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Every Lusin set is undetermined in the point-open game

by

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Abstract. We show that some classes of small sets are topological versions of some combinatorial properties. We also give a characterization of spaces for which White has a winning strategy in the point-open game. We show that every Lusin set is undetermined, which solves a problem of Galvin.

1. Introduction. In Set Theory many infinite combinatorial proofs are "Borel". So we can very often get interesting topological theorems using the same proofs as for combinatorics.

For example, in this paper we consider some classical notions of smallness such as the Hurewicz property, Menger property, C$''$-sets and others (see [M1] and [FM]). We observe that these classes of sets can be expressed by some combinatorial properties used to define some cardinal coefficients, for example $\mathbf{b}$, $\mathbf{d}$, $\mathbf{p}$, cov($\mathbf{M}$). We also investigate some "measurable" versions of additivity of measure and we get some implications using the proofs of Bartoszyński (see [B1]). We apply those methods to investigate the determinacy of point-open games. In particular, we show that every Lusin set is undetermined.

We use the following notation:

- $[\omega]^\omega = \{A \subseteq \omega : A \text{ infinite}\}$,
- $\exists_n^\infty$ — there are infinitely many n,
- $\forall_n^\infty$ — for all but finitely many n,
- $s^\frown t$ — the concatenation of finite sequences s and t,
- $\mathbf{c}$ — the cardinal number continuum,
- $|X|$ — cardinality of X,
- μ — Lebesgue measure,

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- $G_x = \{y : (x, y) \in G\}$,
- $G^y = \{x : (x, y) \in G\}$,
- for $s \in \kappa^{<\omega}$ let $[s] = \{x \in \kappa^\omega : s \subset x\}$,
- $A \subset^* B$ if $|A \setminus B|$ is finite,
- $f <^* g$ if $\{n : g(n) < f(n)\}$ is finite.

We say that $X \subset \omega^\omega$ is *bounded* if there is a function $g \in \omega^\omega$ such that $x <^* g$ for each $x \in X$. We then write $X <^* g$.

We say that $X \subset \omega^\omega$ is *dominating* if for each $g \in \omega^\omega$ there is an $x \in X$ with $g <^* x$.

We say that a subset $L \subseteq \mathbb{R}$ is a *Lusin* (*Sierpiński*) *set* if L is uncountable and its intersection with every set of first category (of measure zero) is countable.

We say that X is *concentrated on* $D \subset X$ if $|X \setminus U| \le \omega$ for every open set $U \supseteq D$.

For a property H let $\mathrm{non}(\mathrm{H}) = \min\{|F| : F$ does not satisfy H and $F \subseteq \mathbb{R}\}$.

2. Games. Let X be a topological space. We recall two infinite games.

The *point-open game* $G(X)$: In the nth move Black plays a point $x_n \in X$ and White plays an open set U_n containing x_n. Black wins if $\bigcup_n U_n = X$. Otherwise White wins.

$G^*(X)$: In the nth move White plays an open cover J_n of X and Black plays an element U_n of J_n. Black wins if $\bigcup_n U_n = X$, otherwise White wins.

Theorem (Galvin [G]). (a) *$G(X)$ and $G^*(X)$ are equivalent.*

(b) (CH) *There is a Lusin set which is undetermined.*

Galvin [G] asked if every Lusin set is undetermined in the point-open game.

Let $\kappa = \omega(X)$ be the weight of the space X with the discrete topology. Then κ^ω is a complete metric space.

Theorem 1. *White has a winning strategy in $G(X)$ iff there is a closed set $D \subseteq X \times \kappa^\omega$ such that D_x is nowhere dense for every x in X, and $\bigcup_{x \in X} D_x = \kappa^\omega$.*

Proof. $\Leftarrow$ We define a winning strategy for White. At each step White chooses two open sets U_n and V_n such that $U_n \subseteq X$, U_n contains x_n, $\overline{V}_n \subseteq V_{n-1} \subseteq \kappa^\omega$, $\mathrm{diam}(V_n) < 1/n$ and $(U_n \times V_n) \cap D = \emptyset$. Then White plays U_n.

Assume that $\bigcup_n U_n = X$. Let $y \in \bigcap_n V_n$. Then $(U_n \times \{y\}) \cap D = \emptyset$ for each n. So $(X \times \{y\}) \cap D = \emptyset$, a contradiction.

$\Rightarrow$ By the Galvin Theorem, White has a winning strategy in the game $G^*(X)$. Let O be a basis of size κ. We can assume that White chooses a function from $\mathcal{A} = \{J : \kappa \to O : \bigcup_{\alpha<\kappa} J(\alpha) = X\}$. So we can assume that Black chooses an $\alpha \in \kappa$.

