FUNDAMENTA
MATHEMATICAE
144 (1994)

Raising dimension under *all* projections

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Abstract. As a special case of the general question—"What information can be obtained about the dimension of a subset of $\mathbb{R}^n$ by looking at its orthogonal projections into hyperplanes?"—we construct a Cantor set in $\mathbb{R}^3$ each of whose projections into 2-planes is 1-dimensional. We also consider projections of Cantor sets in $\mathbb{R}^n$ whose images contain open sets, expanding on a result of Borsuk.

1. Introduction. If C is a Cantor set in $\mathbb{R}^1$ and $f : C \to \mathbb{R}^n$ maps C onto an n-ball, then the graph of f in $\mathbb{R}^1 \times \mathbb{R}^n$ is a Cantor set whose projection into the second factor raises dimension as much as possible; while projection into the first factor preserves dimension, since it is a homeomorphism. We are interested in Cantor sets *all* of whose projections have the same dimension. Our main result (Section 2) is the construction in $\mathbb{R}^3$ of a Cantor set whose projections into planes always have dimension exactly 1. Borsuk ([B]) has constructed, in $\mathbb{R}^n$, Cantor sets whose projections into hyperplanes (= affine subspaces) always contain open sets; in Section 3 we consider some consequences of this example. Finally, in Section 4 we note the existence of Cantor sets with the opposite property: their projections are in some sense as close as possible to being homeomorphisms.

[E] is the standard reference for dimension theory; Sections 1.12 and 4.3 contain material on dimension-raising maps, which does not seem to apply to the present sort of question. The projection of the graph example is well-known; see for example Remark 2 of [M].

2. Raising dimension slightly. In $\mathbb{R}^2$, if L is a line, let $L^\perp$ be a line perpendicular to L, and let $\pi_L : \mathbb{R}^2 \to L^\perp$ be projection parallel to L. (L and $L^\perp$ need only be determined up to parallel families; for convenience $L^\perp$ will be regarded as disjoint from the various subsets we will project into it.)

1991 *Mathematics Subject Classification*: Primary 54F45.

If ε and δ are positive numbers and L is a line, then a bounded set X in $\mathbb{R}^2$ will be said to have *size* (ε,δ) *with respect to* L if $\pi_L(X)$ can be covered by the interiors of a finite collection of pairwise disjoint closed intervals in $L^{\perp}$, each of length less than ε, such that the distance between each pair of these intervals is at least δ. X will have *absolute size* (ε,δ) if it has size (ε,δ) with respect to all lines. Similarly, a set in $\mathbb{R}^3$ will be said to have absolute size (ε,δ) if its projection into each line can be so covered.

In $\mathbb{R}^2$, let F denote the horizontal closed line segment with endpoints $(1,1)$ and $(-1,1)$, and let K denote the trapezoid with vertices $(1,1)$, $(-1,1)$, $(1/2,1/2)$, and $(-1/2,1/2)$. Let S be a partition of F, that is, a finite set of points of F, including the endpoints; and let H be a horizontal strip through K: $H=\mathbb{R}^1\times A$ for some arc A in the interval between $(0,1)$ and $(0,1/2)$. Let the points of S be ordered from left to right: $S=\{s_1,\ldots,s_{k+1}\}$. A set of *baffles* B *for* S *in* H consists of a finite collection of pairwise disjoint horizontal closed segments in H, one for each segment $[s_i,s_{i+1}]$, $1\le i\le k$, with endpoints on the rays Os_i and Os_{i+1}, where O denotes the origin $(0,0)$. Thus each line through O and F hits at least one, and at most two, of the baffles.

The following somewhat technical lemma will be used to assure that images under projections will be large enough to have dimension at least 1, but not large enough to have dimension 2.

