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Ordinal products of topological spaces

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Abstract. The notion of the ordinal product of a transfinite sequence of topological spaces which is an extension of the finite product operation is introduced. The dimensions of finite and infinite ordinal products are estimated. In particular, the dimensions of ordinary products of Smirnov's [S] and Henderson's [He1] compacta are calculated.

Introduction. The necessary information about the notions and notations we use can be found in [A-Pa], [E1], [E2], [K-M] and in the appendix.

One of the main questions in transfinite dimension theory is

THE PROBLEM OF PRODUCT DIMENSION (PPD). Let DIM be a transfinite dimension function, for example: ind, Ind, $\dim_{\mathrm{w}}$, $\dim_{\mathrm{c}}$, D, and suppose U is a fixed class of topological spaces. What can be said about the dimension DIM of the product of two spaces X, Y from the class U if this dimension is defined for the factors?

Let us give some possible concretizations of (PPD):

(1) Does $\mathrm{DIM}\, X \times Y$ exist?

(2) Is there an (optimal) transfinite function $\Phi = \Phi(\alpha, \beta)$ of two transfinite variables such that

$$\mathrm{DIM}\, X \times Y \le \Phi(\mathrm{DIM}\, X, \mathrm{DIM}\, Y)\,?$$

(The function $\Phi(\alpha, \beta)$ is called *optimal* if for every pair of transfinite numbers α, β there are spaces $X = X(\alpha, \beta)$ and $Y = Y(\alpha, \beta)$ in U such that $\mathrm{DIM}\, X = \alpha$, $\mathrm{DIM}\, Y = \beta$, and $\mathrm{DIM}\, X \times Y = \Phi(\alpha, \beta)$.)

(3) What is the value of $\mathrm{DIM}\, X \times Y$?

In this paper we will be interested in questions (2), (3) and their generalizations. In the introduction we discuss the case of metric compacta unless otherwise stated.

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For the traditional transfinite dimensions ind and Ind (see [A-Pa]) the inequality

$$(*) \qquad \operatorname{DIM} X \times Y < \omega_1$$

is well known, and it is equivalent to the existence of $\operatorname{DIM} X \times Y$ for these dimensions. The analogous statement is true for $\dim_{\mathrm{c}}$, the transfinite extension of the Lebesgue covering dimension dim to compact C-spaces ([B3] and [Ha-Y]). A more delicate result for ind has been obtained by Toulmin [T]: $\operatorname{ind} X \times Y \le (\operatorname{ind} X (+) \operatorname{ind} Y) + n$, where n is a finite non-negative integer which depends on the inductive dimensions of X and Y, and $(+)$ is the natural sum of Hessenberg [Hes].

For any metrizable compactum Z we have $\operatorname{Ind} Z \le \omega \cdot \operatorname{ind} Z$ [Le], which leads to a more precise estimate for Ind than $(*)$. Note that an improvement of $(*)$ for Ind for a certain class of topological spaces has also been stated by Polkowski [Po]. For the dimension D the inequality $D(X \times Y) \le DX(+)DY$ has been proved by Henderson [He2].

So PPD(2) for the dimensions indicated above reduces either to obtaining an optimal estimating function Φ or to the proof of optimality of a given one. Note that for $\dim_{\mathrm{w}}$, another transfinite extension of dim to weakly infinite-dimensional compacta [B1], even the rough estimate $(*)$ has not been obtained yet. This is the well-known problem of the weak infinite-dimensionality of the product of weakly infinite-dimensional compacta. Note that PPD(2) coincides with PPD(3) in the part which deals with optimality—one has to calculate the dimension of the product of the chosen pair of compacta. As far as I know the calculation of the dimension of the product of two infinite-dimensional compacta has not been made yet.

In [S] Smirnov constructed compacta S^α with $\operatorname{Ind} S^\alpha = \alpha$, $\alpha < \omega_1$, and from this he deduced that there are no universal spaces in the class of countable-dimensional metric compacta. Smirnov's construction turned out to be very useful. Using its modification Henderson [He1] constructed AR-compacta $H^\alpha \hookleftarrow S^\alpha$ with $\operatorname{Ind} H^\alpha = \alpha$, $\alpha < \omega_1$, and defined for them the notion of an essential mapping. He also proved that $DS^\alpha = \alpha$ [He2]. In [B1], [B2] Borst, having extended the covering dimension dim to ordinals, proved a transfinite analog of Aleksandrov's theorem on essential mappings for locally compact metric spaces, namely: $\dim_{\mathrm{w}} X \ge \alpha$ iff $X \times C$ has an essential mapping onto H^α, where C is the Cantor middle thirds set ($\dim_{\mathrm{w}}$ is the above mentioned transfinite extension of dim). He also proved the equalities $\dim_{\mathrm{w}} S^\alpha = \dim_{\mathrm{w}} H^\alpha = \alpha$ for $\alpha < \omega_1$, from which one sees directly that there is no universal space in the class of weakly infinite-dimensional metric compacta (the weak infinite-dimensionality of a metric compactum X is equivalent to the inequality $\dim_{\mathrm{w}} X < \omega_1$ [B1]). Note that the non-existence of universal spaces in the class of weakly infinite-dimensional compacta also follows from an earlier result of Pol [P] connected with Smirnov compacta. Namely:

If a complete space X topologically contains every Smirnov compactum S^α, then X topologically contains the Hilbert cube. Hence X is not weakly infinite-dimensional.

Naturally PPD(3) arises for useful and easily constructed Smirnov and Henderson compacta. In this paper this question is completely solved. It turns out that

$$\operatorname{DIM} X^\alpha \times X^\beta = \alpha(+)\beta\,,$$

where DIM is Ind, Id (to be defined below), $\dim_{\mathrm{w}}$, or D and X^γ, $\gamma < \omega_1$, are either the Smirnov compacta S^γ or the Henderson compacta H^γ.

The paper consists of three parts. In the first part, starting from Smirnov's construction we suggest the definition of an infinite product of topological spaces—the ordinal $\aleph_0$-product (Definition 1) for which, in contrast to Tikhonov products, there are non-trivial solutions of the natural extension of PPD to an infinite number of non-zero-dimensional factors (Theorem 4).

Let, for example, $S = \{X_\gamma, \gamma < \beta\}$ be a set of compacta indexed by ordinals $< \beta$ (such sets will be called *β-sequences*). Then the compactum

$$\prod_{\gamma<\beta}^{\omega,\mathrm{ord}} X_\gamma = \begin{cases} \text{point} & \text{if } \beta = 0\,;\\ \Big(\prod\limits_{\gamma<\beta-1}^{\omega,\mathrm{ord}} X_\gamma\Big) \times X_{\beta-1} & \text{if } \beta \text{ is a non-limit ordinal;}\\ \text{Aleksandrov compactification of the free sum} & \\ \Big(\mathop{(+)}\limits_{\delta<\beta}\Big(\prod\limits_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma\Big)\Big) \times \mathbb{N} & \text{if } \beta \text{ is a limit ordinal, where}\\ & \mathbb{N} \text{ are the natural numbers,} \end{cases}$$

is the ordinal $\aleph_0$-product of the β-sequence S. If all the X_γ are homeomorphic to X, then $\prod_{\gamma<\beta}^{\omega,\mathrm{ord}} X_\gamma$ is called the *β-ordinal $\aleph_0$-power* and is denoted by $S^\omega_\beta(X)$. In particular, if $\beta < \omega_1$ and I is the interval [0,1] then $S^\beta \hookrightarrow S^\omega_\beta(I) \hookrightarrow S^\beta$, where S^β is the Smirnov compactum ($\hookrightarrow$ denotes closed embedding).

The new product, just as the Tikhonov product, is an extension of the notion of a finite topological product to an infinite number of factors, but in contrast to the latter, it essentially depends on the order of the indexed set of factors. For example, for two different countable ordinals α and β the α- and β-ordinal $\aleph_0$-powers of the interval are not homeomorphic because $\operatorname{Ind} S^\alpha = \alpha \neq \beta = \operatorname{Ind} S^\beta$. Let us state one of the main results of the paper which explains why $\aleph_0$-products are called products.

THEOREM 1. *Let X be an arbitrary topological space and α, β be countable ordinals. Then*

$$S^\omega_\alpha(X) \times S^\omega_\beta(X) = S^\omega_{\alpha(+)\beta}(X)\,.$$

From Theorem 1 which reminds the main property of the power, one directly obtains

Corollary 2. *Let Φ be a numerical function on topological spaces, monotone on closed subsets, for example a dimension* (ind, Ind, $\dim_{\rm w}$, D *or others*). *Then*

$$\Phi(S^{\omega}_{\alpha}(X)\times S^{\omega}_{\beta}(X))=\Phi(S^{\omega}_{\alpha(+)\beta}(X))\,.$$

In particular, for Smirnov compacta one has:

(a) $\mathrm{DIM}\,S^{\alpha}\times S^{\beta}=\alpha(+)\beta$, *where* DIM *is* $\dim_{\rm w}$, Ind, Id *or* D;

(b) $\mathrm{ind}\,S^{\alpha}\times S^{\beta}=\mathrm{ind}\,S^{\alpha(+)\beta}$;

(c) $S^{\alpha}\times S^{\beta}$ *can be continuously mapped into* $[0,1]$ *so that every point of the interval has a finite-dimensional preimage.*

One can easily see that an upper estimate of the dimension of $S^{\alpha}\times S^{\beta}$ is obtained from the inclusion $S^{\alpha}\times S^{\beta}\hookrightarrow S^{\alpha(+)\beta}$, which is not true for Henderson compacta. For these, the necessary estimate is deduced from the second part of the paper, where for the transfinite extension of the finite dimension Id, introduced by Pasynkov [Pa1], we give the optimal solution of PPD(2), namely:

Let X, Y be compacta for which Id *is defined. Then* $\mathrm{Id}\,X\times Y\le \mathrm{Id}\,X(+)\,\mathrm{Id}\,Y$.