Let $S : \kappa^{<\omega} \to \mathcal{A}$ be a strategy for White. We define

$$W = \bigcup_{s\in\kappa^{<\omega}} \bigcup_{\alpha\in\kappa} S(s)(\alpha) \times [s^\frown\langle\alpha\rangle]\,.$$

Since W is open, $D = (X \times \kappa^\omega) \setminus W$ is closed. Let $x \in X$ and $s \in \kappa^{<\omega}$. Then there is an $\alpha \in \kappa$ such that $x \in S(s)(\alpha)$ because $S(s)$ is a cover of X. Then $[s^\frown\langle\alpha\rangle] \cap D_x = \emptyset$. So D_x is nowhere dense. If there were $y = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \in \kappa^\omega$ such that $y \notin \bigcup_{x\in X} D_x$ then Black would win playing $S(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})(\alpha_n)$ in the nth move. Observe that if $x \in X$ then $(x, y) \in W$ and so there are $s \in \kappa^{<\omega}$ and $\alpha \in \kappa$ such that $(x, y) \in S(s)(\alpha) \times [s^\frown\langle\alpha\rangle]$. Since $y \in [s^\frown\langle\alpha\rangle]$, there is an $n \in \omega$ such that $(\alpha_0, \alpha_1, \ldots, \alpha_n) = s^\frown\langle\alpha\rangle$ so $x \in S(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})(\alpha_n)$.

Remarks. It is consistent that for any uncountable metric space White has a winning strategy so every metric space is determined. Telgársky constructed an uncountable, undetermined, Hausdorff, Lindelöf space. For Lindelöf spaces, the κ in Theorem 1 can also be equal to ω.

Lemma 1. *Let $L \subseteq \omega^\omega$ be a Borel image of a Lusin set. Then there is a function $f \in \omega^\omega$ such that* $\forall_{x\in L}\exists^\infty_n(x(n) = f(n) \wedge \forall_{i<n} f(i) < n)$.

Proof. Every Borel image of a Lusin set is concentrated on a countable subset so it is a C″-set (for definition see the next section). So there is a function $g \in \omega^\omega$ such that $\forall_{x\in L}\exists^\infty_n x(n) = g(n)$. Since g itself need not satisfy the assertion of the theorem, we improve it by putting in some places 0 instead of $g(n)$.

Let $\{y_l : l \in \omega\} \subseteq L$ be such that L is concentrated on it. Inductively we can construct an increasing sequence n_k such that $n_{k+1} > \max\{g(n) : n \le n_k\}$ and $\forall_{l\in\omega}\exists^\infty_k y_l(n_k) = g(n_k)$.

Let $K = \{x \in L : \exists^\infty_k x(n_k) = g(n_k)\}$. Observe that K is a relative G_δ in L containing $\{y_l : l \in \omega\}$ so $|L \setminus K| \le \omega$.

We define a function $F : K \to \omega^\omega$ by $F(x)(n) = n$th element of the set $\{k : g(n_k) = x(n_k)\}$. Obviously $\{k : g(n_k) = x(n_k)\}$ is infinite for $x \in K$. Since F is Borel, $F[K]$ is a Borel image of a Lusin set so there is an increasing function $a \in \omega^\omega$ such that $\forall_{x\in K}\exists^\infty_i F(x)(i) < a(i)$. Let $L \setminus K = \{z_l : l \in \omega\}$. We choose m_k such that $m_k > n_{3a(k)+3}$ and $\forall_{l\in\omega}\exists^\infty_k g(m_k) = z_l(m_k)$.

Now we define a function f by

$$f(n) = \begin{cases} 0 & \text{if } n \notin \{m_k : k \in \omega\} \cup \{n_k : k \in \omega\},\\ g(n) & \text{if } \exists_k n = m_k,\\ g(n) & \text{if } \exists_k (n = n_k \wedge \neg\exists_l n_k < m_l < n_{k+1}),\\ 0 & \text{if } \exists_k (n = n_k \wedge \exists_l n_k < m_l < n_{k+1}). \end{cases}$$

Let $x \in L$. First assume that $x \in L \setminus K$. Then $\exists_k^\infty g(m_k) = x(m_k)$. Let $n < m_k$ and let $l \in \omega$ be such that $n_l < m_k < n_{l+1}$. Then if $n \le n_{l-1}$ then $f(n) \le g(n) < n_l < m_k$. If $n_{l-1} < n < m_k$ then $f(n) = 0 < m_k$.

Now assume that $x \in K$. Then $\exists_k^\infty F(x)(k) \le a(k)$. Since $m_{[k/3]} > a(k)$ there is an $l \le k$ such that $\neg((\exists_i n_{F(x)(l)} < m_i < n_{F(x)(l)+1}) \vee (n_{F(x)(l)-1} < m_i < n_{F(x)(l)}))$. Then $f(n_{F(x)(l)}) = g(n_{F(x)(l)})$ and $\forall_{i \le n_{F(x)(l)-1}} f(i) \le g(i) < n_{F(x)(l)}$ and $\forall_i((n_{F(x)(l)-1} < i < n_{F(x)(l)}) \Rightarrow f(i) = 0)$.