LEMMA 1 (Families of baffles). *Given a partition S of F, a finite collection $H_1,\ldots,H_p$ of disjoint horizontal strips through K, and a positive number ε, there exist a subdivision S' of S, positive numbers δ and η, and a family of baffles $\{B_i\}$, $1\le i\le p$, with B_i a set of baffles for S' in H_i, such that the η-neighborhood of the union of all the elements of all the B_i's has absolute size (ε,δ).*

Proof. First, consider the case of only one horizontal strip, $H\equiv H_1$. Let S' be a subdivision of S such that the distance between adjacent points of S' is less than $\varepsilon/3$. Let L_H be a horizontal line through the middle of H, and let $F_1=H\cap L_H$. Let U_{L_H} be an open neighborhood of directions of lines close to L_H such that, for each $L\in U_{L_H}$, $\pi_L(F_1)$ has diameter $<\varepsilon/3$. For a horizontal line L'_H through H slightly below L_H, let $F_2=H\cap L'_H$. Let $S'=\{s_1,\ldots,s_{k+1}\}$, with s_i to the left of s_{k+1}, $1\le i\le k$. For each adjacent pair s_i, s_{i+1} of points of S', let $[s_i,\, s_{i+1}]_{\hat\imath}$ denote the closed segment in $F_{\hat\imath}$, $\hat\imath=1,2$, with endpoints $Os_i\cap F_{\hat\imath}$ and $Os_{i+1}\cap F_{\hat\imath}$. Let $B_1=\{[s_i,\, s_{i+1}]_{\hat\imath}:1\le i\le k\}$, where $\hat\imath=1$ if i is odd and 2 if i is even; thus the elements of B_1 alternate between the upper segment F_1 and the lower segment F_2. B_1 is a set of baffles for S' in H_1; we next show that F_2 can be chosen so close to F_1 that the lemma is satisfied.

Consider pairs i and j, one odd and one even, with $i+1<j$, and consider lines L intersecting both $[s_i,\, s_{i+1}]_{\hat\imath}$ and $[s_j,\, s_{j+1}]_{\hat\jmath}$: if F_1 and F_2

coincided, the line L would be horizontal. The four lines determined by the end points of the two segments yield the extreme slopes, and there are only finitely many pairs i, j; hence F_2 can be chosen so close to F_1 that all lines intersecting any two non-adjacent and non-colinear elements of B_1 will have their directions lying in the open set U_{L_H} previously defined; let F_2 be chosen so close to F_1 that this is satisfied. Thus if a line L with direction not in U_{L_H} intersects two elements of B_1, they must be adjacent; and if any two elements of $\{\pi_L(b) : b \in B_1\}$ intersect, the corresponding b's must be adjacent.

Let L be a line not in U_{L_H}. Claim: No more than three elements of $\{\pi_L(b) : b \in B_1\}$ can form a connected set in $\pi_L(B_1)$. To see this, let s_i, s_{i+1}, s_{i+2}, s_{i+3} be four consecutive points of S', and let b_1, b_2, b_3 denote the elements of B_1 they determine: $b_1 = [s_i, s_{i+1}]_{\hat{\imath}}$, etc. If the direction of L does not lie between Os_{i+1} and Os_{i+2}, then some line parallel to L intersects one of Os_{i+1} and Os_{i+2} between the horizontal segments F_1 and F_2, and missing the corresponding b's; hence $\pi_L(b_2)$ misses at least one of $\pi_L(b_1)$ and $\pi_L(b_3)$. If the direction of L lies between Os_{i+1} and Os_{i+2} and b_2 lies in the lower segment F_2, then $\pi_L(b_2)$ misses both of $\pi_L(b_1)$ and $\pi_L(b_3)$; while if b_2 lies in the upper segment F_1, then $\pi_L(b_2)$ intersects both of $\pi_L(b_1)$ and $\pi_L(b_3)$, but $\pi_L(b_1)$ and $\pi_L(b_3)$ each misses the image under π_L of its other adjacent segment of B_1. Thus for each line L, $\pi_L(B_1)$ consists of a finite number of intervals (or of just two points in case L is horizontal), and each component of $\pi_L(B_1)$ has length less than ε.