This inequality, obtained as a corollary of the more general Theorem 3, makes it possible to give an optimal upper bound for the dimensions Ind, $\dim_{\rm w}$ of products of compacta under natural additional assumptions (Corollary 6). Note that this inequality has been independently obtained by Vinogradov. Also note that the obtained estimate for Id, just as Henderson's inequality for D, is optimal (use Smirnov compacta).

In the third part questions connected with the dimension of infinite ordinal products are discussed. In particular, we prove the following transfinite generalization to ordinal $\aleph_0$-products of the Brower theorem on the n-dimensionality of the cube I^n.

Theorem 5. *Let* DIM *be* Ind, Id *or* $\dim_{\rm w}$, *and let X_γ, $\gamma<\beta$, be one-dimensional metric compacta. Then*

$$\mathrm{DIM}\prod_{\gamma<\beta}^{\omega,\mathrm{ord}}X_{\gamma}=\beta\,.$$

1. Ordinal products of topological spaces (a special case). Let X_α, $\alpha\in A$, $|A|\ge\aleph_0$, be a family of topological spaces. The *one-point Aleksandrov extension* of the free sum of the spaces X_α, $\alpha\in A$, is the space $X=\{*\}\cup(+)_{\alpha\in A}X_\alpha$, formed from $(+)_{\alpha\in A}X_\alpha$ by adding the point $*$ with the following topology:

• X_α is clopen in X for all $\alpha \in A$;

• the sets $X\backslash\{X_{\alpha_1}(+)\dots(+)X_{\alpha_k}\}$, $\alpha_i \in A$, $i = 1,\dots,k$, $k \in \mathbb{N}$, generate the base of the topology at $*$.

Obviously, if all X_α, $\alpha \in A$, are compact, then the one-point Aleksandrov extension coincides with the one-point Aleksandrov compactification.

Let us list some elementary properties of the one-point Aleksandrov extension. The notation $Z \simeq Y$ will mean that the spaces Z and Y are homeomorphic. Let $X = \{*\} \cup (+)_{\alpha\in A} X_\alpha$. Then

• if V is an open subset of $X\backslash\{*\}$ such that $B = \{\alpha \in A : (X\backslash V) \cap X_\alpha \neq \emptyset\}$ has cardinality $\geq \aleph_0$, then $X\backslash V = \{*\} \cup (+)_{\alpha\in B}(X_\alpha\backslash V) \hookrightarrow X$;

• if $Y_\alpha \hookrightarrow X_\alpha$ for all $\alpha \in A$, then $\{*\} \cup (+)_{\alpha\in A} Y_\alpha \hookrightarrow X$;

• if for every $\alpha \in A$ the space X_α is Hausdorff (Tikhonov, pseudocompact, normal, paracompact, compact, S-weakly infinite-dimensional, a C-space), then so is X;

• if $|A| = \aleph_0$ and for every $\alpha \in A$ the space X_α is perfectly normal (metrizable, with a countable base), then so is X;

• $\boldsymbol{\beta}(\{*\} \cup (+)_{\alpha\in A} X_\alpha) = \{*\} \cup (+)_{\alpha\in A} \boldsymbol{\beta} X_\alpha$ where $\boldsymbol{\beta}$ denotes the Čech–Stone compactification.

A family $S = \{X_\gamma, \gamma < \beta\}$ of topological spaces indexed by ordinals $< \beta$ will be called a β-*sequence* of topological spaces.

Definition 1. The *ordinal product* (resp. *ordinal* $\aleph_0$-*product*) of a β-sequence $S = \{X_\gamma,\ \gamma < \beta\}$ of topological spaces is, respectively, the topological space

$$\prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma = \begin{cases} \text{point} & \text{if } \beta = 0;\\ \Big(\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma\Big) \times X_\delta & \text{if } \beta = \delta + 1;\\ \{*\} \cup \underset{\delta<\beta}{(+)} \Big(\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma\Big) & \text{if } \beta \text{ is a limit ordinal,} \end{cases}$$

and

$$\prod_{\gamma<\beta}^{\omega,\mathrm{ord}} X_\gamma = \begin{cases} \text{point} & \text{if } \beta = 0;\\ \Big(\prod_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma\Big) \times X_\delta & \text{if } \beta = \delta + 1;\\ \{*\} \cup (+)\Big\{\Big(\prod_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma\Big)_i : \delta < \beta, i < \omega\Big\} & \text{if } \beta \text{ is limit.} \end{cases}$$

Here $(\prod_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma)_i \simeq \prod_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma$, $i < \omega$. The notation $\prod_{\gamma<\delta}^{(\omega),\mathrm{ord}} X_\gamma$ will mean either $\prod_{\gamma<\delta}^{\omega,\mathrm{ord}} X_\gamma$ or $\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma$.

Let us list some elementary properties of ordinal products. Let $X = \prod_{\gamma<\beta}^{(\omega),\mathrm{ord}} X_\gamma$. Then

• if $Y_\gamma \hookrightarrow X_\gamma$ for every $\gamma < \beta$, then $\prod_{\gamma<\beta}^{(\omega),\mathrm{ord}} Y_\gamma \hookrightarrow X$;

• if $\delta < \beta$, then $\prod_{\gamma<\delta}^{(\omega),\mathrm{ord}} X_\gamma \hookrightarrow X$;

• if for every $\gamma < \beta$ the space X_γ is a Hausdorff (Tikhonov, compact, compact C-) space, then so is X;

• if $\beta < \omega_1$ and for every $\gamma < \beta$ the space X_γ is metrizable (with a countable base), then so is X;

• if X is pseudocompact, then

$$\boldsymbol{\beta} X = \prod_{\gamma<\beta}^{(\omega),\mathrm{ord}} \boldsymbol{\beta} X_\gamma .$$

Recall that the product of two compact C-spaces is a compact C-space [Ha-Y], and if $X \times Y$ is pseudocompact, then $\boldsymbol{\beta}(X \times Y) = \boldsymbol{\beta} X \times \boldsymbol{\beta} Y$ [E1].

2. Ordinal power of a topological space (a special case). Let X be a topological space and $S = \{X_\gamma, \gamma < \beta\}$ be a β-sequence of topological spaces such that $X_\gamma \simeq X$ for every $\gamma < \beta$.

Definition 2. The *β-ordinal power* (resp. *$\aleph_0$-power*) of X is the space $S_\beta(X) = \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma$ (resp. $S_\beta^\omega(X) = \prod_{\gamma<\beta}^{\omega,\mathrm{ord}} X_\gamma$).

The notation $S_\beta^{(\omega)}(X)$ will mean either $S_\beta^\omega(X)$ or $S_\beta(X)$.

It is clear that, if $1 \le \beta < \omega_1$, then

• $S_\beta(I)$ is the Smirnov compactum S^β;

• $C \hookrightarrow S_\beta(C) \hookrightarrow C$, where C is the Cantor set.

Let us state some elementary properties of ordinal powers:

• if $X \hookrightarrow Y$, then $S_\beta^{(\omega)}(X) \hookrightarrow S_\beta^{(\omega)}(Y)$;

• if $\beta < \alpha$, then $S_\beta^{(\omega)}(X) \hookrightarrow S_\alpha^{(\omega)}(X)$.

The following statement will often be used below.

Lemma 1. *Let $\alpha < \omega_1$. Then*

(a) *if $\{\alpha_i\}_{i=1}^\infty$ is a sequence of ordinals such that $\alpha_i < \alpha_{i+1}$ and $\sup_i \alpha_i = \alpha$, then*

$$S_\alpha^{(\omega)}(X) \hookrightarrow \{*\} \cup \bigoplus_{i=1}^{\infty} S_{\alpha_i}^{(\omega)}(X) \hookrightarrow S_\alpha^{(\omega)}(X);$$

(b) $S_\alpha(X) \hookrightarrow S_\alpha^\omega(X) \hookrightarrow S_\alpha(X)$;

(c) *if α is a limit ordinal, then $S_\alpha^\omega(X) \simeq S_\alpha^\omega(X) \setminus \{$a finite number of terms of the free sum defining $S_\alpha^\omega(X)\}$.*

The proof is obvious.

We now prove the finite multiplicativity of $\aleph_0$-powers for countable ordinals.

THEOREM 1. *Let $\alpha, \beta < \omega_1$. Then*

$$S^{\omega}_{\alpha}(X) \times S^{\omega}_{\beta}(X) = S^{\omega}_{\alpha(+)\beta}(X)$$

(here $\alpha(+)\beta$ is the natural sum of the ordinals α and β; see appendix).