Theorem 2. *Let $X \subseteq 2^\omega$ be a Lusin set and $D \subseteq X \times 2^\omega$ be a closed set such that D_x is nowhere dense for each $x \in X$. Then $\bigcup_{x\in X} D_x \ne 2^\omega$.*

Proof. We closely follow the line of reasoning from [M4].

Let $2^{<\omega} = \{s_0, s_1, \ldots\}$. We define $H : X \to \omega^\omega$ by

$$H(x)(n) = \min\{k : \forall_{i_0,i_1,\ldots,i_{n-1}<n}[s_{i_0}{}^\frown s_{i_1}{}^\frown \ldots {}^\frown s_{i_{n-1}}{}^\frown s_k] \cap D_x = \emptyset\}\,.$$

It is easy to see that H is a Borel function. Then there is a function $f \in \omega^\omega$ such that $\forall_{z\in H[X]}\exists_n^\infty(z(n) = f(n) \wedge \forall_{i<n} f(i) < n)$. Let $y = s_{f(0)}{}^\frown s_{f(1)}{}^\frown \ldots {}^\frown s_{f(n)}{}^\frown \ldots$ We will show that $\forall_{x\in X} y \notin D_x$. For $z \in H[X]$, $\exists_n(z(n) = f(n) \wedge \forall_{i<n} f(i) < n)$. Observe that $y \in [s_{f(0)}{}^\frown s_{f(1)}{}^\frown \ldots {}^\frown s_{f(n)}] = [s_{f(0)}{}^\frown s_{f(1)}{}^\frown \ldots {}^\frown s_{H(x)(n)}]$. So since $f(0), f(1), \ldots, f(n-1) < n$ we have $[s_{f(0)}{}^\frown s_{f(1)}{}^\frown \ldots {}^\frown s_{f(n)}] \cap D_x = \emptyset$.

Corollary 1. *Let $X \subseteq 2^\omega$ be a Lusin set and $D \subseteq X \times \omega^\omega$ be a closed set such that D_x is nowhere dense for each $x \in X$. Then $\bigcup_{x\in X} D_x \ne \omega^\omega$.*

Proof. ω^ω is homeomorphic to a subset $Z \subseteq 2^\omega$ such that $2^\omega \setminus Z$ is countable. If $\bigcup_{x\in X} D_x = \omega^\omega$ then by a natural homeomorphism we can construct a set $C \subseteq X \times 2^\omega$ such that $C_x \supseteq Z$. Then adding to X countably many isolated points we can obtain the missing points from 2^ω to get a contradiction with Theorem 2.

So we get a positive answer to the question of Galvin.

Corollary 2. *Every Lusin set is undetermined in the point-open game.*

Proof. By Theorem 1, White does not have a winning strategy for a Lusin set. For subsets of the reals Black has a winning strategy only for countable sets so every Lusin set is undetermined.

Corollary 3. *Let $X \subseteq 2^\omega$ be a Lusin set and $D \subseteq X \times \mathbb{R}$ be a first category set in $X \times \mathbb{R}$ such that D_x is first category for each $x \in X$. Then $\bigcup_{x\in X} D_x \ne \mathbb{R}$.*

Proof. Observe that for every closed set $D \subseteq X \times \mathbb{R}$ with every section nowhere dense, $\bigcup_{x\in X} D_x$ is not residual. Otherwise we could obtain $\mathbb{R}$ adding countably many isolated points to X.

First assume that $D=\bigcup_n D_n$ is such that D_n is closed with every section nowhere dense. Define $C=\{(n,x,y):(x,y)\in D_n \text{ and } n\in\omega\}$. Observe that C is closed, $\{n\}\times X$ is a Lusin set and $C_{(n,x)}=(D_n)_x$.

Now let $D\subseteq X\times\mathbb{R}$ be a first category set in $X\times\mathbb{R}$ such that each D_x is first category. Then D is contained in a F_σ-set F of first category. The set $A=\{x:F_x \text{ is second category}\}$ is countable. Since $X\setminus A$ is also a Lusin set, $\bigcup_{x\in X\setminus A}F_x$ is not residual. Thus $\bigcup_{x\in X}D_x\subseteq\bigcup_{x\in A}D_x\cup\bigcup_{x\in X\setminus A}F_x\neq\mathbb{R}$.

3. Small sets and cardinal coefficients. In this section we will compare some notions of smallness in the sense of topology and combinatorics. Definitions 1–4 can be found in [FM], and Definition 6 in [GM]. The definitions of the coefficients $\mathbf{p}$, $\mathbf{d}$, $\mathbf{b}$ are in [D]. $\operatorname{cov}(\mathbf{M})$ and $\operatorname{add}(\mathbf{N})$ were investigated for example in [B1] and [M5].

DEFINITION 1. A topological space X has the *Hurewicz property* if for every family $\{J_n:n\in\omega\}$ of open covers of X there is a family $\{J'_n:n\in\omega\}$ such that J'_n is a finite subset of J_n and $X\subseteq\bigcup_k\bigcap_{n>k}\bigcup J'_n$.