For each line L, let W_L be a finite set of disjoint closed intervals in $L^\perp$ whose interiors cover $\pi_L(B_1)$, and let δ_L be a positive number less than the distance between any two elements of W_L. By continuity of π_L, there is an $\eta_L > 0$ such that the interiors of W_L also cover the image under π_L of the closed η_L-neighborhood of B_1. By continuity of π_L with respect to L, if L' is any line sufficiently close in direction to L, then $\{\pi_{L'}(w) \ : w \in W_L\}$ will be a collection of pairwise disjoint closed intervals in $(L')^\perp$ whose interiors cover the image under $\pi_{L'}$ of the η_L-neighborhood of B_1; hence $\langle L, W_L, \varepsilon, \delta_L, \eta_L\rangle$ determines an open set U_L of directions of lines about L, such that if $L' \in U_L$ then the η_L-neighborhood of B_1 has size $(\varepsilon, \delta_L\,)$ with respect to L'. Since the space of directions of lines is compact (homeomorphic to a circle), some finite collection of U_L's covers all directions; picking δ as the minimum of the δ's and η as the minimum of the η's, shows the existence of δ and η such that the closed η-neighborhood of $\bigcup B_1$ has absolute size $(\varepsilon, \delta\,)$.

We now proceed by induction on p, the number of horizontal strips. Let $H_1, \ldots, H_p, H_{p+1}$ and ε be given; pick $\varepsilon_1 < \varepsilon$, and inductively let S_1 be a subdivision of S and $\{B_i : 1 \leq i \leq p\}$ be a family of baffles with respect to $H_1, \ldots, H_p$, with corresponding ε_1, δ_1, and η_1. Let $\varepsilon_2 > 0$ be chosen so that $\varepsilon_2 < \delta_1$ and $\varepsilon_1 + 2\varepsilon_2 < \varepsilon$, and let S_2, B_{p+1}, δ_2, and η_2 be a subdivision, a family of baffles, and parameters for the single strip H_{p+1}, partition S_1, and positive number ε_2.

For any line L, let W_1 and W_2 be a collection of intervals for S_1, ε_1, δ_1, η_1 and for S_2, ε_2, δ_2, η_2 corresponding to the family $\{B_1, \ldots, B_p\}$ and to B_{p+1}, respectively. Let W be the collection of components of $(\bigcup W_1) \cup (\bigcup W_2)$. Since no interval in W_2 can intersect two intervals of W_1, each interval of W has length less than ε. Let $\delta > 0$ be less than the distance between any two intervals of W, and let $\eta = \min\{\eta_1, \eta_2\}$. Then the η-neighborhood of $\bigcup\{B_i : 1 \le i \le p+1\}$ has size (ε, δ) with respect to L. As before, by continuity of π_L with respect to L, the same W, δ, η work for an open set of lines with directions sufficiently close to L, and by compactness δ and η may be chosen so that the closed η-neighborhood of $\bigcup\{B_i : 1 \le i \le p+1\}$ has absolute size (ε, δ).

Since S_2 is a subdivision of S_1, some rays Os, $s \in S_2$, may hit intervals b in B_i, $i < p+1$, in interior points of b. Subdivide each such b at the points $b \cap Os$, move down slightly alternate subintervals of b thus created, contracting so that their endpoints remain on the rays Os and remaining in the η-neighborhood of b. Finally, choose a new η so that the new B_i's form a set of baffles with respect to S_2. (This final η can be chosen so small that the closed η-neighborhoods of different intervals of the baffles will be disjoint.) This completes the proof of the lemma. ∎

In $\mathbb{R}^3$, let the z-axis be the vertical axis; let P and Q be the rectangles with corner points $(1,1,1)$, $(1,-1,1)$, $(-1,1,1)$, $(-1,-1,1)$ and $(1/2,1,1/2)$, $(1/2,-1,1/2)$, $(-1/2,1,1/2)$, $(-1/2,-1,1/2)$, respectively; let W be the convex hull of $P \cup Q$; let F, F', and K be the intersections of P, Q, and W with the plane $y = 0$; and let J be the closed segment with endpoints $(0,1,0)$ and $(0,-1,0)$. Let $\varrho : P \to Q$ be the projection that is the product of the identity on the y-coordinate and the radial projection toward the origin O in the xz-coordinate plane.

