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Examples of non-shy sets

by

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Abstract. Christensen has defined a generalization of the property of being of Haar measure zero to subsets of (abelian) Polish groups which need not be locally compact; a recent paper of Hunt, Sauer, and Yorke defines the same property for Borel subsets of linear spaces, and gives a number of examples and applications. The latter authors use the term "shyness" for this property, and "prevalence" for the complementary property. In the present paper, we construct a number of examples of non-shy Borel sets in various groups, and thereby answer several questions of Christensen and Mycielski. The main results are: in many (most?) non-locally-compact Polish groups, the ideal of shy sets does not satisfy the countable chain condition (i.e., there exist uncountably many disjoint non-shy Borel sets); in function spaces $C({}^{\omega}2, G)$ where G is an abelian Polish group, the set of functions f which are highly non-injective is non-shy, and even prevalent if G is locally compact.

If G is a Polish group which is locally compact, then one can define a Borel measure on G, the Haar measure, which is invariant (on one side) and gives finite non-zero measure to non-empty open sets with compact closure. This measure is unique up to a multiplicative constant, so the collection of measure-zero sets is uniquely determined, and gives an invariant (on both sides) property of "smallness" for subsets of G which is probably closer to the intuitive idea of "smallness" than the category analogue, meagerness.

The definition of Haar measure does not extend to groups which are not locally compact, but there is a suitable extension of the property of being of Haar measure zero; this is given in Christensen [1] and again in Hunt, Sauer, and Yorke [3]. In the former paper, a universally measurable subset S of an abelian Polish group G is called a *Haar zero* set if there is a probability measure on G which gives every translate of S measure 0; the latter paper uses an equivalent definition for Borel subsets of separable complete metric linear spaces, and calls such sets *shy*. Topsøe and Hoffmann-Jørgensen [6]

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and Mycielski [5] have observed that the definitions work just as well for non-abelian Polish groups if one requires that all two-sided translates gSh of S have measure zero (in fact, the former authors generalize even further). In particular, the collection of Haar zero or shy sets (we will use the latter term for the rest of this paper, but will apply it to all universally measurable sets rather than just Borel sets) is a σ-ideal, and in the case where G is locally compact a (universally measurable) set is shy iff it has Haar measure zero.

A measure which gives every translate of S measure zero is called *transverse* to S. Also, a set is *prevalent* if its complement is shy.

There are many properties of measure-zero sets in locally compact groups for which one can ask whether the properties hold for shy sets in general Polish groups G. In particular, Christensen [1] asks whether any collection of disjoint universally measurable non-shy sets must be countable (this property of a σ-ideal is called the *countable chain condition*), and whether any two universally measurable non-shy sets must have translates with non-shy intersection; and Mycielski [5] asks whether any universally measurable subset of G invariant under (left) translations by a countable dense subgroup of G must be either shy or prevalent. In the present paper, we will construct examples giving negative answers to these three questions (using an observation of Mycielski for one of them). We will also answer other questions of Mycielski [5] by showing that certain comeager sets in continuous function spaces $C({}^{\omega}2, G)$ (in particular, the set of injective functions) are not prevalent and are even shy in some cases.

Showing that a set is prevalent requires only a single transverse measure for the complement, but showing that a set is not shy requires that all measures fail to be transverse. The latter would seem to be harder, but it turns out to be apparently easier in some cases, even when the set is indeed prevalent. (For an example of this in addition to the examples in the present paper, see Dougherty and Mycielski [2].) The main fact that will be used here is that every probability measure on a Polish space gives some compact set positive measure; a number of subsets S of Polish groups that we will be interested in have the property that any compact set (or any compact set of small diameter, or something similar) has a translate included in S, and therefore the reverse translation moves S to a set covering the given compact set.

Perhaps the simplest example of this is the set of positive numbers in $\mathbb{R}$; any compact set can be translated entirely to the right of the origin. This can be slightly generalized to the following.

PROPOSITION 1. *For any set A of natural numbers, the set $S(A)$ of all sequences $s \in {}^{\omega}\mathbb{R}$ such that $s(n) > 0$ for $n \in A$ and $s(n) < 0$ for $n \notin A$ is non-shy in ${}^{\omega}\mathbb{R}$.*

P r o o f. Let μ be a probability measure on ${}^\omega\mathbb{R}$; we must find a translate of $S(A)$ which has positive μ-measure. Equivalently, it will suffice to find a set C of positive μ-measure which has a translate included in $S(A)$. To do this, choose a number b_n for each n such that $\mu(\{s : |s(n)| > b_n\}) < 2^{-n-1}$, and let $C = \{s \in {}^\omega\mathbb{R} : |s(n)| \le b_n \text{ for all } n\}$; then $\mu(C) > 0$. Define $t \in {}^\omega\mathbb{R}$ by: if $n \in A$, then $t(n) = b_n + 1$; if $n \notin A$, then $t(n) = -b_n - 1$. Then $C + t \subset S(A)$, so we are done. ∎

Since the sets $S(A)$ are disjoint, the ideal of shy subsets of ${}^\omega\mathbb{R}$ does not satisfy the countable chain condition. Also, if T is defined to be the set of $s \in {}^\omega\mathbb{R}$ such that $s(n) > 0$ for infinitely many n, then T is a Borel set which is invariant under a dense subgroup of ${}^\omega\mathbb{R}$ (the eventually-0 sequences), but T is neither prevalent nor shy.

Mycielski observes that, if A and B are sets of natural numbers whose symmetric difference is infinite, then not only are the sets $S(A)$ and $S(B)$ disjoint, but also any translate of $S(A)$ intersects $S(B)$ in a shy set. (To show this, it suffices to prove that, if we have a finite interval I_j in $\mathbb{R}$ for each j in an infinite set J, then the set S' of all $x \in {}^\omega\mathbb{R}$ such that $x(j) \in I_j$ for all $j \in J$ is shy. This can be deduced from Propositions 12 and 8 below, or one can prove it directly as follows: if we choose a finite interval I'_j for each $j \in \omega$ such that I'_j is twice as long as I_j if $j \in J$, and let μ_j be Lebesgue measurex on I'_j normalized to 1, then the product of the measures μ_j is a measure transverse to S'.) This answers Problem 1 of Christensen [1] negatively. In fact, Mycielski notes that one can get $2^{\aleph_0}$ non-shy subsets of ${}^\omega\mathbb{R}$ which mutually have this strong disjointness property, by taking $S(A)$ for $2^{\aleph_0}$ sets $A \subseteq \omega$ which have infinite symmetric differences with each other (e.g., the sets $p(B \times \omega)$ for $B \subseteq \omega$, where p is an injective pairing function from $\omega \times \omega$ to ω).