PROPOSITION 1. *Let X be a 0-dimensional, separable metric space. Then X has the Hurewicz property iff every continuous image of X into ω^ω is bounded.*

Proof. $\Rightarrow$ Every continuous image of a Hurewicz set is a Hurewicz set. Let X be a subset of ω^ω. Let $J_n=\{\{f\in\omega^\omega:f(n)=k\}:k\in\omega\}$. By the Hurewicz property this set must be bounded.

$\Leftarrow$ Let J_n be a family of open covers of X. By 0-dimensionality we can assume that all elements of J_n are clopen and disjoint. Define $h:X\to\omega^\omega$ by $h(x)(n)=k$ if x belongs to the kth element of J_n. Then h is continuous and $h[X]$ is bounded so there is a $\phi\in\omega^\omega$ such that $h[X]\le^*\phi$. Then J'_n is simply the first $\phi(n)$ elements of J_n.

As in Proposition 1, we will consider other pairs of classes of small sets and coefficients.

We say that a family J of subsets of X is an *ω-cover* if for each finite set $A\subset X$ there is a $U\in J$ such that $A\subset U$.

DEFINITION 2. X is a *γ-set* if for every open ω-cover J of X there exists a sequence $(D_n:n\in\omega)$ of elements of J such that $X\subseteq\bigcup_k\bigcap_{n>k}D_n$.

We say that $F\subseteq\omega^\omega$ has *property* P if $|\bigcap F_0|=\omega$ for every finite subset F_0 of F. Then there exists $A\in[\omega]^\omega$ such that $A\subseteq^* B$ for every $B\in F$. We define

$$\mathbf{p}=\min\{|F|:F\in[\omega]^\omega \text{ and } \neg(F \text{ has property P})\}.$$

PROPOSITION 2. *Let X be a 0-dimensional, separable metric space. Then X is a γ-set iff $f[X]$ has property* P *for every continuous function $f:X\to[\omega]^\omega$.*

P r o o f. $\Rightarrow$ Every continuous image of a γ-set is a γ-set. Let X be a subset of $[\omega]^\omega$. Let $O_n = \{A \in [\omega]^\omega : n \in A\}$. Assume that $|\bigcap F_0| = \omega$ for every finite subset F_0 of X. Then the family $J = \{O_n : n \in \omega\}$ is an open ω-cover of X. By the γ-property there is a sequence n_k such that $X \subseteq \bigcup_m \bigcap_{k>m} O_{n_k}$. The case when the sequence n_k has only finitely many values is left to the reader. Assume that n_k is increasing. Then for every $B \in X$ almost every n_k belongs to B. Thus $\{n_k : k \in \omega\} \subseteq^* B$.

$\Leftarrow$ Let $J = \{D_n : n \in \omega\}$ be an open ω-cover of X. By 0-dimensionality we can assume that all elements of J are clopen and every subset of X is contained in infinitely many elements of J. Define $h : X \to [\omega]^\omega$ by $h(x) = A$ iff (for every $n, n \in A$ iff $x \in D_n$). Then h is continuous and $|\bigcap F_0| = \omega$ for every finite subset F_0 of $h[X]$. So there exists $A \in [\omega]^\omega$ such that $A \subseteq^* B$ for every $B \in h[X]$. We can see that $h^{-1}[O_n] = D_n$. So $X \subseteq \bigcup_m \bigcap_{k>m} D_{n_k}$ where $\{n_k : k \in \omega\} = A$.

DEFINITION 3. X has the *Menger property* if for every sequence $(J_n : n \in \omega)$ of open covers there is a sequence $(J'_n : n \in \omega)$ such that $J'_n \subseteq J_n$, J'_n is finite, and $X \subseteq \bigcup_n \bigcup J'_n$.

We set

$$\mathbf{d} = \min\{|F| : F \subseteq \omega^\omega \text{ and } \neg(F \text{ is not dominating})\}.$$

PROPOSITION 3. *Let X be a 0-dimensional, separable metric space. Then X has the Menger property iff for every continuous function $f : X \to \omega^\omega, f[X]$ is not dominating.*

P r o o f. The proof is similar to the proof of Proposition 1.

DEFINITION 4. X has the C$''$ *property* if for every sequence $(J_n : n \in \omega)$ of open covers there is a sequence $(D_n : n \in \omega)$ such that $D_n \in J_n$ and $X \subseteq \bigcup_n D_n$.

We set

$$\mathrm{cov}(\mathbf{M}) = \min\Big\{|F| : F \subseteq \mathbf{M} \text{ and } \bigcup F = \mathbb{R}\Big\}$$

where $\mathbf{M}$ is the σ-ideal of first category sets.

We say that $F \subseteq \omega^\omega$ has *property* CM if there exists $g \in \omega^\omega$ such that for every $f \in F$ there exist infinitely many n such that $f(n) = g(n)$.