Lemma 2. *W contains a subset C homeomorphic to the Cantor set with the properties that* (i) *any straight line in $\mathbb{R}^3$ which hits both P and J will contain a point of C, while* (ii) *the orthogonal projection of C into any 2-plane will contain no open set.*

Proof. For each $n \ge 0$, $\mathcal{A}^n$ and $\mathcal{B}^n$ will be certain subdivisions of P and Q by rectangles with sides parallel to those of P and Q, and with the property that $\mathcal{B}^n = \{\varrho(A) : A \in \mathcal{A}^n\}$; let Ψ^n be the subset of $\mathcal{A}^n \times \mathcal{B}^n$ with the property that, for each $(A, B) \in \Psi^n$, there is some line in $\mathbb{R}^3$ which intersects each of $\mathrm{int}(A)$, $\mathrm{int}(B)$, and J.

Let T denote the vertical interval with endpoints $(0,0,1)$ and $(0,0,1/2)$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\varepsilon_n \to 0$. We will inductively define a sequence $\{\mathcal{A}^n\}$ of subdivisions of P with $\mathcal{A}^n$ a refinement of

$\mathcal{A}^{n-1}$, a nested sequence $\{\mathcal{C}^n\}$ of finite collections of subintervals of T, and functions $\varphi^n : \Psi^n \to \mathcal{C}^n$ satisfying the following:

(i) $\varphi^n : \Psi^n \to \mathcal{C}^n$ is one-to-one and onto;

(ii) If $(A,B) \in \Psi^{n-1}$ and $(A',B') \in \Psi^n$, with $A' \subset A$, $B' \subset B$, then $\varphi^n(A',B') \subset \varphi^{n-1}(A,B)$;

(iii) $\mathrm{diam}(D^n(A,B)) < \varepsilon_n$, where

$$D^n(A,B) = (\mathbb{R}^2 \times \varphi^n(A,B)) \cap (\text{convex hull of } A \cup B)\,;$$

(iv) G^n has absolute size $(\varepsilon_n, \delta_n)$ for some δ_n, where

$$G^n = \bigcup\{D^n(A,B) : (A,B) \in \Psi^n\}\,.$$

We start the induction by supposing that ε_0 is large, and take $\mathcal{A}^0 = P$ and $\mathcal{C}^0 = T$. As a preliminary to defining the $(n+1)$-level, let $\mathcal{A}_1^n$ be a subdivision of $\mathcal{A}^n$ by lines parallel to the sides of P, such that each element of $\mathcal{A}_1^n$ has diameter $< \frac{1}{2}\varepsilon_{n+1}$, and let $\mathcal{B}_1^n$ be the subdivision of $\mathcal{B}^n$ determined by $\mathcal{A}_1^n$ (projected into F'). Ψ_1^n is generated by $\mathcal{A}_1^n \times \mathcal{B}_1^n$ the same way that Ψ^n is generated by $\mathcal{A}^n \times \mathcal{B}^n$. For each $(A,B) \in \Psi^n$, the set $\{(A_1,B_1) \in \Psi_1^n : A_1 \subset A \text{ and } B_1 \subset B\}$ is finite—corresponding to its elements choose pairwise disjoint intervals $\{\varphi_1^n(A_1,B_1)\}$ in the interior of the interval $\varphi^n(A,B)$; this defines $\mathcal{C}_1^n$ and φ_1^n satisfying (i), and also (ii) with Ψ_1^n for Ψ^n and φ_1^n for φ^n. Let S be the partition of F determined by the corners of the elements of $\mathcal{A}_1^n$, projected into F.