Proof. We use induction. Let $\beta = 1$, and let α be an ordinal $< \omega_1$. By the definition we have

$$S^{\omega}_{\alpha}(X) \times X = S^{\omega}_{\alpha(+)1}(X) .$$

Assume that for $\beta < \nu$ and for all $\alpha < \omega_1$ our statement is true, and let $\beta = \nu$. Suppose $\beta = \varepsilon + 1$. Then by the definition and the inductive assumption one can easily check that

$$\begin{aligned} S^{\omega}_{\alpha}(X) \times S^{\omega}_{\beta}(X) &= S^{\omega}_{\alpha}(X) \times S^{\omega}_{\varepsilon}(X) \times X \\ &= S^{\omega}_{\alpha+1}(X) \times S^{\omega}_{\varepsilon}(X) = S^{\omega}_{(\alpha+1)(+)\varepsilon}(X) = S^{\omega}_{\alpha(+)\beta}(X) . \end{aligned}$$

Let now β be a limit ordinal. We now use induction on α. For $\alpha = 1$ the statement is obvious. Suppose that for every $\alpha < \mu$ and the fixed limit β the statement is true, and let $\alpha = \mu$. Assume that $\alpha = \varepsilon + 1$. Then by the definition and the inductive assumption we have

$$S^{\omega}_{\alpha}(X) \times S^{\omega}_{\beta}(X) = S^{\omega}_{\varepsilon}(X) \times S^{\omega}_{\beta}(X) \times X = S^{\omega}_{\varepsilon(+)\beta}(X) \times X = S^{\omega}_{\alpha(+)\beta}(X) .$$

Let now α be a limit ordinal. By Definition 1 one has

$$S^{\omega}_{\alpha}(X) = \{*_1\} \cup (+)\{(S^{\omega}_{\delta}(X))_i : \delta < \alpha,\ i < \omega\} .$$

Note that for a fixed $\delta < \alpha$ the space $S^{\omega}_{\delta}(X)$ appears in the free sum countably many times. Let us number all spaces of the free sum by positive integers:

$$S^{\omega}_{\alpha}(X) = \{*_1\} \cup \bigoplus_{i=1}^{\infty} X_i ,$$

and the same for $S^{\omega}_{\beta}(X)$:

$$S^{\omega}_{\beta}(X) = \{*_2\} \cup \bigoplus_{i=1}^{\infty} Y_i .$$

We also number by positive integers all different ordinals of the form $\delta(+)\eta$, where $\delta < \alpha$ and $\eta < \beta$:

$$\{\delta(+)\eta : \delta < \alpha,\ \eta < \beta\} = \{\gamma_1, \gamma_2, \ldots\} .$$

By the inductive assumption the product $X_k \times Y_l$, $k, l < \omega$, is homeomorphic to S^{ω}_{ξ}, where $\xi = \delta(+)\eta$ for some $\delta < \alpha$ and $\eta < \beta$, and $X_k = S^{\omega}_{\delta}(X)$, $Y_l = S^{\omega}_{\eta}(X)$. Consider an increasing sequence $\{m(i)\}_{i=0}^{\infty}$ of positive integers with $m(0) = 1$ such that for every $i < \omega$ the spaces $S_{\gamma_j}(X)$, $j = 1, \ldots, i$, occur in the free sum

$$(+)\{X_k \times Y_l : m(i) \le k, l < m(i+1)\} .$$

Hence in the free sum

$$(+)\{X_k \times Y_l : m(i) \le k, l < m(i+1),\ i < \omega\}$$

there are countably many spaces $S_{\gamma_j}(X)$, for $j = 1, 2, \ldots$ Clearly,

$$\begin{aligned} S^\omega_\alpha(X) \times S^\omega_\beta(X) = \{*\} \cup (+)\{X_k \times Y_l \\ (+)X_k \times (S^\omega_\beta(X)\backslash(+)\{Y_p : p < m(i+1)\}) \\ (+)(S^\omega_\alpha(X)\backslash(+)\{X_p : p < m(i+1)\}) \times Y_l : \\ m(i) \le k, l < m(i+1),\ i < \omega\}\,. \end{aligned}$$

By Lemma 1(c) for every $i < \omega$ one has

$$S^\omega_\alpha(X)\backslash(+)\{X_p : p < m(i+1)\} = S^\omega_\alpha(X)\,,$$
$$S^\omega_\beta(X)\backslash(+)\{Y_p : p < m(i+1)\} = S^\omega_\beta(X)\,.$$

Hence

$$\begin{aligned} &S^\omega_\alpha(X) \times S^\omega_\beta(X) \\ &\quad = \{*\} \cup (+)\{X_k \times Y_l (+) X_k \times S^\omega_\beta(X)(+)S^\omega_\alpha(X) \times Y_l\ : \\ &\qquad\qquad m(i) \le k, l < m(i+1), i < \omega\}\,. \end{aligned}$$

Recall that $X_k = S^\omega_\delta(X)$ and $Y_l = S^\omega_\eta(X)$ for some $\delta < \alpha$ and $\eta < \beta$, $k, l < \omega$. Hence by the inductive assumption one has

$$X_k \times S^\omega_\beta(X) = S^\omega_\delta(X) \times S^\omega_\beta(X) = S^\omega_{\delta(+)\beta}(X)\,,$$
$$S^\omega_\alpha(X) \times Y_l = S^\omega_\alpha(X) \times S^\omega_\eta(X) = S^\omega_{\alpha(+)\eta}(X)\,.$$

Moreover, by Lemma A1 of the appendix for every $\gamma < \alpha(+)\beta$ there exist ordinals α_1, β_1 such that $\gamma = \alpha_1(+)\beta_1$, $\alpha_1 \le \alpha$, $\beta_1 \le \beta$. So $S^\omega_\alpha(X) \times S^\omega_\beta(X) = \{*\} \cup (+)\{(S^\omega_\nu(X))_i : \nu < \alpha(+)\beta,\ i < \omega\}$, where $(S^\omega_\nu(X))_i \simeq S^\omega_\nu(X)$, $i < \omega$. By Definitions 1, 2 one finally has $S^\omega_\alpha(X) \times S^\omega_\beta(X) = S^\omega_{\alpha(+)\beta}(X)$. The theorem is proved.

QUESTION 1. Can the assumption $\alpha, \beta < \omega_1$ be omitted in Theorem 1?

Remark 1. $S_\omega(I) \times S_\omega(I) \not\simeq S_{\omega(\cdot)2}(I)$, because in $S_\omega(I) \times S_\omega(I)$ there is one isolated point and in $S_{\omega(\cdot)2}(I)$ there are two isolated points.

Note, however, that for the ordinal powers there is a relation very close to equality:

COROLLARY 1. *Let* $\alpha, \beta < \omega_1$. *Then*

(a) $S_{\alpha(+)\beta}(X) \hookrightarrow S_\alpha(X) \times S_\beta(X) \hookrightarrow S_{\alpha(+)\beta}(X)$;

(b) $S^{(\omega)}_\alpha(X \times Y) \hookrightarrow S^{(\omega)}_\alpha(X) \times S^{(\omega)}_\alpha(Y) \hookrightarrow S^{(\omega)}_{p(\alpha)(\cdot)2+n(\alpha)}(X \times Y)$.

Proof. (a) follows directly from Theorem 1 by using Lemma 1(b). Let us prove (b). We shall examine the case of ordinal products and use induction.

Let $\alpha < \omega$. It is clear that $S_\alpha(X \times Y) = S_\alpha(X) \times S_\alpha(Y)$. Moreover, $p(\alpha)(\cdot)2 + n(\alpha) = \alpha$.

Let us prove the embedding $S_\alpha(X \times Y) \hookrightarrow S_\alpha(X) \times S_\alpha(Y)$ for $\alpha \geq \omega$. Assume that for $\alpha < \nu \geq \omega$ the statement is true, and let $\alpha = \nu$. Suppose that $\alpha = \varepsilon + 1$. By the definition and the inductive assumption one easily has

$$\begin{aligned} S_\alpha(X) \times S_\alpha(Y) &= S_\varepsilon(X) \times S_\varepsilon(Y) \times X \times Y \\ &\hookleftarrow S_\varepsilon(X \times Y) \times (X \times Y) = S_\alpha(X \times Y)\,. \end{aligned}$$

Let now α be a limit ordinal. Then there exists an increasing sequence $\{\alpha_i\}_{i=1}^\infty$ of ordinals such that $\sup_i \alpha_i = \alpha$. Consider the chain of embeddings:

$$\begin{aligned} &S_\alpha(X) \times S_\alpha(Y) \\ &\quad \hookleftarrow \Big(\{*_1\} \cup \bigoplus_{i=1}^{\infty} S_{\alpha_i}(X)\Big) \times \Big(\{*_2\} \cup \bigoplus_{i=1}^{\infty} S_{\alpha_i}(Y)\Big) \qquad \text{(by Lemma 1)} \\ &\quad \hookleftarrow \{*\} \cup \bigoplus_{i=1}^{\infty} S_{\alpha_i}(X) \times S_{\alpha_i}(Y) \\ &\quad \hookleftarrow \{*\} \cup \bigoplus_{i=1}^{\infty} S_{\alpha_i}(X \times Y) \qquad \text{(by the inductive assumption)} \\ &\quad \hookleftarrow S_\alpha(X \times Y) \qquad \text{(by Lemma 1)}\,. \end{aligned}$$

Let us prove the inverse embedding

$$S_\alpha(X) \times S_\alpha(Y) \hookrightarrow S_{p(\alpha)(\cdot)2+n(\alpha)}(X \times Y)\,.$$

Assume that for $\alpha < \nu \geq \omega$ the statement is true, and let $\alpha = \nu$. Suppose $\alpha = \varepsilon + 1$. Clearly,

$$S_\alpha(X) \times S_\alpha(Y) \hookrightarrow S_{p(\varepsilon)(\cdot)2+n(\varepsilon)}(X \times Y) \times (X \times Y) = S_{p(\alpha)(\cdot)2+n(\alpha)}(X \times Y)\,.$$

Suppose now that α is a limit ordinal. Then $p(\alpha)(\cdot)2 + n(\alpha) = \alpha(\cdot)2$. Clearly,

$$\begin{aligned} S_\alpha(X) \times S_\alpha(Y) &\hookrightarrow S_\alpha(X \times Y) \times S_\alpha(X \times Y) \\ &\hookrightarrow S_{\alpha(+)\alpha}(X \times Y) \qquad \text{(by (a))} \\ &= S_{\alpha(\cdot)2}(X \times Y)\,. \end{aligned}$$

The corollary is proved.

Now we get the following statement.