Bartoszyński [B2] showed that $\mathrm{cov}(\mathbf{M}) = \{|F| : F \subseteq \omega^\omega$ and $\neg(F$ has property CM$)\}$.

PROPOSITION 4. *Let X be a 0-dimensional, separable metric space. Then X is a C$''$-set iff $f[X]$ has property* CM *for every continuous function $f : X \to \omega^\omega$.*

P r o o f. The proof is similar to the proof of Proposition 1.

We also set

$$\mathrm{add}(\mathbf{N}) = \Big\{|F| : F \subseteq \mathbf{N} \text{ and } \bigcup F \notin \mathbf{N}\Big\}$$

where $\mathbf{N}$ is the σ-ideal of Lebesgue measure zero sets.

Let k_n be an arbitrary increasing sequence of natural numbers. We say that $F \subseteq \omega^\omega$ has *property* AN if there exists $g \in ([\omega]^{<\omega})^\omega$ such that $|g(n)| \le k_n$ for every n, and $f(n) \in g(n)$ for every $f \in F$ and for almost every n.

Bartoszyński [B1] showed that $\mathrm{add}(\mathbf{N}) = \min\{|F| : F \subseteq \omega^\omega$ and $\neg(F$ has property AN$)\}$.

Definition 5. We say that X is $\mathrm{add}(\mathbf{N})$-*small* if there exists an increasing sequence k_n such that for every sequence $(J_n : n \in \omega)$ of open covers there is a sequence $(J'_n : n \in \omega)$ such that $J'_n \subseteq J_n, |J'_n| \le k_n$, and $X \subseteq \bigcup_k \bigcap_{n>k} \bigcup J'_n$.

Proposition 5. *Let X be a* 0*-dimensional*, *separable metric space. Then X is* $\mathrm{add}(\mathbf{N})$*-small iff $f[X]$ has property* AN *for every continuous function $f : X \to \omega^\omega$.*

Proof. The proof is similar to the proof of Proposition 1.

We say that J is a *cover of* $[X]^k$ if for every finite set $A \subset X$ of size k there is an element $O \in J$ with $A \subset O$.

Definition 6. We say that X is a *strong γ-set* iff there exists $(k_n : n \in \omega)$ such that for any sequence $(J_n : n \in \omega)$ where J_n is an open cover of $[X]^{k_n}$ there exists $(C_n : n \in \omega)$ with $C_n \in J_n$ and $X \subseteq \bigcup_n \bigcap_{m>n} C_m$.

Proposition 6. *Every strong γ-set is* $\mathrm{add}(\mathbf{N})$*-small.*

Proof. Let $(I_n : n \in \omega)$ be a family of open covers of a strong γ-set X, and $J_n = \{\bigcup I'_n : I'_n \subseteq I_n$ and $|I'_n| \le k_n\}$. Then J_n is a cover of $[X]^{k_n}$. Since X is a strong γ-set there is a sequence $(D_n : n \in \omega)$ such that $D_n \in J_n$ and $X \subseteq \bigcup_n \bigcap_{m>n} D_m$. Since we know that D_n is a union of at most k_n elements of I_n we conclude that X is $\mathrm{add}(\mathbf{N})$-small.

From the existence of a strong γ-set of size $\mathbf{c}$ under MA we get:

Corollary 4. *Assuming Martin's Axiom there exists an* $\mathrm{add}(\mathbf{N})$*-small set of reals of size* $\mathbf{c}$.

Proof. See [GM] for an example of a strong γ-set.

Remarks. From the results above we find that: $\mathrm{non}(\gamma) = \mathbf{p}$, non(Hurewicz property) $= \mathbf{b}$, non(Menger property) $= \mathbf{d}$, $\mathrm{non}(\mathrm{C}'') = \mathrm{cov}(\mathbf{M})$ and non(add($\mathbf{N}$)-small) $= \mathrm{add}(\mathbf{N})$. These results except the last one were obtained in [FM].

The following relations between the cardinal coefficients mentioned above are known: $\mathrm{add}(\mathbf{N}) \le \mathrm{cov}(\mathbf{M}) \le \mathbf{d}$ and $\mathbf{p} \le \mathbf{b} \le \mathbf{d}$ and $\mathbf{p} \le \mathrm{cov}(\mathbf{M})$. This is a part of Cichoń's Diagram (see [F]).

Similar inclusions exist for classes of sets: (strong $\gamma \to \gamma \to$ Hurewicz property $\to$ Menger property) and ($\gamma \to \mathrm{C}'' \to$ Menger property).

We can see that non(strong γ) $\le \min(\mathrm{add}(\mathbf{N}), \mathbf{p})$. So if $\mathrm{add}(\mathbf{N}) < \mathbf{p}$, what is consistent, then strong $\gamma \ne \gamma$. This fact was observed before by T. Weiss (private communication) in a more particular model of ZFC.