Apply Lemma 1 on families of baffles to K and S, using $\{K \cap (\mathbb{R}^1 \times C)\colon C \in \mathcal{C}_1^n\}$ as the horizontal strips, and $\frac{1}{2}\varepsilon_{n+1}$ as ε. This gives a refinement S' of S, a collection of baffles $\{B_i\}$, and positive numbers δ_{n+1} and η such that the η-neighborhood of $\bigcup B_i$ has absolute size $(\frac{1}{2}\varepsilon_{n+1}, \delta_{n+1})$. Let $\mathcal{A}^{n+1}$ be obtained from $\mathcal{A}_1^n$ by subdividing each element in the x-coordinate at the points of S'; this also generates $\mathcal{B}^{n+1}$ and Ψ^{n+1}. The horizontal projection of $\bigcup B_i$ into T consists of a finite number of points of $\bigcup\{\mathrm{int}(C) : C \in \mathcal{C}_1^n\}$; choose $\mathcal{C}^{n+1}$ refining $\mathcal{C}_1^n$ to consist of one interval about each of these points, each of diameter $< \min\{\frac{1}{2}\eta, \frac{1}{2}\varepsilon_{n+1}\}$, and pairwise disjoint.

For each $(A',B') \in \Psi^{n+1}$, a line intersecting $\mathrm{int}(A')$, $\mathrm{int}(B')$, and J may intersect several of the elements of the family of baffles $\{B_i\}$, but only one of these elements projects horizontally into a $C \in \mathcal{C}^{n+1}$ with the property that $C \subset \varphi^n(A,B)$ for some $(A,B) \in \Psi^n$ with $A' \subset A$ and $B' \subset B$—this C is defined to be $\varphi^{n+1}(A',B')$.

This completes the inductive definition. Conditions (i) and (ii) are satisfied by construction. Condition (iii) is satisfied, because $\mathrm{diam}(D^{n+1}(A,B))$ $<$ (diameter of an element of the family of baffles $\{B_i\}$) + (diameter of an element of $\mathcal{C}^{n+1}$) $< \frac{1}{2}\varepsilon_{n+1} + \frac{1}{2}\varepsilon_{n+1} = \varepsilon_{n+1}$. Condition (iv) is satisfied, because G^n is contained in the η-neighborhood of the baffles $\bigcup B_i$.

$G = \bigcap_{n=1}^{\infty} G^n$ will be the desired Cantor set. To see this, observe that, if L is any line in $\mathbb{R}^3$ hitting J and $\mathrm{int}(P)$, then the image of G^n under π_L will contain a non-degenerate subinterval of $\pi_L(J)$; while for some line L' parallel to L and less than ε_n away, L' will miss G^n, and hence miss G also. Thus $\dim(\pi_L(G)) = 1$. ∎

Theorem 1. *$\mathbb{R}^3$ contains a Cantor set each of whose images under orthogonal projections into 2-planes has dimension exactly 1.*

Proof. If L denotes the line through the centers of P and J, then the image of G under the projection parallel to L, or parallel to any line L' with direction in an open set of directions sufficiently close to that of L, will be 1-dimensional. By rotating L about the origin in $\mathbb{R}^3$, we obtain an open cover of all directions. Taking a finite subcover (by compactness), we obtain a finite collection $G_1, \ldots, G_k$ of copies of G, such that each projection into a plane has image which is 1-dimensional on one of the G's, and at most 1-dimensional on the others; hence the projection is 1-dimensional. Each G is a Cantor set, and their union is the Cantor set required. ∎

3. Raising dimension as much as possible. Borsuk's theorem [B] can be stated as

Theorem 2. *$\mathbb{R}^m$ contains a Cantor set each of whose images under orthogonal projections into proper hyperplanes always contains (relatively) open sets of the same dimension as the hyperplane. In fact, if B is a convex body in $\mathbb{R}^m$ and V is a convex open set whose closure is contained in the interior of B, then B contains a Cantor set C such that each line in $\mathbb{R}^n$ which intersects V will contain a point of C.*

While the projections of the C of Theorem 2 contain relative open sets, their images are probably 0-dimensional near the "edges". How "nice" could all the projections of a Cantor set be?