Corollary 2. *Let Φ be a numerical function on topological spaces, monotone on closed sets, for example a dimension* (ind, Ind, $\dim_{\mathrm{w}}$, *D or others*). *Then*

$$\Phi(S_\alpha^{(\omega)}(X) \times S_\beta^{(\omega)}(X)) = \Phi(S_{\alpha(+)\beta}^{(\omega)}(X))\,.$$

In particular, for Smirnov compacta S^γ, $\gamma < \omega_1$, *one has*

(a) $\mathrm{DIM}\, S^\alpha \times S^\beta = \alpha(+)\beta$, *where* DIM *is* $\dim_{\mathrm{w}}$, Ind, Id *or* D;

(b) $\mathrm{ind}\, S^\alpha \times S^\beta = \mathrm{ind}\, S^{\alpha(+)\beta}$;

(c) $S^\alpha \times S^\beta$ *can be continuously mapped into* I *so that every point of* I *has a finite-dimensional preimage.*

Proof. Recall (see the introduction) that $\dim_{\mathrm{w}} S^\alpha = \mathrm{Ind}\, S^\alpha = DS^\alpha = \mathrm{Id}\, S^\alpha = \alpha$ for $\alpha < \omega_1$ (for Id see Corollary 5 in §3).

Remark 2. Since the product $H^\alpha \times H^\beta$ cannot be continuously mapped into the interval with every point preimage finite-dimensional, and H^γ can be mapped in such a way, the inclusion $H^\alpha \times H^\beta \hookrightarrow H^\gamma$ is not true for any infinite $\alpha, \beta, \gamma < \omega_1$. Therefore for the upper estimate of the dimension of $H^\alpha \times H^\beta$ we need Corollary 5 of §3, namely: $\mathrm{Id}\, H^\alpha \times H^\beta \le \mathrm{Id}\, H^\alpha (+) \mathrm{Id}\, H^\beta = \alpha(+)\beta$.

Since $S^{\alpha(+)\beta} \hookrightarrow S^\alpha \times S^\beta \hookrightarrow H^\alpha \times H^\beta$, the lower estimate is immediate; recall that $\dim X \le \mathrm{Ind}\, X \le \mathrm{Id}\, X$ for every compactum X. In the case of the dimension D, we need the following statements mentioned in the introduction: $DS^\alpha = \alpha$, $\alpha < \omega_1$, $D(H^\alpha \times H^\beta) \le DH^\alpha(+)DH^\beta$ and the equalities $DH^\gamma = \gamma$, $\gamma < \omega_1$, which one easily checks by induction using the following properties of D in the class of metrizable spaces [He2]:

- if either $\mathrm{Ind}\, X$ or DX is finite then they are equal;
- $D(X \times Y) \le DX(+)DY$;
- if X is the union of a locally finite collection of closed subsets each with D-dimension $\le \beta$, then $DX \le \beta$;
- if F is a closed subset of the space X then $DX \le D(X \setminus F) + DF$.

(For the definition of Henderson's compacta H^γ, $\gamma < \omega_1$, see the proof of Corollary 5.)

One finally has $\mathrm{DIM}\, H^\alpha \times H^\beta = \alpha(+)\beta$, where DIM is Ind, Id, $\dim_{\mathrm{w}}$ or D, and $\alpha, \beta < \omega_1$.

In order to simplify formulas we write $\prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma$ and $S_\beta(X)$ for ordinal $\aleph_0$-products and ordinal $\aleph_0$-powers. The notation $(+)_{\gamma<\beta}\alpha_\gamma$ denotes the inductive extension of the natural sum to infinite sequences of ordinals (see appendix).

Theorem 2. *Let* X *be an arbitrary topological space and let ordinals* β *and* $\alpha_\gamma \ge 1$, $\gamma < \beta$, *be countable. Then*

$$(*) \qquad S_{(+)_{\gamma<\beta}\alpha_\gamma}(X) \hookrightarrow \prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(X) \hookrightarrow S_{(+)_{\gamma<\beta}\alpha_\gamma}(X)\,.$$

Proof. We use induction. Let $\beta < \omega$. Then $(*)$ is obvious by Corollary 1. Suppose that the statement is true for all $\beta < \nu \ge \omega$, and let $\beta = \nu \ge \omega$.

Assume that $\beta = \varepsilon + 1$. Then by the inductive assumption one has

$$(**) \qquad S_{(+)_{\gamma<\varepsilon}\alpha_\gamma}(X) \hookrightarrow \prod_{\gamma<\varepsilon}^{\mathrm{ord}} S_{\alpha_\gamma}(X) \hookrightarrow S_{(+)_{\gamma<\varepsilon}\alpha_\gamma}(X)\,.$$

Multiply $(**)$ by $S_{\alpha_\varepsilon}(X)$. By the definition one has $((+)_{\gamma<\varepsilon}\alpha_\gamma)(+)\alpha_\varepsilon = (+)_{\gamma<\beta}\alpha_\gamma$. Now we only need to use either Corollary 1 or Theorem 1.

Let now β be a limit ordinal. Then there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of ordinals such that $\sup_i \beta_i = \beta$. Set $\sigma_i = (+)_{\gamma<\beta_i}\alpha_\gamma$. It is clear that $(+)_{\gamma<\beta}\alpha_\gamma = \sup_i \sigma_i$. By the inductive assumption for every $i < \omega$ one has

$$S_{\sigma_i}(X) \hookrightarrow \prod_{\gamma<\beta_i}^{\mathrm{ord}} S_{\alpha_\gamma}(X) \hookrightarrow S_{\sigma_i}(X)\,.$$

Moreover, obviously

$$\prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(X) \subset \{*\} \cup \mathop{(+)}_{i=1}^{\infty} \prod_{\gamma<\beta_i}^{\mathrm{ord}} S_{\alpha_\gamma}(X) \hookrightarrow \prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(X)$$

and

$$S_{(+)_{\gamma<\beta}\alpha_\gamma}(X) \hookrightarrow \{*\} \cup \mathop{(+)}_{i=1}^{\infty} S_{\sigma_i}(X) \hookrightarrow S_{(+)_{\gamma<\beta}\alpha_\gamma}(X)\,.$$

Our statement follows from these three chains of embeddings. The theorem is proved.

Remark 3. $S_\omega^\omega(S_2^\omega(I)) \neq S_\omega^\omega(I)$, though $2 \times \omega = \omega$.

COROLLARY 3. *Let* $\alpha, \beta < \omega_1$. *Then*

$$S_{\alpha\times\beta}(X) \hookrightarrow S_\beta(S_\alpha(X)) \hookrightarrow S_{\alpha\times\beta}(X)\,.$$

Proof. Let us only note that if we set $\alpha_\gamma = \alpha$ for all $\gamma < \beta$, then by the definition from the appendix we have $\alpha \times \beta = (+)_{\gamma<\beta}\alpha_\gamma$, and by the definition of the power $\prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(X) = S_\beta(S_\alpha(X))$.

COROLLARY 4. *Let Φ be a numerical function on topological spaces, monotone on closed sets, for example a dimension* (ind, Ind, $\dim_{\mathrm{w}}$, *D or others*), *and let* α_γ, $\beta < \omega_1$. *Then*

$$\Phi\Big(\prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(X)\Big) = \Phi(S_{(+)_{\gamma<\beta}\alpha_\gamma}(X))\,.$$

Note that the product $\prod_{\gamma<\beta}^{\mathrm{ord}} S_{\alpha_\gamma}(I)$ can be continuously mapped into an interval with all point preimages finite-dimensional. For the product $\prod_{\gamma<\beta}^{\mathrm{ord}} H_{\alpha_\gamma}(I)$ there is no such mapping.

3. Solution of the product problem for the inductive dimension functions id **and** id_{p}**.** Let X be a normal space.

Definition 3. Let $\operatorname{Id} X = -1$ iff $X = \emptyset$. Let $\operatorname{Id} X \leq \alpha$, where α is an ordinal, if there are collections $\sigma_{-1}, \sigma_0, \ldots, \sigma_\delta$, $\delta \leq \alpha$, of closed subsets of X such that

(a) $\sigma_{-1} = \{\emptyset\}$, $\sigma_\delta \ni X$, $\sigma_\beta \subseteq \sigma_\gamma$ for all β, γ with $-1 \leq \beta \leq \gamma \leq \delta$;

(b) for all $0 \leq \gamma \leq \delta$ and $F \in \sigma_\gamma$ and every pair of disjoint closed subsets A and B of X there exist $\beta < \gamma$ and $\psi \in \sigma_\beta$ such that $\psi \subset F$ and ψ is a partition in F between $A \cap F$ and $B \cap F$;

(c) for every $\gamma \leq \delta$ and every pair F_1, F_2 of elements of σ_γ there exists $F \in \sigma_\gamma$ such that $F \hookleftarrow F_1 \cup F_2$.

Let us put $\operatorname{Id} X = \min\{\alpha : \operatorname{Id} X \leq \alpha\}$.

If $\operatorname{Id} X \leq \alpha$ for no ordinal α, then we define $\operatorname{Id} X = \infty$ (which means that $\operatorname{Id} X$ does not exist).

Note that for every $F \in \sigma_\gamma$, $\gamma \leq \delta$, we have $\operatorname{Id} F \leq \gamma$.

If one of the sets from Definition 3, say A, is a single point, we get the definition of the dimension id.

If both A, B are singletons, we get the definition of the dimension id_p. If both A, B are compacta we get the definition of the dimension cId.

Clearly, $\mathrm{id}_\mathrm{p} X \leq \mathrm{id}\, X \leq \operatorname{Id} X$ and $\mathrm{id}_\mathrm{p} X \leq \mathrm{cId}\, X$ for every space X.

Remark 4. (a) The definitions of the finite dimensions id and Id and the statements given below for id and Id in the finite-dimensional case are due to Pasynkov [Pa1].

(b) The transfinite extensions of id and Id were independently considered by Vinogradov. He has independently proved the elementary properties of these dimensions given below, as well as Theorem 3 (for id) and Corollaries 5, 6.