Now that we know that the ideal of shy sets satisfies the countable chain condition for some Polish groups but not for others, the next question to consider is how to characterize the groups for which the ccc does hold, and the natural conjecture is that the ccc holds only in the locally compact case. The following three results show that this is true for most natural examples of Polish groups, but it remains open whether it is true for all Polish groups.

The proof of Proposition 1 can be generalized to give:

PROPOSITION 2. *If $\langle G_n : n \in \omega\rangle$ is a sequence of locally compact but non-compact Polish groups, and $G = \prod_{n\in\omega} G_n$, then the ideal of shy subsets of G does not satisfy the countable chain condition.*

P r o o f. For each n, let $\langle O_{nk} : k \in \omega\rangle$ be a sequence of open sets of G_n with compact closure whose union is G_n; we may assume $O_{nk} \subseteq O_{n,k+1}$. Since G_n is not compact, the set $(\overline{O}_{nk})^{-1}\overline{O}_{nk}$ is not all of G_n, so we can choose $r(n,k) \in G_n$ outside it; then $\overline{O}_{nk} r(n,k)$ is disjoint from $\overline{O}_{nk}$. For $g \in G_n$, let $f_n(g)$ be the least k such that $g \in O_{nk}$.

For $A \subseteq \omega$, let $S(A)$ be the set of all $s \in G$ such that $f_{2n}(s(2n)) > f_{2n+1}(s(2n+1))$ for $n \in A$ and $f_{2n}(s(2n)) < f_{2n+1}(s(2n+1))$ for $n \notin A$. The sets $S(A)$ are clearly Borel and disjoint; we will now see that $S(A)$ is not shy. Let μ be a probability measure on G. For each n, the sets $\{s \in G : f_n(s(n)) \le k\}$ for $k \in \omega$ have union G, so we can choose a number b_n such that $\mu(\{s \in G : f_n(s(n)) > b_n\}) < 2^{-n-1}$. Let $C = \{s \in G : f_n(s(n)) \le b_n$ for all $n\}$; then $\mu(C) > 0$. Define $t \in G$ as follows: for $n \in A$, let $t(2n) = r(2n, \max(b_{2n}, b_{2n+1}))$ and $t(2n+1) = 0$; for $n \notin A$, let $t(2n) = 0$ and $t(2n+1) = r(2n+1, \max(b_{2n}, b_{2n+1}))$. Then $Ct \subseteq S(A)$, so $\mu(S(A)t^{-1}) > 0$, as desired. ■

Another method of producing non-shy sets gives the following general result.

THEOREM 3. *Let G be a Polish group which has an invariant metric and is not locally compact. Suppose that there exist a neighborhood U of the identity and a dense subgroup G_* of G such that, for any finitely generated subgroup F of G_*, $\overline{F \cap U}$ is compact. Then the ideal of shy subsets of G does not satisfy the countable chain condition.*

Proof. Let d be an invariant complete metric for G (by Exercise 1.9 of Kechris [4], any invariant metric for G is complete), and let e be the identity of G. First, we show that there are arbitrarily small positive numbers δ such that, for any finitely generated subgroup H of G_*, there exists $x \in G_*$ such that $d(x,e) \le \delta$ and $d(x, \overline{H}) \ge \delta/2$. Let ε be a positive number, which we may assume is small enough that the open ball $B(e, 2\varepsilon)$ is included in the given neighborhood U.

If ε is not a suitable value for δ, then there is a finitely generated subgroup H_0 of G_* such that every point of G_* within distance ε of e is at distance less than $\varepsilon/2$ from H_0. Next, if $\varepsilon/2$ is not a suitable value for δ, then there is a finitely generated subgroup H_1 of G_* such that every point of G_* within distance $\varepsilon/2$ of e is at distance less than $\varepsilon/4$ from H_1; we may assume $H_0 \subseteq H_1$. But then, since d is left-invariant, every point of G_* within distance $\varepsilon/2$ of a point of H_0 is at distance less than $\varepsilon/4$ from H_1; in particular, every point of G_* within distance ε of e is at distance less than $\varepsilon/4$ from H_1.

Continued iteration shows that, if none of the values $\varepsilon, \varepsilon/2, \ldots, \varepsilon/2^n$ is suitable for δ, then there is a finitely generated subgroup H_n of G_* such that every point of $G_* \cap B(e,\varepsilon)$ is at distance less than $\varepsilon/2^{n+1}$ from H_n, and hence, clearly, from $H_n \cap B(e, 2\varepsilon)$. Since $H_n \cap B(e, 2\varepsilon)$ has compact closure, it can be covered by finitely many balls of radius $\varepsilon/2^{n+1}$, so $G_* \cap B(e,\varepsilon)$ can be covered by finitely many balls of radius $\varepsilon/2^n$. Since n is arbitrary, $\overline{G_* \cap B(e,\varepsilon)} = \overline{B(e,\varepsilon)}$ is compact, which is impossible since G was assumed not to be locally compact. Therefore, at least one of the numbers $\varepsilon/2^n$ must be a suitable value for δ.

Choose a sequence $\delta_0, \delta_1, \delta_2, \ldots$ of numbers δ as above such that $\delta_{i+1} \le \delta_i/8$ for all i. Let $\{r_n : n \in \omega\}$ be a countable dense subset of G_*, and let F_n be the subgroup generated by $\{r_k : k < n\}$. We may assume that the points r_n are chosen iteratively so that the following is true: if n has the form $2^i 3^j$, then $d(r_n, e) \le \delta_i$ and $d(r_n, \overline{F}_n) \ge \delta_i/2$. (Use the points r_n for n not of the form $2^i 3^j$ to ensure density.) For any natural number i and any $x \in G$, let $M_i(x)$ be the least M such that $d(x, \overline{F}_M) < \delta_i/4$. Now, for any set $A \subseteq \omega$, define $S(A)$ to be the set of $x \in G$ such that $M_{2i}(x) = M_{2i+1}(x)$ for all $i \in A$ and $M_{2i}(x) < M_{2i+1}(x)$ for all $i \notin A$. The sets $S(A)$ are clearly disjoint; we will show that, for each A, $S(A)$ is not shy.