We will show that in the case of projections from $\mathbb{R}^2$ into lines they can all be convex, while in higher dimensions this is impossible. However, all projections can be the closures of their interiors.

Example. In $\mathbb{R}^2$, let D^2 denote a square. Place along the diagonals of D^2 (exclusive of the corner points) a countably infinite, locally finite collection of Cantor sets given by Theorem 2, such that their diameters decrease to keep them inside D^2, and such that their V's (from Theorem 2) cover the diagonals. The union of these Cantor sets, along with the four corner points of D^2, forms a Cantor set whose projections into lines coincide with the projections of D^2, hence are all convex. However,

Theorem 3. *There is no Cantor set in $\mathbb{R}^m$, $m \geq 3$, all of whose orthogonal projections into $(m-1)$-planes are convex bodies.*

This is an immediate consequence of the following lemma.

LEMMA 3. *In* $\mathbb{R}^m$, $m \geq 3$, *let* B *be a convex body* (*a compact convex set with non-empty interior*). *Then there is a line* L *in* $\mathbb{R}^m$ *and an arc* A *in* $\operatorname{bdy}(B)$ *such that each line through* A *parallel to* L *hits* B *in only one point.*

Proof of Lemma 3. By induction on m; first, let $m = 3$.

There are at most countably many planes P in $\mathbb{R}^3$ which hit $\operatorname{bdy}(B)$ in an open subset of P; let L be a line not parallel to any such plane. Let π be the orthogonal projection parallel to L of B into some plane (call it $\mathbb{R}^2$) orthogonal to L, and let $C = \operatorname{bdy}(\pi(B))$. If C contains a segment D, then no point of D could have non-degenerate inverse image under π (for if $\pi^{-1}(p)$ is a non-degenerate segment, its join with D in $\operatorname{bdy}(B)$ would be an open subset of a plane); hence A could be chosen to be $\pi^{-1}(D)$.

Suppose E is an arc in C containing no segment. Either $\pi|\pi^{-1}(E)$ is one-to-one (in which case $\pi^{-1}(E)$ will serve for A), or there exists a point $p \in E$ such that $\pi^{-1}(p)$ is a segment in $\operatorname{bdy}(B)$. Let L' be a line in $\mathbb{R}^2$ "tangent" to C at p, in the sense that $L' \cap C = p$ and L' and C do not cross; take $A = \pi^{-1}(p)$, and use L' for L.

Now let $m \geq 4$ and let $\pi : B \to \mathbb{R}^{m-1}$ be projection parallel to a line L_1. By induction, there is an arc A in $\pi(B)$ and a line L_2 in $\mathbb{R}^{m-1}$ such that each line in $\mathbb{R}^{m-1}$ through A and parallel to L_2 hits $\pi(B)$ in only one point. If $\pi|\pi^{-1}(A)$ is one-to-one, we are done. Otherwise there is a point $p \in A$ such that $\pi^{-1}(p)$ contains a segment A'. Now any line in $\mathbb{R}^m$ through A' and parallel to L_2 hits B in a single point—for if it hit in a second point q, then some line in $\mathbb{R}^{m-1}$ parallel to L_2 would hit $\pi(B)$ in both p and $\pi(q)$. ■

