \downarrow \tau_4 & \downarrow \tau_5 \\
 & & A_{1,0} & A_{1,1} & A_{1,2} & \begin{array}{c} A_{1,1} \\ \oplus \\ A_{2,1} \end{array}
\end{array}$$

We define the bracket map as follows. For $i = 0, 1$ and $(a, b) \in A_i \times A_{i+1}$, we let $\langle a, b \rangle_{\mathfrak{A}} = \langle a, b \rangle_{\mathcal{A}}$. Suppose $i \geq 2$ and that we have shown how to define the bracket map for all $j < i$. Given (a, b) as above, write $b = c + d$ where $c \in A_{i-1}$ and $d \in A'_{i-1}$. Now let

$$\langle a, b \rangle_{\mathfrak{A}} = \langle c, a \rangle_{\mathfrak{A}} + \langle \tau_i(a), d \rangle_{\mathfrak{A}_1}$$

where $\mathfrak{A}_1$ is the model being constructed from $\mathcal{A}'$. Finally, we let $\sigma_{\mathfrak{A},0} = \sigma_{\mathcal{A},0}$ and $\sigma_{\mathfrak{A}_i}$ be inclusion for $i \geq 1$. This completes the construction technique.

We now show that given a long sequence $\mathfrak{A}$, if we apply the construction procedure to the derived sequence $\mathcal{A}$, we obtain a sequence isomorphic to $\mathfrak{A}$. This will establish Lemma 3.4.

Given a standard long sequence $\mathfrak{A}$, a *k-truncation* of $\mathfrak{A}$ is the structure

$$(A_n\ (n < k);\ \langle -, - \rangle;\ \sigma_n\ (n < k - 2);\ \mathbb{Z})\,.$$

We show by induction that for all k the k-truncation of $\mathfrak{A}$ is isomorphic to the k-truncation of the sequence resulting from applying the construction procedure to $\mathcal{A}$. Since the isomorphisms for various k are compatible, the result follows immediately.

The cases for $k = 1, 2$ or 3 are trivial, by definition of the construction procedure. Suppose that $l \geq 3$, and that the result holds for every integer

$1 \leq k < l$. Now the derived sequence $\mathcal{A}'$ of the quotient model $\mathfrak{A}_1$ is equal (via an obvious shift) to the structure

$$(A_{n,i}\ (n \geq 1, i < 3);\ \langle -, - \rangle, \sigma_n, \pi_n, \mathbb{Z}),$$

and so applying the induction hypothesis, the construction procedure reconstructs the $(l-1)$-truncation of $\mathfrak{A}_1$. Now Lemma 3.3 says that $A_{l-2} \oplus A'_{l-2}$ is isomorphic to A_l, so that the construction procedure applied to $\mathcal{A}$ does give the correct group in the lth position. One may also use the lemma to easily check that the procedure gives the correct σ and bracket maps. ∎

We may now prove Theorem 3.2.

Proof (necessity). Suppose that the dual groups A_0 and B_0 have the same length rank. Form the two corresponding standard models $\mathfrak{A}$ and $\mathfrak{B}$. To show that these two models are $L_{\infty,\omega}$-equivalent we will first show that their derived sequences $\mathcal{A}$ and $\mathcal{B}$ are $L_{\infty,\omega}$-equivalent, by an argument similar to that for the short sequences in Section 1. Then by taking a forcing extension which collapses the cardinalities of $\mathfrak{A}$ and $\mathfrak{B}$ to $\aleph_0$, we will know by Lemma 3.4 that they are isomorphic in the extended universe. It follows (via a standard argument, or see [6]) that they are $L_{\infty,\omega}$-equivalent in the ground model. (This forcing argument may be replaced with a back-and-forth argument, but at the cost of greater complexity of proof.)

In the nontrivial case, we assume that A_0 and B_0 are of infinite rank.

It follows directly from the definition of length rank that

$$\operatorname{rank}(A_{n,2}/\sigma_n[A_{n,0}]) = \operatorname{rank}(B_{n,2}/\sigma_n[B_{n,0}])$$

for all n. We show that we can build a set of partial isomorphisms between substructures of $\mathcal{A}$ and $\mathcal{B}$ which satisfy the back-and-forth property, primarily by appealing to the short sequence case.