Let μ be a probability measure on G. For each n, the sets $\{x : M_n(x) \le k\}$ for $k \in \omega$ have union G, so we can choose a number $b(n)$ such that $\mu(\{x : M_n(x) \le b(n)\}) > 1 - 2^{-n-1}$. Let $C = \{x : M_n(x) \le b(n) \text{ for all } n\}$; then $\mu(C) > 0$. Given A, we will find an element g of G such that $Cg \subseteq S(A)$.

For $k = 0, 1, \ldots$, define $c(k) \in \omega$ as follows. If $k \in A$, let $i = 2k$; if $k \notin A$, let $i = 2k + 1$. Now find a number n of the form $2^i 3^j$ for this i so that $n > c(k-1)$ (if $k > 0$) and $n > b(2k+2)$, and let $c(k) = n$. Since $d(r_{c(k)}, e) \le \delta_{2k}$ and d is invariant, the infinite product $r_{c(0)} r_{c(1)} r_{c(2)} \cdots$ converges; let g be its value. Then, for any $x \in C$, we have

$$d(x, \overline{F}_{c(k)}) = d(x r_{c(0)} \ldots r_{c(k-1)}, \overline{F}_{c(k)}) < \delta_{2k+2}/4\,,$$
$$d(r_{c(k+1)} r_{c(k+2)} \ldots, e) \le (8/7)\delta_{2k+2}\,,$$

and $\delta_i/2 \le d(r_{c(k)}, \overline{F}_{c(k)}) \le \delta_i$, so we get

$$\delta_i/4 < d(xg, \overline{F}_{c(k)}) < 2\delta_i \quad \text{and} \quad d(xg, \overline{F}_{c(k)+1}) < 2\delta_{2k+2} \le \delta_{2k+1}/4\,.$$

Therefore, we have $M_{2k}(xg) = M_{2k+1}(xg) = c(k) + 1$ if $k \in A$, and $M_{2k}(xg) \le c(k) < c(k) + 1 = M_{2k+1}(xg)$ if $k \notin A$. This shows that $xg \in S(A)$; since $x \in C$ was arbitrary, we are done. ∎

The supposition in Theorem 3 holds with $G_* = G$ for standard examples of Polish groups, including all separable Banach spaces (since, in a finite-dimensional subspace, closed bounded sets are compact) and all abelian torsion groups (since finitely generated subgroups are finite). I do not have an example with an invariant metric where the supposition fails (without the invariant metric, one can consider the semidirect product $\mathbb{Z} \times_\theta {}^{\mathbb{Z}}(\mathbb{R} \times_\psi \mathbb{R})$, where $\theta(n)(s)(m) = s(m-n)$ and $\psi(x)(y) = e^x y$), but here is an example where the supposition cannot hold with $G_* = G$: Let $\mathbb{T}$ be the circle group $\mathbb{R}/\mathbb{Z}$, and take the infinite product ${}^{\omega}\mathbb{T}$ with the sup- or ℓ^∞-metric; this is a complete but non-separable metric space. Let F be the subgroup generated by the single element s defined by $s(n) = 2^{-n^2}$ for all n. Then the closure of

F is an abelian Polish group which is not locally compact but has a dense cyclic subgroup.

Often a counterexample to the countable chain condition for the shy sets in a group can be transferred to an extension of that group, as follows.

PROPOSITION 4. *Suppose* $(G, +)$ *is an abelian Polish group and* H *is a closed subgroup of* G. *If the ideal of shy subsets of* H *does not satisfy the countable chain condition* (*within the algebra of Borel subsets of* H), *then the ideal of shy subsets of* G *does not satisfy the countable chain condition.*

Proof. Let $s : G/H \to G$ be a Borel selector for the cosets of H (see Theorem 1.21 of Kechris [4]). Define $f : G \to H$ by $f(g) = g - s(g + H)$; then f is Borel, and we have $f(g+h) = f(g)+h$ for $h \in H$. For any $S \subseteq H$, we get a corresponding set $f^{-1}(S) \subseteq G$, and $f^{-1}(S)$ is shy in G iff S is shy in H.

To see this, first suppose $f^{-1}(S)$ is shy in G, as witnessed by the measure μ, and use f to map μ to a measure μ' on H; then, for any $h \in H$, $\mu'(S+h) = \mu(f^{-1}(S + h)) = \mu(f^{-1}(S) + h) = 0$, so S is shy. Conversely, if μ is a measure on H witnessing that S is shy, then μ can be extended to a measure on G which concentrates on H. Then, for any $g \in G$, $\mu(f^{-1}(S) + g) = \mu((f^{-1}(S)+g)\cap H)$. But, for $h \in H$, we have $f(h-g) \in S$ iff $h+f(-g) \in S$; hence, $(f^{-1}(S)+g)\cap H = S-f(-g)$, so $\mu(f^{-1}(S)+g) = \mu(S-f(-g)) = 0$. Therefore, $f^{-1}(S)$ is shy.

It follows that any counterexample to the ccc for the shy sets in H is mapped by f^{-1} to a counterexample to the ccc for the shy subsets of G, so we are done. ∎

This works for non-abelian G if one assumes, not only that H is a closed normal subgroup of G, but also that for every $g \in G$ there is $h \in H$ such that gh commutes with every element of H (equivalently, conjugation by members of G gives no more automorphisms of H than conjugation by members of H does).

Now for some more specific examples involving spaces of the form $C({}^\omega 2, G)$, where G is a Polish group. This will give negative answers to some questions from Mycielski [5], by showing that injectivity is not a prevalent property.

For continuous functions f from ${}^\omega 2$ to a metric space X, define a *modulus of continuity* to be a sequence $b \in {}^\omega\omega$ such that, if $x, y \in {}^\omega 2$ agree on their first $b(n)$ coordinates, then $d(f(x), f(y)) \le 2^{-n}$.

LEMMA 5. *If* μ *is a probability measure on* $C({}^\omega 2, X)$ *where* X *is a Polish space with complete metric* d, *then there exist a compact set* $K \subseteq X$ *and a sequence* $b \in {}^\omega\omega$ *such that*, *if* C *is the set of functions in* $C({}^\omega 2, X)$ *with range included in* K *and having modulus of continuity* b, *then* $\mu(C) > 0$.

Proof. Let $\{r_0, r_1, r_2, \ldots\}$ be a countable dense subset of X. For any function $f \in C({}^\omega 2, X)$, the range of f is compact; hence, for any $n \in \omega$, the range of f is included in the union of finitely many of the open balls $B(r_j, 2^{-n})$, say the first $a_{f,n}$ of them. Also, f must be uniformly continuous, so there exists a number $b_{f,n}$ such that, if x and y are members of ${}^\omega 2$ which agree on the first $b_{f,n}$ coordinates, then $d(f(x), f(y)) \le 2^{-n}$.