EXAMPLE. However, in the spirit of [M], in $\mathbb{R}^m$ there is a Cantor set whose projections into *coordinate* hyperplanes coincide with projections of the unit cube. First, thanks to [R], $I \times I^{m-1}$ contains a Cantor set C with the property that each line which intersects both $\{0\} \times I^{m-1}$ and $\{1\} \times I^{m-1}$ contains a point of C. [To see this, let K be a Cantor set in I $(= [0,1])$ containing both endpoints, let $\varphi : K \to I^{m-1} \times I^{m-1}$ be continuous and onto with $\varphi(t) = (u(t), v(t))$, and let C be the Cantor set in I^m consisting of the intersection of $\{t\} \times I^{m-1}$ with the line joining $\{0\} \times \{u(t)\}$ and $\{1\} \times \{v(t)\}$, for $t \in K$. If a line L intersects I^m in $(0, p)$ and $(1, q)$, there is a $t_0 \in K$ such that $\varphi(t_0) = (p, q)$, so L contains the point $C \cap \{t_0\} \times I^{m-1}$.] Now in I^m take m copies of this C, one with respect to each coordinate axis; their union is the desired Cantor set.

EXAMPLE. Each $\mathbb{R}^m$ *does* contain a compact $(m-2)$-dimensional set all of whose projections to proper hyperplanes coincide with the projections of I^m. Let X^{m-2} be the union of the $(m-2)$-faces of I^m, p be a point of $\operatorname{int}(I^m)$, pX be the cone, and $W = pX \setminus X$. As before, cover W with a locally finite collection of Theorem 2 type Cantor sets whose projections will equal the projections of pX; adding X to the union of these Cantor sets provides the example.

Question. Does each $\mathbb{R}^m$ contain a compact set of dimension $\leq m-2$ whose projections to proper hyperplanes have the same images as the ball B^m?

Theorem 4. *Each* $\mathbb{R}^m$, $m \geq 2$, *contains a Cantor set whose projection into each* (*proper*) *hyperplane is the closure of its interior.*

Proof. We will inductively show the existence of two countable sequences $\mathcal{D}_1, \mathcal{D}_2, \ldots$ and $C_1, C_2, \ldots$ such that

(i) $\mathcal{D}_i$ is a finite collection of open sets in $\mathbb{R}^m$, each of diameter $< 1/i$, and the closures of the elements of $\mathcal{D}_i$ are pairwise disjoint;

(ii) $\mathcal{D}_{i+1}$ is a refinement of $\mathcal{D}_i$;

(iii) Each C_i is a Cantor set, with $C_i \subset C_{i+1}$ and $C_i \subset \mathcal{D}_i^*$ $(\equiv \bigcup_{D\in\mathcal{D}_i} D)$, and each member of $\mathcal{D}_i$ contains points of C_i;

(iv) C_{i+1} is the union of C_i with a finite collection of the C's of Theorem 2, whose associated V's cover C_i.

First, note that this suffices for the proof, by letting $C = \bigcap_{i=1}^{\infty} \mathcal{D}_i^*$. By properties (i) and (ii), C is a Cantor set; by (i) and (iii), $\bigcup_{i=1}^{\infty} C_i$ is dense in C; and by (iv), $\pi(\bigcup_{i=1}^{\infty} C_i)$ is contained in the interior of $\pi(C)$, where π is the projection of C into any proper hyperplane in $\mathbb{R}^m$.

To establish the existence of the C_i's and $\mathcal{D}_i$'s, start by letting C_1 be any Cantor set in $\mathbb{R}^m$; the existence of $\mathcal{D}_1$ is an elementary property of Cantor sets.

Now suppose $C_1, \ldots, C_i$ and $\mathcal{D}_1, \ldots, \mathcal{D}_i$ satisfy (i)–(iv). For each point $x \in C_i$, there is a unique $D \in \mathcal{D}_i$ with $x \in D$; and there is a small Theorem 2 type Cantor set C_x with $C_x \subset D$ and $x \in V_x$. Since C_i is compact, some finite collection of the V_x's covers C_i—these determine the C_x's which are added to C_i to give C_{i+1}, as required for (iv). Again, the existence of $\mathcal{D}_{i+1}$ satisfying (i)–(iii) follows from the fact that C_{i+1} is a Cantor set. ∎

Question. Could there be Cantor sets all of whose projections are connected, or even cells? Can Theorem 1 be generalized—given integers $m > n > k > 0$, is there a Cantor set in $\mathbb{R}^m$ each of whose projections into n-hyperplanes will be exactly k-dimensional? Theorem 1 is the case (3, 2, 1).