Let DM be id_p, id, Id or cId, and DIM be ind_p, ind, Ind or cInd (in the definition of ind_p the partitions are taken between points and in the definition of cInd—between compacta [Ha]).

Statement 1. *$\mathrm{DIM}\, X \leq \alpha$ iff there exist collections $\sigma_{-1}, \ldots, \sigma_\delta$, $\delta \leq \alpha$, of closed subsets of X satisfying conditions* (a), (b) *of Definition* 3.

Proof. Let $\mathrm{DIM}\, X = \delta \leq \alpha$. Put $\sigma_\gamma = \{Y \subseteq X : Y$ is closed in X and $\mathrm{DIM}\, Y \leq \gamma\}$, $\gamma = -1, \ldots, \delta$. It is clear that (a) and (b) are satisfied. Let us prove the converse. We use induction. Let $\alpha = -1$; then $X = \emptyset$ and, hence, $\mathrm{DIM}\, X = -1$. Suppose that for $\alpha < \nu$ the statement is true, and let $\alpha = \nu$. By (a), $X \in \sigma_\delta$, $\delta \leq \nu$, and by (b) for every pair A, B of disjoint closed subsets of X there exist $\beta < \delta \leq \nu$ and $\psi \in \sigma_\beta$ such that ψ is a partition in X between A and B. Consider the collections $\sigma_\gamma^1 = \{C \cap \psi : C \in \sigma_\gamma\}$, $\gamma \leq \beta$, of subsets of ψ. It is clear that they satisfy conditions (a) and (b). By the inductive assumption we have $\mathrm{DIM}\, \psi \leq \beta < \nu$. Hence, $\mathrm{DIM}\, X \leq \nu$.

A collection $\mathcal{B}$ of closed subsets of X is called *monotone* if every closed subset of any element of $\mathcal{B}$ also belongs to $\mathcal{B}$. A collection $\mathcal{B}$ of closed subsets of X is called *additive* if for any $A, B \in \mathcal{B}$ we have $A \cup B \in \mathcal{B}$.

STATEMENT 2. $\mathrm{DM}\, X \le \alpha$ *iff there exist collections* $\sigma_{-1}, \ldots, \sigma_\delta$, $\delta \le \alpha$, *of closed subsets of* X *satisfying conditions* (a), (b) *of Definition* 3 *and the condition*

(c)^1 σ_γ *is monotone and additive for every* $\gamma \le \delta$.

Proof. The "if" part is clear. Let us prove the "only if" part. Let $\mathrm{DM}\, X \le \alpha$. Then there exist collections $\sigma_{-1}, \ldots, \sigma_\delta$, $\delta \le \alpha$, of closed subsets of X satisfying conditions (a), (b) and (c) of Definition 3. Put

$$\sigma_\gamma^1 = \{A \subseteq B : A \text{ is closed in } X \text{ and } B \in \sigma_\gamma\}, \quad \gamma \le \delta\,.$$

It is clear that these collections satisfy (a), (b) and (c)^1.

Note that in Statement 2 one can always put $\delta = \alpha$. From Statements 1 and 2 we get

STATEMENT 3. *If* $\mathrm{DM}\, X$ *exists*, *then* $\mathrm{DIM}\, X$ *exists. Moreover*, $\mathrm{DIM}\, X \le \mathrm{DM}\, X$.

STATEMENT 4. (i) *For every compactum* X *we have* $\mathrm{id_p}\, X = \mathrm{id}\, X = \mathrm{Id}\, X = \mathrm{cId}\, X$.

(ii) *For every normal space* X *we have* $\mathrm{cId}\, X = \mathrm{id_p}\, X$.

Proof. (i) Let $\mathrm{id_p}\, X \le \alpha$. There exist collections $\sigma_{-1}, \sigma_0, \ldots, \sigma_\alpha$ of closed subsets of X, satisfying (a), (b) and (c)^1 (with A and B being points). By additivity and monotonicity (Statement 2), the same systems σ_γ, $\gamma \le \alpha$, satisfy (a), (b) and (c)^1 for A and B arbitrary closed disjoint sets. Thus (i) is proved. The proof of (ii) is analogous.

STATEMENT 5. (i) *If* F *is closed in* X, *then* $\mathrm{DM}\, F \le \mathrm{DM}\, X$.

(ii) *Suppose* $\mathrm{DIM}\, X$ *exists and the finite sum theorem for* DIM *holds in* X, *for example*, X *is a metric compactum with* $\mathrm{Ind}\, X \le \omega$. *Then* $\mathrm{DM}\, X$ *exists and* $\mathrm{DM}\, X = \mathrm{DIM}\, X$.

(iii) *Let* X_α *be a compactum with* $\mathrm{DM}\, X_\alpha = \beta_\alpha$, $\alpha \in A$, $|A| \ge \omega$, *and* $\sup_{\alpha \in A} \beta_\alpha = \beta$. *Let* $X = \{*\} \cup (+)_{\alpha \in A} X_\alpha$ *be the one-point Aleksandrov compactification of the free sum of the spaces* X_α, $\alpha \in A$. *Then* $\mathrm{DM}\, X$ *exists and* $\mathrm{DM}\, X = \beta$.

Proof. (i) is clear.

(ii) Let $\mathrm{DIM}\, X = \alpha$. Then for $\gamma = -1, 0, \ldots, \alpha$ we put

$$\sigma_\gamma = \{Y \subseteq X : Y \text{ is closed in } X \text{ and } \mathrm{DIM}\, Y \le \gamma\}\,.$$

By the monotonicity of DIM on closed sets and the finite sum theorem one can easily check that the collections σ_γ, $\gamma \le \alpha$, satisfy conditions (a), (b) and (c)[1]. So $\mathrm{DM}\, X \le \mathrm{DIM}\, X$. From Statement 3 one finally has $\mathrm{DM}\, X = \mathrm{DIM}\, X$.

(iii) Let $\sigma_\gamma^{(\alpha)}$, $\gamma \le \beta$, be collections of closed subsets of X satisfying (a), (b) and (c)[1], for $\alpha \in A$. Put

$$\sigma_\gamma = \{A_1 \cup \ldots \cup A_k : A_i \in \sigma_\gamma^{(\alpha_i)},\ \alpha_i \in A,\ k \in \mathbb{N}\}, \qquad \gamma < \beta\,,$$

and $\sigma_\beta = \{A \subseteq X : A \text{ is compact}\}$. It is clear that for $\sigma_{-1}, \ldots, \sigma_\beta$ conditions (a), (b) and (c)[1] are satisfied.

Hence, $\mathrm{DM}\, X \le \beta$. Since $\mathrm{DM}\, X_\alpha \le \mathrm{DM}\, X$ for every $\alpha \in A$, one finally obtains $\mathrm{DM}\, X = \beta$.

Questions. 2. Does $\mathrm{DM}\, X$ exist if $\mathrm{DIM}\, X$ exists?

3. It is known that for Smirnov compacta with $\mathrm{Ind} > \omega$ the finite sum theorem for Ind does not hold [Le]. Does there exist a compactum X with $\mathrm{Ind}\, X > \omega$ in which the finite sum theorem for Ind holds?

4. By Filippov's [F] and Pasynkov's [Pa1] results there exists a compactum X with $\mathrm{Ind}\, X = 2$ and $\mathrm{Id}\, X \ge 3$. How large may be the difference between Ind and Id for infinite-dimensional spaces (for DIM and DM)?

Lemma 2 (B. A. Pasynkov). *Let $X = F_1 \cup \ldots \cup F_n$ be a normal space, where $F_i, i = 1, \ldots, n$, are closed in X. Let A and B be two disjoint closed subsets of X, and C_i be a partition in F_i between $A \cap F_i$ and $B \cap F_i$, $i = 1, \ldots, n$. Then there exists a partition C in X between A and B such that*

$$C \subseteq \Big(\bigcup_{i=1}^{n} C_i\Big) \cup \bigcup_{i<j} (F_i \cap F_j)\,.$$

The notation $\mathrm{id}_{(\mathrm{p})}$ will mean either id_{p} or id. The main statement of this section is

Theorem 3. *Let $X_1 \times X_2$ be a normal space and suppose $\mathrm{id}_{(\mathrm{p})}\, X_i$ exists for $i = 1, 2$. Then*

$$\mathrm{id}_{(\mathrm{p})}\, X_1 \times X_2 \le \mathrm{id}_{(\mathrm{p})}\, X_1 (+)\, \mathrm{id}_{(\mathrm{p})}\, X_2\,.$$

P r o o f. Let us show the inequality

$$(\#) \qquad \mathrm{id}_{\mathrm{p}}\, X_1 \times X_2 \le \mathrm{id}_{\mathrm{p}}\, X_1 (+)\, \mathrm{id}_{\mathrm{p}}\, X_2\,.$$

For id the proof of the corresponding inequality is analogous.

Let $\mathrm{id}_{\mathrm{p}}\, X_1 = \xi$, $\mathrm{id}_{\mathrm{p}}\, X_2 = \zeta$ and let $\sigma_{-1,1} \subseteq \ldots \subseteq \sigma_{\xi,1}$ be collections of closed subsets of X_1, and $\sigma_{-1,2} \subseteq \ldots \subseteq \sigma_{\zeta,2}$ collections of closed subsets of X_2 satisfying (a), (b) and (c)[1] for A and B being points. Put

$$\Sigma = \xi(+)\zeta, \qquad \sigma_{-1} = \{\emptyset\}\,,$$

$$\sigma_\gamma^1 = \{A \times B : A \in \sigma_{\alpha,1},\ B \in \sigma_{\beta,2} \text{ and } \alpha(+)\beta \le \gamma\},$$

$$\sigma_\gamma^2 = \{C_1 \cup \ldots \cup C_k : C_i \in \sigma_\gamma^1,\ i = 1, \ldots, k,\ k \in \mathbb{N}\},$$

$$\sigma_\gamma = \{P \subseteq C : P \text{ is closed in } X_1 \times X_2 \text{ and } C \in \sigma_\gamma^2\}, \quad \gamma \le \Sigma.$$

Obviously, the collections σ_γ, $\gamma \le \Sigma$, satisfy (a) and (c)[1]. Let us show that (b) holds. Let $P \in \sigma_\gamma$ and A, B be a pair of different points in P. We have to show that there exist $\nu < \gamma$ and a set $C \in \sigma_\nu$ with $C \subseteq P$ which is a partition in P between A and B.