We can now choose numbers $a(n)$ and $b(n)$ so large that the set $C_n = \{f : a_{f,n} \le a(n) \text{ and } b_{f,n} \le b(n)\}$ satisfies $\mu(C_n) > 1 - 2^{-n-1}$. Let

$$K = \bigcap_{n\in\omega} \overline{\bigcup_{j\le a(n)} B(r_j, 2^{-n})}\,.$$

Then K is compact (complete and totally bounded). If C is the set of functions in $C({}^\omega 2, X)$ with range included in K and having modulus of continuity b, then C includes $\bigcap_{n\in\omega} C_n$, so $\mu(C) > 0$. ∎

PROPOSITION 6. *Suppose K is a compact metric space, and $b \in {}^\omega\omega$. Then there is a function $F \in C({}^\omega 2, K)$ such that, for every $f \in C({}^\omega 2, K)$ with modulus of continuity b, f agrees with F on a perfect subset of ${}^\omega 2$. In fact, there exist an infinite set $I \subseteq \omega$ and a function $F \in C({}^\omega 2, K)$ such that, for every $f \in C({}^\omega 2, K)$ with modulus of continuity b, there exists $z \in {}^\omega 2$ such that $f(x) = F(x)$ for all x such that $x(j) = z(j)$ for $j \notin I$.*

Proof. Fix a metric d for K, and, for each n, let S_n be a finite 2^{-n}-dense subset of K. Define sequences b' and c recursively as follows. Let $b'(0) = b(0)$; for $n > 0$, let $b'(n) = \max(b(n), c(n-1))$. Given $b'(n)$, choose $c(n)$ so large that $2^{c(n)-b'(n)-1} \ge |S_n|^{2^{b'(n)}}$.

Now define continuous functions $F_n : {}^\omega 2 \to K$ as follows. Fix a mapping Q_n from the set ${}^{(b'(n),c(n))}2$ (essentially the set of binary sequences of length $c(n) - b'(n) - 1$) onto ${}^{{}^{b'(n)}2}S_n$. Now, for any $x \in {}^\omega 2$, let

$$\widehat{F}_n(x) = Q_n(x\restriction(b'(n), c(n)))(x\restriction b'(n))\,.$$

Let $F_0 = \widehat{F}_0$. For $n > 0$, let

$$F_n(x) = \begin{cases} \widehat{F}_n(x) & \text{if } d(\widehat{F}_n(x), F_{n-1}(x)) \le 6 \cdot 2^{-n}, \\ F_{n-1}(x) & \text{otherwise.} \end{cases}$$

Then $\widehat{F}_n(x)$ and $F_n(x)$ depend only on $x\restriction c(n)$, so $\widehat{F}_n$ and F_n are continuous. Also, $d(F_{n-1}(x), F_n(x)) \le 6 \cdot 2^{-n}$ for all x, so F_n converges uniformly to a function $F \in C({}^\omega 2, K)$ as $n \to \infty$. Let $I = \{b'(n) : n \in \omega\}$.

Let $f \in C({}^\omega 2, K)$ be a function with modulus of continuity b. For each n and each binary sequence σ of length $b'(n)$, choose an element $r_n(\sigma)$ of S_n such that $d(r_n(\sigma), f(\sigma^\frown\overline{0})) \le 2^{-n}$, and hence $d(r_n(\sigma), f(x)) \le 2\cdot 2^{-n}$ for any x which starts with σ, since $b'(n) \ge b(n)$. Choose $y_n \in {}^{(b'(n),c(n))}2$ such that $Q_n(y_n)(\sigma) = r_n(\sigma)$ for all σ. Now find $z \in {}^\omega 2$ such that $z\restriction(b'(n), c(n)) =$

y_n for all n. If $x \in {}^\omega 2$ satisfies $x(j) = z(j)$ for $j \notin I$, then we have $x{\restriction}(b'(n), c(n)) = y_n$ for all n, so $\widehat{F}_n(x) = Q_n(y_n)(x{\restriction}b'(n)) = r_n(x{\restriction}b'(n))$. But we have $d(\widehat{F}_n(x), f(x)) = d(r_n(x{\restriction}b'(n)), f(x)) \leq 2\cdot 2^{-n}$ for all n; this implies that $d(\widehat{F}_n(x), \widehat{F}_{n-1}(x)) \leq 6 \cdot 2^{-n}$ for all $n > 0$, so we have $F_n(x) = \widehat{F}_n(x)$ for all n. Hence, $d(F_n(x), f(x)) \leq 2 \cdot 2^{-n}$ for all n, so $F(x) = f(x)$, as desired. ∎

PROPOSITION 7. *Let K_0 be a compact subset of a Polish group G. Then the set of functions $f \in C({}^\omega 2, G)$ such that every point in K_0 is the image of $2^{\aleph_0}$ points from ${}^\omega 2$ is non-shy in $C({}^\omega 2, G)$.*

Proof. Suppose that μ is a probability measure on $C({}^\omega 2, G)$. Let K, b, and C be as in Lemma 5, and then let F and I be as in Proposition 6. We will find a function $h \in C({}^\omega 2, G)$ such that every element of Ch maps $2^{\aleph_0}$ points of ${}^\omega 2$ to each point in K_0; since $\mu(C) > 0$, this shows that the set in question is not shy.

Since K_0 is compact (and, we may assume, non-empty), there is a continuous function p which maps ${}^\omega 2$ onto K_0. Let $b'(0) < b'(1) < \ldots$ be the increasing enumeration of I, and let $q(x) = \langle x(b'(2n)) : n \in \omega\rangle$. Now let $h(x) = F(x)^{-1}p(q(x))$ for all x. If $f \in C$, then there is $z \in {}^\omega 2$ such that $f(x) = F(x)$ for all x which agree with z on coordinates outside I. But, for any $w \in K_0$, we can find $2^{\aleph_0}$ x's such that $p(q(x)) = w$: find $y \in {}^\omega 2$ such that $p(y) = w$, and let $x(j)$ be $z(j)$ if $j \notin I$, $y(n)$ if $j = b'(2n)$, and arbitrary if $j = b'(2n+1)$. For such an x, we have $fh(x) = F(x)F(x)^{-1}p(q(x)) = w$. Therefore, fh maps $2^{\aleph_0}$ points of ${}^\omega 2$ to each point of K_0, as desired. ∎

The same method gives a number of similar results, such as: for any $f_0 \in C({}^\omega 2, G)$, the set of $f \in C({}^\omega 2, G)$ which agree with f_0 on a perfect set is non-shy.