4. "Measurable" additivity and non-covering. In [B1] Bartoszyński introduced several equivalent conditions for additivity of measure. We will translate some of them to "measurable" versions. We will also show some implications between them. A first condition can be found in Definition 5. Below we consider some other properties of a set X:

(∗) Let $V \subseteq \mathbb{R}^2$ be such that $V \subseteq \bigcap_n U_n$ where U_n is open and $\mu((U_n)_x) < 2^{-n}$ for all $x \in X$. Then $\mu(\bigcup_{x\in X} V_x) = 0$.

It is easy to see that every set $X \subseteq \mathbb{R}$ with property (∗) also has the following property: For every G of measure zero, $X + G$ is also of measure zero. Sets with this property were investigated for example in [GM] and [FJ].

(∗∗) For every sequence of continuous functions $f_n : X \to \mathbb{R}$ such that the series $\sum f_n$ is converging there is a convergent series $\sum a_n$ eventually dominating $(f_n : n \in \omega)$ (that is, $\forall_x \exists_k \forall_{n>k} |f_n(x)| < a_n$).

Proposition 7. (∗∗)⇒(∗).

Proof. The proof uses similar arguments to a proof in [B1].

Let $U_n = \bigcup_k P_{nk} \times Q_{nk}$ be such that the $P_{nk} \times Q_{nk}$ are pairwise disjoint for each n, the P_{nk} are clopen and the Q_{nk} are intervals. Let us enumerate $\{P_{nk} \times Q_{nk} : n, k \in \omega\}$ as $\{P_l \times Q_l : l \in \omega\}$. Then $V \subseteq \bigcap_k \bigcup_{l>k} P_l \times Q_l$.

Let $f_l : X \to \mathbb{R}, f_l = \chi_{P_l} \cdot \mu(Q_l)$. Since $\sum_l f_l < \infty$ there is a convergent series $\sum a_l$ such that $\forall_x \exists_k \forall_{l>k} |f_l(x)| < a_l$. We define

$$R_l = \begin{cases} Q_l & \text{if } \mu(Q_l) < a_l, \\ \emptyset & \text{otherwise.} \end{cases}$$

Observe that $\bigcap_k \bigcup_{l>k} R_l$ is a null set. We know that $V_x \subseteq \bigcap_k \bigcup_{l>k, x\in P_l} Q_l$ for every x. Observe that for every x and almost every l if $x \in P_l$ then $Q_l = R_l$ so $V_x \subseteq \bigcap_k \bigcup_{l>k} R_l$.

We next define two properties of a set X:

(∗∗∗) $\forall_{f_n : X \to \mathbb{R} \text{ Borel}}$
$$\Big(\sum f_n < \infty\Big) \Rightarrow \exists_{a_n} \Big(\sum a_n < \infty \text{ and } \forall_n^\infty |f_n(x)| < a_n\Big).$$

(∗∗∗∗) $\forall_{g : X \to \omega^\omega \text{ Borel}} \exists_{I_n \subset \omega} |I_n| < n^2 \wedge \forall_{h \in g[X]} \forall_n^\infty h(n) \in I_n\,.$

It is obvious that $(***)\Rightarrow(**)$.

PROPOSITION 8. $(****)\Rightarrow(***)$.

Proof. The proof is a slight modification of a proof from [B1].

Let $g : X \to \omega^\omega$, $g(x)(k) = \min\{n : \sum_{l>n} |f_l(x)| < 2^{-k}\}$. Then g is a Borel function. So there is a sequence b_k such that $\forall_k^\infty \forall_{n\ge b_k} \sum_{l>n} |f_l(x)| < 2^{-k}$. We can assume that $f_n[X] \subseteq \mathbb{Q}$ for each n. Let $g_1 : X \to (\mathbb{Q}^{<\omega})^\omega$ where we take $\mathbb{Q}^{<\omega}$ with discrete topology such that $g_1(x)(k) = (f_{b_k}(x), f_{b_k+1}(x), \ldots, f_{b_{k+1}-1}(x))$. Since g_1 is a Borel function, there are $I_k \subseteq \mathbb{Q}^{<\omega}$ such that $|I_k| < k^2$ and $\forall_{h\in g_1} \forall_k^\infty h(k) \in I_k$. Define $a_n = \sup\{f_n(x) : (f_{b_k}(x), f_{b_k+1}(x), \ldots, f_{b_{k+1}-1}(x)) \in I_k$ and $\sum_{l=b_k}^{b_{k+1}-1} |f_l(x)| < 2^{-k}$ and $x \in X$ and $b_k \le n \le b_{k+1}-1\}$. It is easy to see that $\sum_{l=b_k}^{b_{k+1}-1} a_l \le k^2 2^{-k}$ so a_n is convergent and eventually dominates each f_n.

FACT. $\mathrm{non}((*)) = \mathrm{non}((**)) = \mathrm{non}((***)) = \mathrm{non}((****)) = \mathrm{add}(\mathbf{N})$.

Proof. It is enough to show that $\mathrm{add}(\mathbf{N}) \ge \mathrm{non}((*))$.