The main case. Let $P \subseteq D_1 \times D_2$, where $D_1 \in \sigma_{\alpha,1}$, $D_2 \in \sigma_{\beta,2}$ and $\alpha(+)\beta \le \gamma$. From the monotonicity of the collections σ_γ, $\gamma \le \Sigma$, one can easily check that it is only necessary to consider the case $P = D_1 \times D_2$.

Let $\pi : X_1 \times X_2 \to X_1$ be the projection. Since $A = B$, without loss of generality one can assume that $\pi A = \pi B$. There exists a partition C_1 in D_1 between the points πA, πB such that $C_1 \in \sigma_{\lambda,1}$, $\lambda < \alpha$. Then clearly $\pi_1^{-1} C_1$ is a partition in P between A and B, and $\pi_1^{-1} C_1 \in \sigma_{\lambda(+)\beta}$. Note that $\lambda(+)\beta < \alpha(+)\beta$ (see appendix).

The general case. Without loss of generality, by monotonicity and additivity of σ_γ, $\gamma \le \Sigma$, and by Lemma 2 one can assume that $P = (D_1^{(1)} \times D_2^{(1)}) \cup (D_1^{(2)} \times D_2^{(2)})$, where $D_1^{(i)} \in \sigma_{\alpha_i,1}$, $D_2^{(i)} \in \sigma_{\beta_i,2}$ and $\alpha_1(+)\beta_1 \le \gamma$, $\alpha_2(+)\beta_2 \le \gamma$. Two cases are possible:

(I) $\alpha_1(+)\beta_1 < \alpha_2(+)\beta_2 \le \gamma$. By the main case there exists a partition C_1 in $D_1^{(2)} \times D_2^{(2)}$ between A and B such that $C_1 \in \sigma_\mu$ for some $\mu < \gamma$. By Lemma 2 one can choose a partition C in P between A and B such that $C \subseteq C_1 \cup (D_1^{(1)} \times D_2^{(1)})$. Let $\nu = \max(\alpha_1(+)\beta_1, \mu) < \gamma$. Then $C \in \sigma_\nu$.

(II) $\alpha_1(+)\beta_1 = \alpha_2(+)\beta_2$. The following subcases are possible.

$(\mathrm{II})_1$ Let $\alpha_1 = \alpha_2 = \alpha$. Then (see appendix) $\beta_1 = \beta_2 = \beta$. By additivity, $D_1 = D_1^{(1)} \cup D_1^{(2)} \in \sigma_{\alpha,1}$, $D_2 = D_2^{(1)} \cup D_2^{(2)} \in \sigma_{\beta,1}$ and $P \subseteq D_1 \times D_2$. So the conditions of the main case are satisfied.

$(\mathrm{II})_2$ Let $\alpha_1 < \alpha_2$. Then (see appendix) $\beta_1 > \beta_2$. In this case by monotonicity one has $D_1^{(1)} \cap D_1^{(2)} \in \sigma_{\alpha_1,1}$, $D_2^{(1)} \cap D_2^{(2)} \in \sigma_{\beta_2,2}$ and

$$L = (D_1^{(1)} \times D_2^{(1)}) \cap (D_1^{(2)} \times D_2^{(2)}) = (D_1^{(1)} \cap D_1^{(2)}) \times (D_2^{(1)} \cap D_2^{(2)}).$$

Moreover, $\alpha_1(+)\beta_2 < \alpha_1(+)\beta_1 = \alpha_2(+)\beta_2 \le \gamma$ and $L \in \sigma_{\alpha_1(+)\beta_2}$. By the main case there exists a partition C_i in $D_1^{(i)} \times D_2^{(i)}$ between A and B such that $C_i \in \sigma_{\mu_i}$ for some $\mu_i < \gamma$, $i = 1, 2$. By Lemma 1 there exists a partition C in P between A and B such that $C \subseteq L \cup C_1 \cup C_2$. Let $\nu = \max(\alpha_1(+)\beta_2, \mu_1, \mu_2) < \gamma$. By the additivity and monotonicity of σ_γ, $\gamma \le \Sigma$, we get $C \in \sigma_\nu$. The theorem is proved.

In the sequel, dim stands for Borst's [B1] transfinite extension $\dim_{\mathrm{w}}$ of the Lebesgue covering dimension dim. Recall that $\dim X \le \operatorname{Ind} X$ for every normal space X [B1].

COROLLARY 5. (i) *Let* X_1, X_2 *be compacta for which* Id *exists. Then*

$$\dim X_1 \times X_2 \le \operatorname{Ind} X_1 \times X_2 \le \operatorname{Id} X_1 \times X_2 \le \operatorname{Id} X_1 (+) \operatorname{Id} X_2 .$$

In particular, *if* $\operatorname{Id} X_2 = n < \omega$, *for example*, $X_2 = I^n$, *then*

$$\dim X_1 \times X_2 \le \operatorname{Ind} X_1 \times X_2 \le \operatorname{Id} X_1 \times X_2 \le \operatorname{Id} X_1 + n .$$

(ii) $\operatorname{Id} S^\alpha = \alpha$ *for* $\alpha < \omega_1$.
(iii) $\operatorname{Id} H^\alpha = \alpha$ *for* $\alpha < \omega_1$.

Proof. (i) is evident.

(ii) One can easily check by induction using (i) and Statement 5(iii) that $\operatorname{Id} S^\alpha = \alpha$ for $\alpha < \omega_1$ (recall that $\operatorname{Ind} S^\alpha = \alpha$ for $\alpha < \omega_1$, see [S]).

(iii) Recall Henderson's description of H^α. For $\alpha < \omega_1$, we define H^α and p_α as follows: $H^0 = \{0\}$, $H^1 = [0,1] = I$ and $p_1 = 0$; $H^{\alpha+1} = H^\alpha \times I$ and $p_{\alpha+1} = p_\alpha \times \{0\}$; if α is a limit ordinal, for $\beta < \alpha$ let A^β_α be a half-open arc with $H^\beta \cap A^\beta_\alpha = \{p_\beta\}$; then $H^\alpha = \{*\} \cup (+)_{\beta<\alpha}(H^\beta \cup A^\beta_\alpha)$ is the one point-compactification of the free sum where $p_\alpha = *$ is the compactification point. Since $S^\alpha \hookrightarrow H^\alpha$ and $\operatorname{Id} S^\alpha = \alpha$ for $\alpha < \omega_1$ (see (ii)), we need to prove that $\operatorname{Id} H^\alpha \le \alpha$ for $\alpha < \omega_1$. If α is a non-limit ordinal we can use (i).

Let now α be a limit ordinal and suppose that $\operatorname{Id} H^\beta \le \beta$ for all $\beta < \alpha$. By Statement 4, Id can be replaced by id_{p} in the above inequality. Let $\sigma^{(\beta)}_\gamma$, $\gamma \le \alpha$, be collections of closed subsets of H^β satisfying (a), (b) and (c)[1] (see Statement 2 for id_{p}) for $\beta < \alpha$. Put

$$M = \{\emptyset\} \cup \Big\{\text{finite subsets of } \bigcup\{A^\beta_\alpha : \beta < \alpha\}\Big\}, \qquad \sigma_{-1} = \{\emptyset\} ,$$
$$\sigma_\gamma = \{A_1 \cup \ldots \cup A_k \cup P : A_i \in \sigma^{(\beta_i)}_\gamma, \beta_i < \alpha, k \in \mathbb{N}, P \in M\}, \qquad \gamma < \alpha ,$$

and $\sigma_\alpha = \{A \subseteq X : A \text{ is compact}\}$. Obviously, $\sigma_{-1}, \ldots, \sigma_\alpha$ satisfy (a), (b) and (c)[1].

COROLLARY 6. *Let* X_1 *and* X_2 *be compacta for which* Ind *exists and in which the finite sum theorem for* Ind *holds. Then*

$$\dim X_1 \times X_2 \le \operatorname{Ind} X_1 \times X_2 \le \operatorname{Ind} X_1 (+) \operatorname{Ind} X_2 .$$

In particular, *if* $\operatorname{Ind} X_2 = n < \omega$, *for example*, $X_2 = I^n$, *then* $\dim X_1 \times X_2 \le \operatorname{Ind} X_1 \times X_2 \le \operatorname{Ind} X_1 + n$.

QUESTION 5. Are there two metric compacta X_1 and X_2 such that $\operatorname{Ind} X_1 \times X_2 > \operatorname{Ind} X_1 (+) \operatorname{Ind} X_2$? The same question may be asked for $\dim_{\mathrm{w}}$, $\dim_{\mathrm{c}}$ and ind.