One can extend this to other function spaces by using the following result:

PROPOSITION 8. *If $\varphi : G \to H$ is a continuous epimorphism of Polish groups, and $S \subseteq H$ is universally measurable, then S is shy in H iff $\varphi^{-1}(S)$ is shy in G.*

Proof. First suppose that S is shy in H, as witnessed by the probability measure ν. Let ψ be a Borel right inverse of φ. (To see that ψ exists, use the fact that the canonical projection from G to $G/\ker(\varphi)$ is an open map, and apply Theorem 1.13a and Exercise 1.21a from Kechris [4].) Use ψ to transfer ν to a measure μ on G. For any $g_1, g_2 \in G$, if $\psi(h) \in g_1\varphi^{-1}(S)g_2$, then $h = \varphi(\psi(h)) \in \varphi(g_1)S\varphi(g_2)$. Hence, $\psi^{-1}(g_1\varphi^{-1}(S)g_2) \subseteq \varphi(g_1)S\varphi(g_2)$, so $\mu(g_1\varphi^{-1}(S)g_2) = \nu(\psi^{-1}(g_1\varphi^{-1}(S)g_2)) = 0$. Therefore, μ witnesses that $\varphi^{-1}(S)$ is shy in G.

Now suppose that S is not shy in H. Let μ be a probability measure on G, and use φ to get a corresponding measure ν on H. Find $h_1, h_2 \in H$ such that $\nu(h_1 S h_2) > 0$, and find $g_1, g_2 \in G$ such that $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. If $\varphi(g) \in h_1 S h_2$, then $\varphi(g_1^{-1} g g_2^{-1}) \in h_1^{-1} h_1 S h_2 h_2^{-1} = S$, so $g \in g_1 \varphi^{-1}(S) g_2$. Therefore, $\mu(g_1 \varphi^{-1}(S) g_2) \geq \mu(\varphi^{-1}(h_1 S h_2)) = \nu(h_1 S h_2) > 0$. Since μ was arbitrary, φ^{-1} is not shy in G. ∎

For example, in the space $C([0,1], \mathbb{R}^n)$ where n is finite or ω, the set of (highly) non-injective functions is non-shy, because, if A is the Cantor middle-thirds set in $[0,1]$, then restriction to A is a continuous epimorphism from $C([0,1], \mathbb{R}^n)$ to $C(A, \mathbb{R}^n)$.

For non-locally-compact G, the non-shy sets above turn out to be non-prevalent as well.

PROPOSITION 9. *If G is a Polish group which is not locally compact, and A is a σ-compact subset of G, then the set of $f \in C({}^\omega 2, G)$ such that f is injective and the range of f is disjoint from A is not shy.*

Proof. Let μ be a probability measure on $C({}^\omega 2, G)$, and find K and C as in Lemma 5. We will show that there is $h \in C({}^\omega 2, G)$ such that, for any $f \in C$, fh is injective and has range disjoint from A.

Let A_n, $n \in \omega$, be compact sets with union A. Since G is not locally compact, the compact sets $K^{-1}K$ and $K^{-1}A_n$ are nowhere dense in G. Working in the space $K(G)$ of compact subsets of G, the sets $\{E : KE \cap A_n = \emptyset\} = \{E : E \cap K^{-1}A_n = \emptyset\}$ are open dense, as are the sets

$$\{E : \forall x, y \in E,\ d(x,y) \geq 2^{-n} \Rightarrow Kx \cap Ky = \emptyset\}$$
$$= \{E : \forall x, y \in E,\ d(x,y) \geq 2^{-n} \Rightarrow xy^{-1} \notin K^{-1}K\}\,.$$

Therefore, the intersection of all of these open sets is comeager in $K(G)$, so there is a perfect set E which is in all of these sets. Then $KE \cap A = \emptyset$ and, for any $x \neq y$ in E, $Kx \cap Ky = \emptyset$. Let $h \in C({}^\omega 2, G)$ be a continuous injection from ${}^\omega 2$ to E; then, for any $f \in C({}^\omega 2, G)$ with range included in K, fh is injective and has range disjoint from A, so we are done. ∎

This gives more examples of sets invariant under a countable dense subgroup which are neither prevalent nor shy, such as $\{f \in C({}^\omega 2, G) : \text{the range of } f \text{ avoids } D\}$, where G is as in Proposition 9 and D is a countable dense subgroup of G, or

$$\{f \in C({}^\omega 2, G) : f \text{ is locally injective}\}$$

(i.e., f is injective on sufficiently small neighborhoods in ${}^\omega 2$; this is invariant under translation by locally constant functions). A Borel variant of the latter is

$$\{f \in C({}^\omega 2, {}^\omega \mathbb{R}) : f \text{ is locally distance-non-decreasing}\}\,.$$

(To see that this is non-shy, note that any compact subset K of ${}^\omega\mathbb{R}$ is included in $\{s : \forall n, |s(n)| \le a(n)\}$ for some fixed sequence a; if $h \in C({}^\omega 2, {}^\omega\mathbb{R})$ is defined by $h(x)(n) = (2a(n)+1)x(n)$, then any function $f+h$ where the range of f is included in K does not decrease distances.)

The case where G is locally compact is quite different; here, at least if G is abelian, one can show that the set of highly non-injective functions is prevalent in $C({}^\omega 2, G)$. The proof of this involves the following variant of the Law of Large Numbers (which I assume is a corollary of other well-known variants).