Let $\{G_x : x \in X\}$ be a family of G_δ-sets of measure zero such that $\mu^*(\bigcup_{x\in X} G_x) > 0$. Let $\{U_x^n : n \in \omega\}$ be a family of open sets such that $G_x = \bigcap_n U_x^n$ and $\mu(U_x^n) < 1/n$. Now we can find a separable, 0-dimensional metric topology such that $\bigcup_{x\in X} \{x\} \times U_x^n$ is an open set in $X \times \mathbb{R}$ for each n (see [BBM]). So we see that X does not satisfy $(*)$.

PROPOSITION 9. *Suppose that every Borel image of a set X into ω^ω is bounded. Let $B \subseteq X \times \mathbb{R}$ be a Borel set. Then there is a sequence of open sets $U_n \subseteq X \times \mathbb{R}$ such that $B \subseteq \bigcap_n U_n$ and $\forall_{x\in X} \forall_n^\infty \mu((U_n \setminus B)_x) < 2^{-n}$.*

Proof. First we show that the family $\mathcal{A}$ of all Borel sets $A \subseteq X \times \mathbb{R}$ for which there is a sequence of open subsets U_n of $X \times \mathbb{R}$ such that $A \subseteq \bigcap_n U_n$ and $\forall_{x\in X} \forall_n^\infty \mu((U_n \setminus A)_x) < 2^{-n}$ is a monotonic family. Let $A = \bigcap_k A^k$ and let U_n^k witness that $A^k \in \mathcal{A}$.

We define $f : X \to \omega^\omega$ by $f(x)(k) = \min\{n : \forall_{l\ge n} \mu((U_l^k \setminus A_k)_x) < 2^{-k}\}$. Since f is Borel, there is an $h \in \omega^\omega$ with $f[X] <^* h$. Let $g : X \to \omega^\omega$ be defined by $g(x)(i) = \min\{n : \forall_{k\ge n} \mu((A^k \setminus A)_x) < 2^{-i}\}$. Then g is Borel so there is an $\phi \in \omega^\omega$ with $g[X] <^* \phi$. Then $U_{h(\phi(k))}^{\phi(k)}$ has the required properties.

Now let $A = \bigcup_k A^k$ with $A^k \subseteq A^{k+1}$ and let U_n^k witness that $A^k \in \mathcal{A}$. We define $f : X \to \omega^\omega$ by $f(x)(k) = \min\{n : \forall_{l\ge n} \mu((U_l^k \setminus A_k)_x) < 2^{-k}\}$. Since f is Borel, there is an $h \in \omega^\omega$ with $f[X] <^* h$. Then the family $V_n = \bigcup_{l\ge n} U_{h(l)}^l$ has the required properties.

It is easy to show that the algebra generated by rectangles (open interval cross open interval) is contained in $\mathcal{A}$ so all Borel sets belong to $\mathcal{A}$.

COROLLARY 5. *If X satisfies $(****)$ then $\mu(\bigcup_{x\in X} B_x) = 0$ for every Borel set $B \subseteq X \times \mathbb{R}$ with $\mu(B_x) = 0$ for each $x \in X$.*

P r o o f. Obviously property $(****)$ is hereditary and $(****)$ for X implies that every Borel image of X into ω^ω is bounded. So there is a family of open sets U_n such that $\forall_{x\in X}\forall_n^\infty \mu(U_n) < 2^{-n}$ and $B \subseteq \bigcap_n U_n$. Let $X_k = \{x \in X : \forall_{n\ge k}\mu((U_n)_x) < 2^{-n}\}$. Then $\bigcup_k X_k = X$. By Propositions 7 and 8, $\mu(\bigcup_k \bigcup_{x\in X_k} B_x) = 0$.

EXAMPLES. Every set of size less than add($\mathbf{N}$) satisfies $(****)$.

Another example comes from a result of Todorčević. Since it has not been published we will sketch the proof.

THEOREM 3 (Todorčević). *Assuming CH there is a set of size* $\mathbf{c}$ *whose every Borel image is a strong* γ*-set.*

LEMMA 2. *Let* $\{B_n : n \in \omega\}$ *be a family of disjoint perfect sets*, A *a countable set and* $(J_n : n \in \omega)$ *a sequence of countable Borel families such that* J_n *is a cover of* $[\bigcup_l D_l \cup A]^{k_n}$. *Then there is a family* $\{B'_n : n \in \omega\}$ *of perfect sets with* $B'_n \subseteq B_n$ *and a sequence* $\{D_n : n \in \omega\}$ *with* $D_n \in J_n$ *such that* $\bigcup_n B'_n \cup A \subseteq \bigcup_m \bigcap_{n>m} D_n$.

P r o o f. The proof uses similar argument to the proof of a lemma in [GM].