4. Preserving dimension. In the opposite direction, there are Cantor sets in $\mathbb{R}^m$ all of whose projections have 0-dimensional images—a Cantor set in a segment is an example; most of its projections are embeddings, while the others have "large" fibers. Of course there cannot be a Cantor set C

with every projection an embedding—for take three points of C and project parallel to a 2-plane containing them; or take two pairs of points and project parallel to a 2-plane parallel to each pair. We will show that there are Cantor sets which have at worst the later sort of singularities under projections.

Let $m > k > 0$; in $\mathbb{R}^m$ let H^k be a k-hyperplane, H^{m-k} an $(m-k)$-hyperplane orthogonal to H^k, and $\pi_k : \mathbb{R}^m \to H^{m-k}$ projection parallel to H^k. Denote the fibers of $\pi_k|X$ by $F(x) = (\pi_k|X)^{-1}(\pi_k(x))$, for $x \in X$. A set $X \subset \mathbb{R}^m$ is said to be *in general position with respect to* π_k or H^k if $\pi_k|X$ has only finitely many non-degenerate fibers, each of which has $\leq k+1$ points which form the vertices of a simplex of dimension $|F(x)| - 1$, and $\sum_{x\in X}(|F(x)| - 1) \leq k$. X is said to be *in general position with respect to all projections* if, for *each* k (with $m > k > 0$) and *each* H^k, X is in general position with respect to H^k.

Clearly, if X consists of the $m+1$ vertices of an m-simplex in $\mathbb{R}^m$, X is in general position with respect to all projections, and there is an H^k (parallel to each of some collection of lower dimensional faces) for which equality ($\sum(|F(x)| - 1) = k$) is attained.

THEOREM 5. *Each* $\mathbb{R}^m$, $m \geq 2$, *contains a Cantor set in general position with respect to all projections.*

Before considering the proof, we extend the definition of general position to collections of sets: a finite collection $\mathcal{S}$ of pairwise disjoint subsets of $\mathbb{R}^m$ will be said to be in general position with respect to H^k if $\mathcal{S}$ can be partitioned into subcollections $\{\mathcal{S}_i\}$ with the property that $\sum(|\mathcal{S}_i| - 1) \leq k$, and no k-hyperplane parallel to H^k hits elements from two different $\mathcal{S}_i$'s; and $\mathcal{S}$ is in general position with respect to all projections if it is so for each H^k.

Note that (i) each finite set X of points in $\mathbb{R}^m$ can be put into general position with respect to all projections by arbitrarily small displacements, and (ii) if a finite set X is in general position with respect to all projections, then for sufficiently small $\delta > 0$, the collection of δ-balls centered at points of X is in general position with respect to all projections.

Sketch of proof of Theorem 5. Start with the vertices of an m-simplex in $\mathbb{R}^m$ and choose small balls about them by (ii); the collection of the interiors of these balls is the first stage in the construction of the desired Cantor set. Next pick two points inside each ball, use (i) to put them in general position with respect to all projections, and then use (ii) to get a collection of smaller balls about them for the second stage. Continuing in this fashion yields a Cantor set C, the intersection of the unions of the balls at each stage. That C is in general position with respect to all projections follows from the properties of the defining collections of balls. ■

Question. Cantor sets that raise dimension under all projections and those in general position with respect to all projections are both dense in the Cantor sets in $\mathbb{R}^m$—which (if either) is more common, in the sense of category or dimension or anything?

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[R] Referee's comment.

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Received 2 July 1992;
in revised form 2 June 1993