LEMMA 10. *For any natural number N and any $\varepsilon > 0$, there is a $\delta > 0$ such that the following is true: If one is given numbers $c_1, \ldots, c_n$ such that $0 \le c_i \le \delta$ for all i and $\sum_{i=1}^n c_i = 1$, and if one assigns these numbers to N sets $A_1, \ldots, A_N$ randomly (with c_i being assigned to A_j with probability $1/N$, and the assignments of the various c_i's being independent), then the probability that all of the sets A_i have sums at least $1/(2N)$ is greater than $1-\varepsilon$.*

Proof. Fix a number $j \le N$, and let X be a random variable giving the sum of the elements c_i which are assigned to A_j. Then X is the sum of n independent random variables X_i, where X_i takes the value c_i if c_i is assigned to A_j, 0 otherwise. The variable X_i has mean c_i/N and variance less than c_i^2, so the sum X has mean $\sum_{i=1}^n c_i/N = 1/N$ and variance less than $\sum_{i=1}^n c_i^2 \le \sum_{i=1}^n c_i\delta = \delta$. Therefore, by Chebyshev's inequality, the probability that X differs from $1/N$ by more than $1/(2N)$ is at most $\delta(1/(2N))^{-2}$. Hence, if δ is chosen to be so small that $4N^2\delta < \varepsilon/N$, then the probability that A_j will have sum less than $1/(2N)$ is less than ε/N. This is true for any $j \le N$, so the probability that all of the sets A_j have sums at least $1/(2N)$ is greater than $1 - N\varepsilon/N = 1-\varepsilon$. ∎

THEOREM 11. *If G is a locally compact abelian Polish group, then the set of all functions $f \in C({}^\omega 2, G)$ such that the range of f has non-empty interior is prevalent.*

Proof. Since the domain of a function in $C({}^\omega 2, G)$ is compact, its range is also compact and hence closed. Therefore, it will suffice to show that the set of f such that the range of f is dense in some non-empty open set is prevalent.

Let λ be the standard symmetric probability measure on ${}^\omega 2$. Let $+$ and $\mathbf{0}$ be the group operation and identity element of G, and let d be an invariant complete metric for G. Let $R > 0$ be so small that some (and hence any) closed ball of radius $5R$ is compact. For each n, let S_n be a finite $2^{-n-1}R$-dense subset of the open ball $B(\mathbf{0}, 2^{-n}R)$. Also, let k be a natural number such that there exists an R-dense subset of $B(\mathbf{0}, 5R)$ of size k. Then any open ball of radius $5R$ can be covered by k open balls of radius R, and the open ball $B(x, 2^{-n}R)$ is covered by the $|S_n|$ open balls $B(x+y, 2^{-n-1}R)$, $y \in S_n$. It follows that, for $m \le n$, the open ball $B(x, 2^{-m}R)$ is covered by the open balls $B(x+y, 2^{-n}R)$ for y in $S_m + S_{m+1} + \ldots + S_{n-1}$ (which is defined to be $\{\mathbf{0}\}$ for $m = n$).

For each n, let $p_n = \prod_{i=0}^{n-1} |S_i|$ and $a_n = 2^n k p_n^3$. Now define a sequence of natural numbers $0 = b(0) < b(1) < b(2) < \ldots$ recursively as follows: given $b(n)$, choose $b(n+1)$ so large that $2^{-(b(n+1)-b(n))} a_n |S_n|$ is less than the number δ given by Lemma 10 for $N = |S_n^2|$ and $\varepsilon = 2^{-n} p_n^{-1}$.

We now define a probability measure on $C({}^\omega 2, G)$ by specifying a process for randomly choosing an element f of $C({}^\omega 2, G)$ according to this measure. For each $n \in \omega$, define a function $f_n \in C({}^\omega 2, G)$ as follows. For each $\sigma \in {}^{[b(n), b(n+1))}2$ (i.e., for each binary sequence of length $b(n+1) - b(n)$), choose a pair $(r(\sigma), r'(\sigma))$ of elements of S_n at random, with equal probability for each of the $|S_n|^2$ possibilities; these choices should be independent for the various σ's. Now let $f_n(x) = r(\sigma) - r'(\sigma)$, where $\sigma = x \restriction [b(n), b(n+1))$. Let μ_n be the probability measure corresponding to this method of randomly choosing f_n (so μ_n concentrates on a finite set). Clearly the randomly chosen function f_n is always continuous, and we always have $d(f_n(x), \mathbf{0}) \le 2^{1-n}R$. Therefore, if we choose such an f_n independently for each n, then the series $\sum_{n=0}^{\infty} f_n$ converges uniformly; let f be its sum. Then the probability measure μ under which f is chosen is just the infinite convolution of the measures μ_n.

We will show that, for any fixed $h \in C({}^\omega 2, G)$, if f is chosen randomly according to μ, then with probability 1 the function $f + h$ has a range which is dense in some non-empty open set U. Hence, the measure μ witnesses that the set of functions with this property is prevalent.

Let $h \in C({}^\omega 2, G)$ be fixed. Let $\overline{n}$ be a natural number such that the range of h can be covered by $\overline{n}$ open balls of radius R; since the range of h is compact, any sufficiently large $\overline{n}$ will do. If f_i is chosen randomly according to the measure μ_i for $i < \overline{n}$, then we have $d(f_i(x), \mathbf{0}) \le 2^{1-i}R$ for all x, so $d(\sum_{i=0}^{\bar{n}-1} f_i(x), \mathbf{0}) < 4R$. Therefore, if we let $h_{\bar{n}} = h + \sum_{i=0}^{\bar{n}-1} f_i(x)$, then the range of $h_{\bar{n}}$ will be covered by $\overline{n}$ open balls of radius $5R$, and hence by $k\overline{n}$ open balls of radius R and by $k\overline{n}p_{\bar{n}}$ balls of radius $2^{-\bar{n}}R$. Since $a_{\bar{n}} \ge k\overline{n}p_{\bar{n}}$, we can find an open ball $U = B(x, 2^{-\bar{n}}R)$ such that $\lambda(h_{\bar{n}}^{-1}(U)) \ge a_{\bar{n}}^{-1}$.

We now prove the following claim by induction on m: For any $m \ge \overline{n}$, if the functions f_i are chosen randomly according to μ_i for $\overline{n} \le i < m$, and we let $h_n = h_{\bar{n}} + \sum_{i=\bar{n}}^{n-1} f_i$ for $\overline{n} \le n \le m$, then the probability that $\lambda(h_n^{-1}(B(x+y, 2^{-n}R))) \ge a_n^{-1}$ for all possible choices of $n \le m$ and $y \in S_{\bar{n}} + S_{\bar{n}+1} + \ldots + S_{n-1}$ is at least $1 - 2^{1-\bar{n}} + 2^{1-m}$. (Note that the functions f_i for $i < \overline{n}$ were chosen previously and are being held fixed here.)