As we did in the case of short sequences, we shall list various properties ("axioms") that any sequence derived from a long sequence must have. In our proof we shall use only these properties, a fact we will appeal to in the next section.

(i) Each $A_{n,i}$ is an $\aleph_1$-free Abelian group.

(ii) If $\langle a \rangle$ is pure and nonzero in $A_{n,i}$, then there exists $f \in A_{n,i+1}$ so that $\langle a, f \rangle = 1$ (for $i = 0, 1$).

(iii) If $\langle f \rangle$ is pure and nonzero in $A_{n,i+1}$, then there exists $f \in A_{n,i}$ so that $\langle a, f \rangle = 1$ (for $i = 0, 1$).

(iv) For $a \in A_{n,i}$, $f, g \in A_{n,i+1}$, $\langle a, f - g \rangle = \langle a, f \rangle - \langle a, g \rangle$ (for $i = 0, 1$).

(v) For $a, b \in A_{n,i}$, $f \in A_{n,i+1}$, $\langle a - b, f \rangle = \langle a, f \rangle - \langle b, f \rangle$.

(vi) $\langle a, f \rangle = \langle \sigma_n(a), f \rangle$ for all $a \in A_{n,i}$, $f \in A_{n,i+1}$.

(vii) π_n is a surjection with kernel $\sigma_n[A_{n,0}]$.

Our set P of partial isomorphisms will consist of isomorphisms ϕ between *substructures* $\mathcal{F}$ of $\mathcal{A}$ (with universe $(F_{n,i}\ (n \in \omega, i < 3))$ and $\mathcal{G}$ of $\mathcal{B}$ (with universe $(G_{n,i}\ (n \in \omega, i < 3))$, which have the following properties:

(A) Each group $F_{n,i}$ is a finite rank pure subgroup of $A_{n,i}$ (and likewise for $\mathcal{G}$).

(B) The groups $F_{n,0}$ and $F_{n,1}$ have dual bases, for all n (and likewise for $\mathcal{G}$).

(C) $\sigma_n[A_{n,0}] \cap F_{n,2} = \sigma_n[F_{n,0}]$.

(D) The map π_n maps $F_{n,2}$ onto $F_{n+1,0}$, and likewise for $\mathcal{G}$.

We claim that the set P of isomorphisms between substructures with the properties above has the back-and-forth property. Since the cases are symmetric we prove only the case when a new element of $\mathcal{A}$ is chosen.

Suppose that we have an isomorphism ϕ between substructures $\mathcal{F}$ and $\mathcal{G}$ with the properties above. We will use the proof we gave for short sequences, referring to the cases covered there as Case S-I, S-II and S-III. The corresponding cases here are:

Case I: We choose $a \in A_{n,0} - F_{n,0}$. If $n = 0$ then we may proceed as in Case S-I, adding new elements in positions $(0,0), (0,1)$ and $(0,2)$, and extending ϕ as we do there. There is no need to add elements at lower levels (that is, at levels k where $k > n$).

If $n \geq 1$ we proceed as in Case S-I, extending the groups on level n as we do there (and keeping the same notation). There is no need to add elements on lower levels (by axiom (vii)) but we must extend the groups in position $(n-1, 2)$: Let $k_1 \in A_{n-1,2}$ be a representative for a_2, and $l_1 \in B_{n-1,2}$ be a representative for b_2. The choice of a_2 and b_2 and hypothesis (D) ensure that k_1 and l_1 are chosen as in Case S-III(b) (on level $n-1$). So we can continue to follow S-III(b) on level $n-1$ to incorporate k_1 and l_1. Note that the choice of a_2 and b_2 also implies that we need not extend the groups in position $(n-1, 0)$. Thus we are done, and ϕ may be extended in the obvious manner to an isomorphism with a in its domain. We can check as we did in Section 2 that hypotheses (A) to (D) hold for the extended structures.

Case II: We choose $f \in A_{n,1} - F_{n,1}$. We follow Case S-II to make the necessary extensions on level n, and Case I for level $n-1$ (if $n \geq 1$). No extensions are necessary for lower levels.