Now let us recall the definition of metrizable spaces M_α, α an ordinal, constructed by Hattori [Ha]. $M_0 = \{*\}$ is a one-point space. Suppose that the metrizable spaces M_β are defined for all $\beta < \alpha$. If $\alpha = \beta + 1$, define $M_\alpha = M_\beta \times I$. If α is a limit ordinal, let M_α be the tolopogical sum of copies of all M_β, $\beta < \alpha$, together with a new point x_α, with the following topology: for each $\beta < \alpha$, let $h_{\beta\alpha}$ be the natural embedding of M_β in M_α. Then $M_\alpha = \{x_\alpha\} \cup (+)\{h_{\beta\alpha}(M_\beta) : \beta < \alpha\}$. A subset U of $(+)\{h_{\beta\alpha}(M_\beta) : \beta < \alpha\}$ is open in M_α if $U \cap h_{\beta\alpha}(M_\beta)$ is open in $h_{\beta\alpha}(M_\beta)$ ($h_{\beta\alpha}(M_\beta)$ is homeomorphic to M_β) for all $\beta < \alpha$. For every $n, m \in \mathbb{N}$ let

$$V_m(\alpha) = (+)\{h_{\gamma+m,\alpha}(M_{\gamma+m}) : \gamma < \alpha \text{ is a limit ordinal}\},$$
$$U_n(x_\alpha) = \{x_\alpha\} \cup \bigcup\{V_m(\alpha) : m \geq n\}.$$

Let $\{U_n(x_\alpha) : n \in \mathbb{N}\}$ be a base of neighborhoods at x_α. This completes the inductive construction. In [Ha] Hattori showed the inequalities $\operatorname{cInd} X \leq \omega \operatorname{ind} X$ for every metric space X and $\operatorname{cInd} M_\alpha \geq \alpha$ for any α, from which he got an affirmative answer to question 3.11 of [E3]: for every ordinal α there exists a metrizable space X_α with $\operatorname{ind} X_\alpha = \alpha$. Note that Pasynkov [Pa2] had earlier stated the same result.

We have the following

Corollary 7. (i) $\operatorname{cId} M_\alpha = \operatorname{id_p} M_\alpha = \alpha$;

(ii) $\operatorname{cId} M_\alpha \times M_\beta = \operatorname{id_p} M_\alpha \times M_\beta \leq \alpha(+)\beta$.

Proof. Let us only note (see Statements 3 and 4) that $\alpha \leq \operatorname{cInd} M_\alpha \leq \operatorname{cId} M_\alpha = \operatorname{id_p} M_\alpha$ and $\operatorname{id_p} I^n = n$.

4. Dimension of ordinal products. Let $\mathcal{B} = \{X_\gamma, \gamma < \beta\}$ be a β-sequence of topological spaces. In this section, $\prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma$ stands for either $\prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma$ or $\prod^{\omega,\mathrm{ord}}_{\gamma<\beta} X_\gamma$.

Theorem 4. *Suppose $\prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma$ is a normal space* (*for example*, *all X_γ, $\gamma < \beta$, are metric and $\beta < \omega_1$*). *Then*

$$\operatorname{id}_{(\mathrm{p})} \prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma \leq \underset{\gamma<\beta}{(+)} \operatorname{id}_{(\mathrm{p})} X_\gamma.$$

Proof. The statement directly follows from Theorem 3, Statement 5 and the definition of the natural sum of a β-sequence of ordinals (see appendix).

Corollary 8. *Let X_γ, $\gamma < \beta$, be compact spaces. Then*

$$\operatorname{Id} \prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma \leq \underset{\gamma<\beta}{(+)} \operatorname{Id} X_\gamma.$$

Proof. Use Statement 4.

COROLLARY 9. (i) *Suppose* $S_\beta(X)$ *is a normal space and* $\mathrm{id}_{(\mathrm{p})} X = \alpha$. *Then* $\mathrm{id}_{(\mathrm{p})} S_\beta(X) \le \alpha \times \beta$.

(ii) $\mathrm{ind}\, S_\alpha(\mathbb{R}^k) \le \mathrm{id}\, S_\alpha(\mathbb{R}^k) \le p(\alpha) + k \cdot n(\alpha)$, *where* $\mathbb{R}$ *is the real line* (note that $\mathrm{ind}\, S_{\omega+3}(\mathbb{R}) = \omega + 2 < \omega + 3$, see [L]).

(iii) *If* X_γ, $\gamma < \beta$, *are finite-dimensional metric compacta*, *then*

$$\dim \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma \le \mathrm{Ind} \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma \le \mathrm{Id} \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma \le \underset{\gamma<\beta}{(+)} \dim X_\gamma \,.$$

(iv) *If all* X_γ, $\gamma < \beta$, *are finite-dimensional in the sense of* $\mathrm{ind}_{(\mathrm{p})}$, *the finite sum theorem is satisfied for this dimension and* $\prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma$ *is a normal space*, *then*

$$\mathrm{ind}_{(\mathrm{p})} \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma \le \mathrm{id}_{(\mathrm{p})} \prod_{\gamma<\beta}^{\mathrm{ord}} X_\gamma \le \underset{\gamma<\beta}{(+)} \mathrm{ind}_{(\mathrm{p})} X_\gamma \,.$$

Proof. Just note that $\alpha \times \beta = (+)_{\gamma<\beta} \alpha_\gamma$ where $\alpha_\gamma = \alpha$ for all γ.

For L an arbitrary set, $\mathrm{Fin}\, L$ is the family of all finite non-empty subsets of L. The notation $L(X)$ stands for the family of all pairs of disjoint closed subsets of the topological space X. A finite set $\{(A_i, B_i)\}_{i=1}^n \in \mathrm{Fin}\, L(X)$ is called *essential* if every collection of partitions $\{C_i\}_{i=1}^n$, where C_i is a partition between A_i and B_i, has non-empty intersection. Put $M_L = \{\sigma \in \mathrm{Fin}\, L : \sigma \text{ is essential}\}$ for $L \subseteq L(X)$.

LEMMA 3 (A. N. Dranishnikov [D]). *Let* $\{(A_i, B_i)\}_{i=1}^n$ *be an essential set in a metric compactum* X, Z *a metric continuum and* z^+, z^- *different points in* Z. *Then the set* $\{(A_i \times Z, B_i \times Z)\}_{i=1}^n \cup (X \times z^+, X \times z^-)$ *is essential in* $X \times Z$.

Let X_γ, $\gamma < \beta$, be metric one-dimensional continua and z_γ^+, z_γ^- be different points in X_γ. For every $\delta \le \beta$ we define a subset P_δ of $L(\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma)$ by

$$\begin{aligned} P_1 &= \{(z_0^-, z_0^+)\}\,, \\ P_{\delta+1} &= \{(F \times X_\delta, G \times X_\delta) : (F, G) \in P_\delta\} \\ &\quad \cup \Big(\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma \times z_\delta^-, \prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma \times z_\delta^+ \Big)\,, \end{aligned}$$

and if δ is a limit number, then

$$P_\delta = \{(i_\delta^\mu(F), i_\delta^\mu(G)) : (F, G) \in P_\mu,\ \mu < \delta\}\,,$$

where i_δ^μ is the natural embedding of $\prod_{\gamma<\mu}^{\mathrm{ord}} X_\gamma$ into $\prod_{\gamma<\delta}^{\mathrm{ord}} X_\gamma$.

Using Lemma 3 one can show (see [B2]) that for $\alpha \le \beta$,

$$(**) \qquad \mathrm{Ord}\, M_{P_\alpha} \ge \alpha\,,$$

where Ord is Borst's transfinite function. Note that for every $L \subseteq L(X)$ and every normal space X the following inequalities are satisfied [B2]:

$$(***) \qquad \operatorname{Ord} M_L \le \operatorname{Ord} M_{L(X)} = \dim X \le \operatorname{Ind} X\,.$$

THEOREM 5. *Let* DIM *be* dim, Ind *or* Id, *and let* X_γ, $\gamma < \beta$, *be one-dimensional metric compacta. Then* $\operatorname{DIM} \prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma = \beta$.

Proof. The estimate $\operatorname{Id} \prod^{\mathrm{ord}}_{\gamma<\beta} X_\gamma \le \beta$ follows from Corollary 8. The rest follows from (**) and (***).

COROLLARY 10. *Let* DIM *be* dim, Ind *or* Id, *let* X *be a one-dimensional metric compactum and let* α_γ, $\beta < \omega_1$. *Then*

$$\operatorname{DIM} \prod^{\mathrm{ord}}_{\gamma<\beta} S_{\alpha_\gamma}(X) = \underset{\gamma<\beta}{(+)} \alpha_\gamma\,.$$

In particular, *if*, *for each* $\alpha < \omega_1$, X^α *is the Smirnov compactum* S^α *or the Henderson compactum* H^α, *then*

$$\operatorname{DIM} \prod^{\mathrm{ord}}_{\gamma<\beta} X^{\alpha_\gamma} = \underset{\gamma<\beta}{(+)} \alpha_\gamma\,.$$

Proof. By Corollary 4 we have

$$\operatorname{DIM}\Big(\prod^{\mathrm{ord}}_{\gamma<\beta} S_{\alpha_\gamma}(X)\Big) = \operatorname{DIM}(S_{(+)_{\gamma<\beta}\alpha_\gamma}(X))\,.$$

So by Theorem 5, $\operatorname{DIM}(S_{(+)_{\gamma<\beta}\alpha_\gamma}(X)) = (+)_{\gamma<\beta}\alpha_\gamma$. Recall that $S_\gamma(I) = S^\gamma$, $\gamma < \omega_1$.

In the case of Henderson compacta we have

$$\operatorname{Id} \prod^{\mathrm{ord}}_{\gamma<\beta} H^{\alpha_\gamma} \le \underset{\gamma<\beta}{(+)} \operatorname{Id} H^{\alpha_\gamma}$$

(see Corollary 8),

$$\underset{\gamma<\beta}{(+)} \operatorname{Id} H^{\alpha_\gamma} = \underset{\gamma<\beta}{(+)} \alpha_\gamma$$

(recall that $\operatorname{Id} H^{\alpha_\gamma} = \alpha_\gamma$, see Corollary 5), $\prod^{\mathrm{ord}}_{\gamma<\beta} S^{\alpha_\gamma} \subseteq \prod^{\mathrm{ord}}_{\gamma<\beta} H^{\alpha_\gamma}$, and the inequalities $\dim X \le \operatorname{Ind} X \le \operatorname{Id} X$ hold for every compactum X.