The base case $m = \overline{n}$ is immediate from the choice of U and x. Now suppose the claim is true for m. If f_i has been chosen for $i < m$, and

$y \in S_{\bar n}+S_{\bar n+1}+\ldots+S_{m-1}$ is such that $\lambda(h_m^{-1}(B(x+y,2^{-m}R))) \ge a_m^{-1}$, then there must exist a $z \in S_m$ such that $M = \lambda(h_m^{-1}(B(x+y+z,2^{-m-1}R))) \ge a_m^{-1}/|S_m|$. Fix such z and M, and let

$$c_\sigma = \lambda(\{\alpha : \alpha\restriction[b(m),b(m+1)) = \sigma \text{ and } h_m(\alpha) \in B(x+y+z,2^{-m-1}R)\})/M$$

for $\sigma \in {}^{[b(m),b(m+1))}2$. Then the numbers c_σ add up to 1 and we have

$$c_\sigma \le 2^{-(b(m+1)-b(m))}/M \le 2^{-(b(m+1)-b(m))}a_m|S_m|$$

for all σ, so, by the definition of $b(m+1)$ and Lemma 10, the probability is at least $1-2^{-m}p_m^{-1}$ that, for all $z' \in S_m$, the sum of the c_σ's for which $r(\sigma)=z$ and $r'(\sigma)=z'$ (where r and r' are as in the definition of μ_m) is at least $1/(2|S_n|^2)$. But, if $r(\sigma)=z$ and $r'(\sigma)=z'$ then

$$h_m^{-1}(B(x+y+z,2^{-m-1}R)) \cap \{\alpha : \alpha\restriction[b(m),b(m+1)) = \sigma\} \subseteq (h_m+f_m)^{-1}(B(x+y+z',2^{-m-1}R));$$

hence, the probability is at least $1-2^{-m}p_m^{-1}$ that, for all $z' \in S_m$,

$$\lambda(h_{m+1}^{-1}(B(x+y+z',2^{-m-1}R))) \ge M/(2|S_m|^2) \ge a_{m+1}^{-1}\,.$$

Applying this to all y in $S_{\bar n}+S_{\bar n+1}+\ldots+S_{m-1}$ (a set of size at most p_m) shows that, if the assertion in the claim holds for m, then the probability that it fails for $m+1$ is at most $p_m 2^{-m}p_m^{-1} = 2^{-m}$. By the inductive hypothesis, the probability that the assertion holds for m is at least $1-2^{1-\bar n}+2^{1-m}$, so the probability that the assertion holds for $m+1$ is at least $1-2^{1-\bar n}+2^{1-m}-2^{-m} = 1-2^{1-\bar n}+2^{1-(m+1)}$. This completes the inductive proof of the claim.

Therefore, the probability that $\lambda(h_n^{-1}(B(x+y,2^{-n}R))) \ge a_n^{-1}$ for all $n \ge \bar n$ and $y \in S_{\bar n}+S_{\bar n+1}+\ldots+S_{n-1}$ is at least $1-2^{1-\bar n}$. For any particular n, the open balls $B(x+y,2^{-n}R)$ for $y \in S_{\bar n}+S_{\bar n+1}+\ldots+S_{n-1}$ cover the original ball $U = B(x,2^{-\bar n}R)$. Now, if V is a non-empty open subset of U, choose $w \in V$, and let $n \ge \bar n$ be so large that $B(w,6\cdot 2^{-n}R) \subseteq V$. If $w \in B(x+y,2^{-n}R)$, then for any α such that $h_n(\alpha) \in B(x+y,2^{-n}R)$ we have

$$d\Big(w, h_n(\alpha)+\sum_{i=n}^{\infty} f_i(\alpha)\Big) < 2\cdot 2^{-n}R + \sum_{i=n}^{\infty} 2^{1-i}R = 6\cdot 2^{-n}R\,,$$

so, if $f=\sum_{i=0}^{\infty} f_i$, then $h(\alpha)+f(\alpha) \in V$. Hence, $\lambda(h_n^{-1}(B(x+y,2^{-n}R))) > 0$ implies $\lambda((h+f)^{-1}(V)) > 0$ and hence $(h+f)^{-1}(V) \ne \emptyset$. Therefore, the probability that $(h+f)^{-1}(V) \ne \emptyset$ for all non-empty open $V \subseteq U$ (i.e., the range of $h+f$ is dense in U) is at least $1-2^{1-\bar n}$. Since, as we noted earlier, $\bar n$ could be chosen to be an arbitrarily large natural number, the probability that there is some non-empty open U such that the range of $h+f$ is dense in U is 1, as desired. ∎

We now argue further that, for "most" (in the sense of prevalence) functions f, there exists a non-empty open set U such that every point of U is the image of $2^{\aleph_0}$ points of ${}^\omega 2$. To see this, first note that G, being homogeneous, must be either discrete or perfect; the discrete case is trivial, so assume G is perfect. One can define a bicontinuous isomorphism Φ from $C({}^\omega 2, G)$ to $C({}^\omega 2, G^2)$ by the formula $\Phi(f)(x) = (f(0x), f(1x))$. For "most" functions $h \in C({}^\omega 2, G^2)$, the range of h includes an open set, which in turn includes a basic open set of the form $U \times V$. Therefore, if $f = \Phi^{-1}(h)$ (note that "most" $f \in C({}^\omega 2, G)$ have this form for some h as above), then for any $y \in U$ and $z \in V$ there exists $x \in {}^\omega 2$ such that $f(0x) = y$ and $f(1x) = z$; hence, for each $y \in U$ there are $2^{\aleph_0}$ x's such that $f(0x) = y$.

Some of the results here can be worked with more easily in terms of variants of non-shyness, a number of which are mentioned in Hunt, Sauer, and Yorke [3]. In the following list of properties of a universally measurable subset S of a Polish group G, ε will vary over positive real numbers, μ over probability measures on G, and t over translation functions $g \mapsto g_1 g g_2$.

(1)	$\exists\mu\forall t\ \mu(t(S)) = 1$	[prevalent]
(2)	$\forall\varepsilon\exists\mu\forall t\ \mu(t(S)) > 1-\varepsilon$	[lower density 1]
(3)	$\exists\varepsilon\exists\mu\forall t\ \mu(t(S)) > \varepsilon$	[positive lower density]
(4)	$\exists\mu\forall t\ \mu(t(S)) > 0$	[observable]
(1′)	$\forall\mu\exists t\ \mu(t(S)) = 1$	
(2′)	$\forall\varepsilon\forall\mu\exists t\ \mu(t(S)) > 1-\varepsilon$	[upper density 1]
(3′)	$\exists\varepsilon\forall\mu\exists t\ \mu(t(S)) > \varepsilon$	[positive upper density]
(4′)	$\forall\mu\exists t\ \mu(t(S)) > 0$	[non-shy]

(Perhaps, for completeness, one should introduce a word for (1′), such as "ubiquitous.") These come in four dual pairs: S satisfies (1) iff the complement of S does not satisfy (4′), and so on. The implications $(j) \to (k)$ and $(j') \to (k')$ for $j < k$ are trivial. An argument using convolution and Fubini's theorem shows that $(j) \to (j')$ for each j.