Case III: We choose $k \in A_{n,2} - F_{n,2}$. There are two cases:

Case III(a): $k \in \sigma_n[A_{n,0}] + F_{n,2}$. In this case we may follow the procedure of Case I on level n to add the necessary element of $A_{n,0}$. The image of k under π_n is in $F_{n+1,0}$ by axiom (vii), so there is no need to extend the groups at lower levels.

Case III(b): $k \notin \sigma_n[A_{n,0}] + F_{n,2}$. Let $a = \pi_n(k)$, an element of $A_{n+1,0} - F_{n+1,0}$. Now follow Case S-I on level $n+1$, extending the groups as we do there. This process produces elements $a_2 \in A_{n+1,0}$ and $b_2 \in B_{n+1,0}$; pick $k_1 \in A_{n,2}$ and $l_1 \in B_{n,2}$ such that $\pi_n(k_1) = a_2$ and $\pi_n(l_1) = b_2$. The properties of a_2 and b_2 ensure that k_1 and l_1 have the properties required by Case S-III(b); so we may continue with the methods of that case on level n, to incorporate k_1 and l_1. Finally, we may need to add elements to $F_{n,0}$ to ensure that k itself is incorporated; to do this we may use Case I.

This establishes the "necessity" case of Theorem 3.2. For sufficiency, we first notice that by the same sort of forcing argument as above, two $L_{\infty,\omega}$-equivalent sequences $\mathfrak{A}$ and $\mathfrak{B}$ have $L_{\infty,\omega}$-equivalent derived sequences. It is then clear from the definitions that A_0 and B_0 must have the same length rank. ∎

4. Countable models. We show that countable models of the theory of long sequences exist, and discuss the connection with the "models" of [5]. We finish by showing that the first order theory of long sequences is undecidable.

LEMMA 4.1. *For any length rank, there is a countable model of the $L_{\infty,\omega}$-theory of long sequences of the given rank.*

Proof. Fix a length rank, and let $\mathfrak{A}$ be a standard model of that rank. (The existence of this is established in [4], Theorem XIV.4.9.) We may form its derived sequence $\mathcal{A}$. Now it follows directly from the form of the listed axioms (see the proof of Theorem 3.2) that we may use a Löwenheim–Skolem type argument to find a countable $L_{\infty,\omega}$ submodel $\mathcal{S}$ of $\mathcal{A}$, of the same length rank, and thus by (the proof of) Theorem 3.2, $L_{\infty,\omega}$-equivalent to $\mathcal{A}$. Now we may extend the set-theoretic universe to a new universe in which the cardinality of $\mathcal{A}$ is countable. It follows that $\mathcal{S}$ and $\mathcal{A}$ are isomorphic in this new universe. In the ground universe we may carry out the reconstruction procedure (see the proof of Lemma 3.4) on both $\mathcal{A}$ and $\mathcal{S}$ to obtain $\mathfrak{A}$ (by Lemma 3.4) and some structure $\mathfrak{S}$, and since the reconstruction procedure is absolute, we know that in the extended universe $\mathfrak{A}$ and $\mathfrak{S}$ are isomorphic. It follows that they are $L_{\infty,\omega}$-equivalent in the ground model. So $\mathfrak{S}$ is a countable model for the theory of sequences of dual groups (of the given length rank). ∎

In [5] the authors employed structures similar to our long sequences; these are $\omega+1$-sorted structures

$$\mathfrak{F} = (F_n\ (n \in \omega);\ \langle -,- \rangle;\ \sigma_n\ (n \in \omega);\ \varrho_n\ (n \geq 2);\ \mathbb{Z})$$

for the language $(\langle \boldsymbol{-},\boldsymbol{-} \rangle, \boldsymbol{\sigma}_n\ (n \in \omega), \boldsymbol{\varrho}_n\ (n \geq 2))$. The reader should refer to the definitions and results given there; we will comment briefly on the connection with our notion of long sequence.