P r o o f o f T h e o r e m 3. We construct an Aronszajn tree built from perfect sets of $\mathbb{R}$ ordered by reverse inclusion. First we order all pairs $(f_\alpha, (I_n^\alpha : n \in \omega))$ where $f_\alpha : \mathbb{R} \to \mathbb{R}$ is a Borel function and each I_n is a family of open subsets of $\mathbb{R}$. Let $J_n^\alpha = \{f_\alpha^{-1}(O) : O \in I_n^\alpha\}$. On each level α we extend the tree by constructing a countable family of perfect sets and a countable subset X_α of their union either using Lemma 2 for $(J_n^\alpha : n \in \omega)$ or choosing X_α such that J_n^α is not a cover of $[X_\alpha]^{k_n}$ for some n. Then $X = \bigcup_{\alpha<\omega_1} X_\alpha$ has the required properties. For details see [GM].

COROLLARY 6. *Assuming CH there is a set of reals of size* $\mathbf{c}$ *with property* $(****)$.

COROLLARY 7. *Assuming CH there is a set of reals of size* $\mathbf{c}$ *such that* $\bigcup_{x\in X} \mu(B_x) = 0$ *for every Borel set* $B \subseteq X \times \mathbb{R}$ *with* $\mu(B_x) = 0$ *for each* $x \in X$.

D. H. Fremlin and J. Jasiński [FJ] showed that if Martin's Axiom holds and there exists $\kappa < \mathbf{c}$ such that $P(\kappa)$ contains a proper uniform ω_1-saturated κ-additive ideal then there exists a set X of reals of cardinality $\mathbf{c}$ containing a subset D of cardinality less than $\mathbf{c}$, Borel-dense in X. It is easy to show that this set also satisfies $(****)$.

It is obvious that $(****)$ implies that X is add($\mathbf{N}$)-small.

DEFINITION 7. A set $X \subseteq \mathbb{R}$ is *strong first category* (= *strongly meagre*) (see [M1]) if for every set $G \subseteq \mathbb{R}$ of measure zero there is a $t \in \mathbb{R}$ such that $X \cap (G + t) = \emptyset$ (or equivalently $X + G \neq \mathbb{R}$).

We now consider another property of a set X:

(+) For every Borel set $H \subseteq \mathbb{R}^2$ such that $\mu(H_x) = 0$ for every $x \in X$, we have $\bigcup_{x\in X} H_x \neq \mathbb{R}$.

Observe that (+) $\Rightarrow$ strong first category.

There is a question of Galvin (see [M1]) whether every Sierpiński set is strong first category. We can also raise the question whether every Sierpiński set has (+).

It is known that under CH there is a Sierpiński set which is strong first category. We show a stronger example:

PROPOSITION 10. *Under the Continuum Hypothesis there is a Sierpiński set with* (+).

Proof. Let $H_\alpha, \alpha < \mathbf{c}$, be all Borel sets on the plane such that $\mu((H_\alpha)_x) = 0$ for every $x \in \mathbb{R}$, and let $G_\alpha, \alpha < \mathbf{c}$, be all Borel subsets of the reals of measure zero. We define inductively sequences $\{x_\alpha, y_\alpha : \alpha < \mathbf{c}\}$.

Let $y_\alpha \notin \bigcup_{\beta<\alpha}(H_\alpha)_{x_\beta}$ be such that $\mu(\mathbb{R} \setminus (H_\alpha)^{y_\alpha}) = 0$. Let $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha}(G_\beta \cup (H_\beta)^{y_\beta})$. Observe that $y_\alpha \notin \bigcup_{\beta<\mathbf{c}}(H_\alpha)_{x_\beta}$ for every α.

Remark. It is easy to see that no set with (+) can be mapped onto the reals by a Borel function. In [R] under MA there is constructed a strong first category set which can be mapped onto the reals by a continuous function. So under MA strong first category does not imply (+).

Let $\mathbf{N}_2$ be the ideal of null sets on the plane and $\mathbf{M}_2$ the ideal of meagre sets on the plane.

PROBLEM. (CH) *Do there exist functions* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies* $(f, g)[\mathbf{N}_2] = \mathbf{M}_2$?

A simpler question is also interesting: (CH) Let $G \subseteq \mathbb{R}^2$ be a set of measure zero. Do there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $(f, g)[G]$ is first category and $f[\mathbf{N}] = \mathbf{M}$ and $g[\mathbf{N}] = \mathbf{M}$?

Observe that the positive answer to one of the above questions and Corollary 3 show that every Sierpiński set has (+) so give the solution to the problem of Galvin about Sierpiński sets.

Remark. In [PR] the authors show that for every $X \subseteq \mathbb{R}$ if $\mu(\bigcup_{x\in X} G_x) = 0$ for every Borel set $G \subseteq \mathbb{R}^2$ with $\mu(G_x) = 0$ for each $x \in X$, then $\bigcup_{x\in X} F_x$ is meagre for every Borel set $F \subseteq \mathbb{R}^2$ such that each F_x is meagre.

Remark. Recently J. Pawlikowski has solved the problem of Galvin by showing that every Sierpiński set is strongly meagre. He also showed that every Sierpiński set has (+).

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