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Appendix. Let us recall (see [K-M]) some notions and statements from set theory. For every ordinal α the power with base ω and exponent α is defined by $\omega^0 = 0$, $\omega^{\xi+1} = \omega^\xi \cdot \omega$, and $\omega^\lambda = \sup_{\gamma<\lambda} \omega^\gamma$ if λ is a limit ordinal.

Some properties of the power:

- if $\alpha < \beta$, then $\omega^\alpha < \omega^\beta$;
- $\omega^{\xi+\eta} = \omega^\xi \cdot \omega^\eta$;

- $(\omega^\xi)^\eta = \omega^{\xi\cdot\eta}$;
- if $\eta > \xi_1 > \ldots > \xi_p$ and $n_i \in \mathbb{N}$, $i = 1, \ldots, p$, then

$$\omega^\eta > \omega^{\xi_1} \cdot n_1 + \ldots + \omega^{\xi_p} \cdot n_p\,;$$

- every ordinal α can be uniquely represented as

$$\alpha = \omega^{\eta_1} \cdot n_1 + \ldots + \omega^{\eta_k} \cdot n_k, \qquad n_i \in \mathbb{N}\,,$$

where $\eta_1 > \ldots > \eta_k \geq 0$ are ordinals.

Let α and β be ordinals and

$$\alpha = \omega^{\eta_1} \cdot n_1 + \ldots + \omega^{\eta_k} \cdot n_k, \qquad \beta = \omega^{\zeta_1} \cdot m_1 + \ldots + \omega^{\zeta_l} \cdot m_l\,.$$

Adding powers with zero coefficients we get expansions with the same powers of ω:

$$(*) \qquad \alpha = \omega^{\xi_1} \cdot p_1 + \ldots + \omega^{\xi_h} \cdot p_h, \qquad \beta = \omega^{\xi_1} \cdot q_1 + \ldots + \omega^{\xi_h} \cdot q_h\,.$$

The ordinal

$$\alpha(+)\beta = \omega^{\xi_1} \cdot (p_1 + q_1) + \ldots + \omega^{\xi_h} \cdot (p_h + q_h)$$

is called the *natural sum* of α and β. Their *natural product* $\alpha(\cdot)\beta$ is obtained by multiplying the expansions $(*)$ as polynomials in ω, i.e. multiplying two powers of ω we take the natural sum of the exponents and arrange the monomials in the decreasing order of exponents.

Let $\Phi(\alpha, \beta)$ denote either $\alpha(+)\beta$ or $\alpha(\cdot)\beta$. Then

- $\Phi(\alpha, \beta) = \Phi(\beta, \alpha)$;
- if $\alpha_1 < \alpha_2$, then $\Phi(\alpha_1, \beta) < \Phi(\alpha_2, \beta)$;
- if $\Phi(\alpha_1, \beta_1) = \Phi(\alpha_2, \beta_2)$, then $\alpha_1 = \alpha_2$ implies $\beta_1 = \beta_2$, and $\alpha_1 < \alpha_2$ implies $\beta_1 > \beta_2$;
- $\alpha(+)\ldots(+)\alpha$ (n times) $= \alpha(\cdot)n$ for $n < \omega$;
- $\alpha(+)n = \alpha + n$ for $n < \omega$.

A function φ defined on a completely ordered set $W(\alpha) = \{\gamma : \gamma < \alpha\}$ of type α is called a *transfinite sequence* of type α or an α-*sequence*. If the values of this sequence are ordinals and if $\gamma < \beta < \alpha$ implies $\varphi(\gamma) < \varphi(\beta)$ then this sequence is called *increasing*.

Let λ be a limit ordinal and φ be an increasing λ-sequence. Then

$$\sup_{\xi<\lambda}(\alpha + \varphi(\xi)) = \alpha + \sup_{\xi<\lambda} \varphi(\xi), \qquad \sup_{\xi<\lambda}(\alpha \cdot \varphi(\xi)) = \alpha \cdot \sup_{\xi<\lambda} \varphi(\xi)\,,$$

$$\sup_{\xi<\lambda} \omega^{\alpha\cdot\varphi(\xi)} = \omega^{\alpha\cdot\sup_{\xi<\lambda}\varphi(\xi)}\,.$$

Lemma A1 [H]. *Let $\alpha, \beta < \omega_1$. Then for every $\gamma < \alpha(+)\beta$ there exist finitely many pairs α_1, β_1 such that $\gamma = \alpha_1(+)\beta_1$ and $\alpha_1 \leq \alpha$, $\beta_1 \leq \beta$.*

EXAMPLE A1. Let $\alpha = \beta = \omega^2$. Then

$$\alpha(+)\beta = \omega^2 \cdot 2 \neq \sup_{n,m}(\omega \cdot n(+)\omega \cdot m) = \omega^2 .$$

Let $\mathcal{B} = \{\alpha_\gamma, \gamma < \beta\}$ be a β-sequence of ordinals. The *sum* of all numbers of $\mathcal{B}$ is defined by

$$\sum_{\gamma<\beta} \alpha_\gamma = \begin{cases} \alpha_0 & \text{if } \beta = 1; \\ (\sum_{\gamma<\delta} \alpha_\gamma) + \alpha_\delta & \text{if } \beta = \delta + 1; \\ \sup_{\delta<\beta}(\sum_{\gamma<\delta} \alpha_\gamma) & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

(see, for example, [H]).

The *natural sum* of all numbers of $\mathcal{B}$ is the number

$$\underset{\gamma<\beta}{(+)} \alpha_\gamma = \begin{cases} \alpha_0 & \text{if } \beta = 1; \\ ((+)_{\gamma<\delta}\alpha_\gamma)(+)\alpha_\delta & \text{if } \beta = \delta + 1; \\ \sup_{\delta<\beta}(+)_{\gamma<\delta}\alpha_\gamma & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

Clearly, if $\gamma < \delta < \beta$, then $\sum_{\mu<\gamma} \alpha_\mu \leq \sum_{\mu<\delta} \alpha_\mu$ and $(+)_{\mu<\gamma}\alpha_\mu \leq (+)_{\mu<\delta}\alpha_\mu$, moreover, $\sum_{\gamma<\beta} \alpha_\gamma \leq (+)_{\gamma<\beta}\alpha_\gamma$.

It is well known that every ordinal α may be represented in the form $\alpha = p(\alpha) + n(\alpha)$, where $p(\alpha)$ is a limit ordinal (0 is considered to be a limit ordinal) and $n(\alpha) < \omega$.

One can easily check the following properties:

- If all α_γ, $\gamma < \beta$, are equal to 1, then $\sum_{\gamma<\beta} \alpha_\gamma = (+)_{\gamma<\beta}\alpha_\gamma = \beta$.
- If $1 \leq \alpha_\gamma < \omega$ for $\gamma < \beta$, then

$$\sum_{\gamma<\beta} \alpha_\gamma = \underset{\gamma<\beta}{(+)} \alpha_\gamma = \begin{cases} p(\beta) + \alpha_{p(\beta)} + \ldots + \alpha_{\beta-1} & \text{if } \beta \text{ is a non-limit ordinal,} \\ \beta & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

- $$\sum_{\gamma<\beta} \min_{\gamma<\beta} \alpha_\gamma \leq \sum_{\gamma<\beta} \alpha_\gamma \leq \sum_{\gamma<\beta} \sup_{\gamma<\beta} \alpha_\gamma ,$$
  $$\underset{\gamma<\beta}{(+)} \min_{\gamma<\beta} \alpha_\gamma \leq \underset{\gamma<\beta}{(+)} \alpha_\gamma \leq \underset{\gamma<\beta}{(+)} \sup_{\gamma<\beta} \alpha_\gamma .$$
- $$\sum_{\gamma<\beta} \alpha_\gamma = \alpha \cdot \beta \text{ if all } \alpha_\gamma = \alpha,\ \gamma < \beta .$$

In view of the last equality, it is natural to give another definition of the product of two ordinals α and β:

$$\alpha \times \beta = \underset{\gamma<\beta}{(+)} \alpha_\gamma ,$$

where $\alpha_\gamma = \alpha$ for $\gamma < \beta$. It is clear that $\alpha \cdot \beta \leq \alpha \times \beta$ and there exist pairs α, β such that $\alpha \cdot \beta < \alpha \times \beta$, for example,

$$(\omega + 1) \cdot 2 = \omega \cdot 2 + 1 < (\omega + 1) \times 2 = \omega \cdot 2 + 2 .$$

It is not difficult to establish a relation between the new product and the usual operations on ordinals. We indicate it without proof:

Let $\alpha = \omega^{\eta_1} \cdot n_1 + \ldots + \omega^{\eta_k} \cdot n_k$, $n_i \in \mathbb{N}$, and $\eta_1 > \ldots > \eta_k$. Then

$$\alpha \times \beta = \omega^{\eta_1} \cdot p(\beta) + \alpha(\cdot)n(\beta) .$$

Note that $\alpha \times \beta \le \alpha(\cdot)\beta$ and there exist pairs α, β such that $\alpha \times \beta < \alpha(\cdot)\beta$, for example $2 \times \omega = \omega < 2(\cdot)\omega = \omega \cdot 2$. Note also that

- if $\beta_1 < \beta_2$, then $\alpha \times \beta_1 < \alpha \times \beta_2$, $\alpha > 0$;
- $\alpha \times (\beta_1(+)\beta_2) = (\alpha \times \beta_1)(+)(\alpha \times \beta_2)$;
- if λ is a limit ordinal and φ is an increasing λ-sequence, then $\sup_{\xi<\lambda}(\alpha \times \varphi(\xi)) = \alpha \times \sup_{\xi<\lambda} \varphi(\xi)$.

For the new product one may also introduce the notion of power.

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