In a *standard* sequence of this type, F_0 is a dual group, each group F_{n+1} is the dual of F_n, the bracket and σ maps are interpreted as usual, and the map ϱ_n is interpreted as restriction, for $n \geq 3$. That is,

$$\langle a, \varrho_n(f)\rangle = \langle \sigma_{n-3}(a), f\rangle$$

for all $a \in F_{n-3}$ and $f \in F_n$. The map ϱ_2 is chosen to be the restriction map arising from the existence of a predual B for A_0. The existence of the ϱ_2 maps changes the situation considerably from what we have considered here: for the reasons discussed in Example 1, Theorem 3.2 does not hold for these structures, and of course, having a specified predual B for a given dual group A_0 misses the point of the program we discussed in the introduction.

However, if one removes the ϱ_2 map from the language, then the [5] models are really the same as those considered here, since the ϱ_n maps for $n \geq 3$ are definable in the reduct of a model to the language of Definition 1.2. One could hope that the axioms for sequences given in [5] axiomatize the $L_{\infty,\omega}$-theory of long sequences, but the following example shows otherwise:

EXAMPLE 2. Let C be a group which is torsionfree but not $\aleph_1$-free, and F a separable group so that $F^{**}/\sigma[F] \cong C$ ([8]). Let $\mathfrak{F}$ be the sequence (of the [5] type without ϱ_2) generated by F. Then $\mathfrak{F}$ satisfies the axioms for sequences listed in [5], but in any model $\mathfrak{G}$ of the theory of long sequences, one must have $G_2/\sigma_0[G_0]$ $\aleph_1$-free. So $\mathfrak{F}$ does not model the theory of long sequences.

One can in fact extend the axioms given in [5] to obtain an axiomatization of the theory of long sequences, but since this is somewhat clumsy, and not particularly useful, we do not discuss this here.

We finish by establishing the following theorem.

THEOREM 4.2. *The first order theory of any nontrivial sequence of dual groups is undecidable.*

Proof. To prove the theorem we embed the first order theory of the ring $\mathbb{Z}$ into the theory of a (long or short) sequence $\mathfrak{A}$. We will assume that A_0 has rank at least 1. Fix the element 1 in $\mathbb{Z}$, $a \in A_0$ and $f \in A_1$ so that $\langle a, f\rangle = 1$. Notice that we can say that "g is a multiple of f" and "b is a multiple of a" by saying that $\ker g = \ker f$ and $\ker a = \ker b$. Given this observation it is easy to define addition and multiplication. First to define addition. Given n and m choose g, h in A_1 so that $\langle a, g\rangle = n$ and $\langle a, h\rangle = m$; then $n + m = \langle a, g + h\rangle$. To define multiplication is a little trickier. Given n and m choose b a multiple of a so that $\langle b, f\rangle = n$ and choose g a multiple of f so that $\langle a, g\rangle = m$. Then $\langle b, g\rangle = nm$.

The proof may seem circular in that in the multiplicative case we choose b so that $b = na$. The point is that there is a definable function φ (in the parameters a, f) so that for $n \in \mathbb{Z}$, $\varphi(n) = na$; namely, $\varphi(n)$ is the element b such that $\ker b = \ker a$ and $\langle f, b \rangle = n$.

References

[1] J. Barwise, *Back and forth through infinitary logic*, in: Studies in Model Theory, MAA Stud. Math. 8, Math. Assoc. Amer., 1973, 5–34.

[2] S. Chase, *Function topologies on abelian groups*, Illinois J. Math. 7 (1963), 593–608.

[3] K. Eda and H. Ohta, *On abelian groups of integer-valued continuous functions*, *their $\mathbb{Z}$-duals and $\mathbb{Z}$-reflexivity*, in: Abelian Group Theory, Proc. Third Conf. Oberwolfach, Gordon & Breach, 1987, 241–257.

[4] P. Eklof and A. Mekler, *Almost Free Modules*, North-Holland, 1990.

[5] P. Eklof, A. Mekler and S. Shelah, *On strongly-non-reflexive groups*, Israel J. Math. 59 (1987), 283–298.

[6] E. Ellentuck, *Categoricity regained*, J. Symbolic Logic 41 (1976), 639–643.

[7] G. Reid, *Almost Free Abelian Groups*, lecture notes, Tulane University, unpublished, 1968.

[8] G. Schlitt, *Sheaves of abelian groups and the quotients A^{**}/A*, J. Algebra 158 (1993), 50–60.

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