N o t e. The application of the above definitions to any universally measurable set S follows Christensen [1] and Kechris [4] but differs from Hunt, Sauer, and Yorke [3], in which the definitions are given for Borel sets and then extended to other sets by inclusion. For an example where the difference is significant, one can use the fact that, if the continuum hypothesis (or just Martin's axiom) holds, then any Polish space X has a subset S such that $S \cap A$ has cardinality less than $2^{\aleph_0}$ whenever A is σ-compact but $S \cap A$ has cardinality $2^{\aleph_0}$ whenever A is a Borel set which is not included in a

σ-compact set. If X is a Polish group with an invariant metric which is not locally compact, then this set S is universally measurable and shy (since any probability measure on X is based on a σ-compact subset of X, and Martin's axiom implies that sets of cardinality less than $2^{\aleph_0}$ have measure 0 under any atomless measure, any atomless probability measure will be transverse to S), but any Borel set which includes X must be the complement of a σ-compact set and therefore prevalent by Proposition 12 below. [This gives a conditional negative answer to problem (P_1) from Mycielski [5].] However, this set S has a high complexity, if it is definable at all; as far as I know, it is open whether any, say, analytic shy set must be included in a Borel shy set, or any analytic non-shy set must include a Borel non-shy set.

On the other hand, one cannot apply the above definitions to *arbitrary* sets S, if one wants to maintain desirable properties such as additivity of shyness. Again assuming the continuum hypothesis or Martin's axiom, if G is an uncountable Polish group and $\preceq$ is a well-ordering of G in minimal order type, then both $\preceq$ and its complement are subsets of G^2 which have transverse Borel measures (namely product measures $\mu \times \mu'$, where one of μ, μ' is an atomless measure on G and the other is a measure which concentrates on a single point).

Hence, it is important to note that, in each case where we have constructed a non-shy or prevalent set which is not obviously universally measurable (e.g., the set of functions $f \in C({}^\omega 2, G)$ for which every point in some non-empty open subset of G is the f-image of $2^{\aleph_0}$ points from ${}^\omega 2$), we have done so by showing that the set includes a non-shy or prevalent set which *is* universally measurable (e.g., the preimage under the isomorphism Φ defined after Proposition 11 of the set of $f \in C({}^\omega 2, G^2)$ such that the range of f is dense in some non-empty open neighborhood; this set is Borel).

For compact G, (1), (2), (1′), and (2′) are equivalent to having full Haar measure, while the remaining four are equivalent to having positive Haar measure. For locally compact G, (1) and (1′) are equivalent to having full Haar measure and (4) and (4′) to having positive Haar measure, but the remaining four can be distinguished: a finite interval in $\mathbb{R}$ satisfies (4) but not (3′), its complement satisfies (2) but not (1′), and the set of positive reals satisfies (2′) but not (3).

All of the examples of non-shy sets we have constructed here are actually sets of upper density 1. To see this, note that, given a measure, we constructed certain intermediate sets of large measure and intersected them to get a set C of positive measure; by making the intermediate sets a little larger, we may ensure that C has measure greater than $1 - \varepsilon$ for a given $\varepsilon > 0$. In certain cases, we can do even better. For instance, take the set P of $s \in {}^\omega\mathbb{R}$ such that $s(n) > 0$ for all but finitely many n. If the numbers

b_n are constructed from a given measure μ as in the proof of Proposition 1, then $\mu(\{s \in {}^\omega\mathbb{R} : |s(n)| \le b_n \text{ for all } n \ge N\}) > 1 - 2^N$ for all N, so the set $\{s \in {}^\omega\mathbb{R} : |s(n)| \le b_n \text{ for almost all } n\}$ has measure 1. This set has a translate included in P, so P satisfies (1′). The complement of P also satisfies (1′) for the same reasons, so neither P nor its complement is observable. Therefore, all eight of these properties are distinct in general.

All eight properties are preserved under finite products (considered as subsets of the corresponding finite products of groups). However, it is not hard to see that the stronger properties (1), (2), (1′), and (2′) are preserved under countably infinite products as well. Hence, once one has a pair of disjoint subsets of G satisfying (1′) or (2′) (such as the positive and negative reals, or the set P and its complement from the preceding paragraph), one can immediately get $2^{\aleph_0}$ disjoint subsets of ${}^\omega G$ satisfying the same property. One cannot do this with properties (1) and (2), of course, since these properties define (countably closed) filters.

Finally, a more general version of Fact 8 from Hunt, Sauer, and Yorke [3] and Exercise 1.56 from Kechris [4]:

PROPOSITION 12. *Let G be a non-locally-compact Polish group with an invariant metric. Then any compact subset* (*and hence any K_σ subset*) *of G is shy.*

Proof. Let d be the invariant metric, and let e be the identity element of G. Let K be a compact subset of G. Construct positive numbers δ_n, ε_n and finite subsets S_n of G for $n \in \omega$ as follows. Let $\delta_0 = 1$. Given δ_n, choose $\varepsilon_n > 0$ so small that there are infinitely many points in $B(e, \delta_n/2)$ which are at distance at least ε_n from each other. Since K is compact, there is a finite subset of K which is $\varepsilon_n/6$-dense in K. Let N be the size of such a subset; then K cannot contain more than N points at distance at least $\varepsilon_n/3$ from each other. Let S_n be a subset of $B(e, \delta_n/2)$ of cardinality $\max(2, N)$ whose points are at distance at least ε_n from each other. Now let $\delta_{n+1} = \min(\delta_n/2, \varepsilon_n/3)$.

Let μ_n be the measure on the finite set S_n which assigns measure $1/|S_n|$ to each point in S_n, and let μ be the infinite convolution of the measures μ_n. Then, for any n, μ concentrates on the union of $\prod_{i=0}^n |S_i|$ closed balls of radius δ_{n+1}, giving equal measure to each, and these balls are at distance at least $\varepsilon_n/3$ from each other. Therefore, any translate of K can meet at most $|S_n|$ of these balls; since $\prod_{i=0}^n |S_i| \ge 2^n|S_n|$, any translate of K has μ-measure at most 2^{-n}. Since n was arbitrary, all translates of K have μ-measure 0, as desired. ■

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