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Examples for Souslin forcing

by

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Abstract. We give several examples of Souslin forcing notions. For instance, we show that there exists a proper analytical forcing notion without ccc and with no perfect set of incompatible elements, we give an example of a Souslin ccc partial order without the Knaster property, and an example of a totally nonhomogeneous Souslin forcing notion.

1. Introduction. In this paper we continue with our study of forcing notions having a simple definition. We began this study in [JS1] and [JS2]. In [BJ] we gave more results about Souslin forcing notions and in this paper we will give some examples of Souslin forcing notions answering a question of [JS1] and a question of H. Woodin.

A forcing notion $\mathbf{P}$ is *Souslin* if $\mathbf{P} \subseteq \mathcal{R}$ is a Σ^1_1-set, $\{(p,q) : p \leq_{\mathbf{P}} q\}$ is a Σ^1_1-set and $\{(p,q) : p$ is incompatible with $q\}$ is a Σ^1_1-set.

More information on Souslin forcing notions can be found in [JS1]. A related work is [BJ]. In [JS1] we prove that if $\mathbf{P}$ is Souslin ccc and $\mathbf{Q}$ is any forcing notion then $\mathbf{V}^{\mathbf{Q}} \models$ "$\mathbf{P}$ satisfies ccc". A natural question was: does "$\mathbf{P}$ is Souslin ccc" imply "$\mathbf{P}$ has the Knaster property"? Recall that $\mathbf{P}$ *has the Knaster property* if and only if

$$(\forall A \in [\mathbf{P}]^{\omega_1})(\exists B \in [A]^{\omega_1})(\forall p, q \in B)(p \text{ is compatible with } q)\,.$$

In the second section we will give a model where there is a ccc Souslin forcing which does not satisfy the Knaster condition. Recall that under the assumption of **MA** every ccc notion of forcing has the Knaster property.

Many simple forcing notions $\mathbf{P}$ satisfy the following condition:

$$\Vdash_{\mathbf{P}} \text{“}\widehat{\mathbf{P}} \text{ is } \sigma\text{-centered”}.$$

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This property is connected with the homogeneity of the forcing notion. The example of a totally nonhomogeneous Souslin forcing will be constructed in the third section.

In the next section we present a model where there is a σ-linked not σ-centered Souslin forcing such that all its small subsets are σ-centered but the Martin Axiom fails for this order.

In Section 5 we will give an example of a σ-centered Souslin forcing notion and a model of the negation of **CH** in which the union of less than continuum meager subsets of $\mathcal{R}$ is meager but the Martin Axiom fails for this notion of forcing.

In the last session of the MSRI Workshop on the continuum (October 1989) H. Woodin asked if "**P** has a simple definition and does not satisfy ccc" implies that there exists a perfect set of mutually incompatible conditions. Clearly the Mathias forcing satisfies such a requirement. In Section 6 we will find a Souslin forcing which is proper but not ccc and does not contain a perfect set of mutually incompatible conditions.

The last section will show that ccc Σ^1_2-notions of forcing may not be indestructible ccc.

Our notation is standard and derived from [Je]. There is one exception, however. We write $p \le q$ to say that q is a stronger condition than p.

2. On the Knaster condition. In this section we will build a Souslin forcing satisfying the countable chain condition but which fails the Knaster condition.

Fix a sequence $\langle \sigma_i : i \in \omega \rangle$ of functions from ω into ω such that

$(*)$ if $N < \omega$, $\phi_i : N \to \omega$ (for $i < N$) then there are distinct $n_0, \ldots, n_{N-1}$ such that

$$(\forall i, j_0, j_1 < N)(\phi_i(j_0) = j_1 \Rightarrow \sigma_i(n_{j_0}) = n_{j_1})\,.$$

Note that there exists a sequence $\langle \sigma_i : i \in \omega \rangle$ satisfying $(*)$.

Indeed, suppose we have defined $\sigma_i | m_0 : m_0 \to m_0$ for $i < m_0$. We want to ensure $(*)$ for $n_0 + 1$ and ϕ_i $(i \le n_0)$. Define $\sigma_i(m_0 + j_0) = m_0 + \phi_i(j_0)$ for $i, j_0 \le n_0$. Take large m_1 and extend all σ_i $(i \le n_0)$ on m_1 in such a way that $\mathrm{rng}(\sigma_i) \subseteq m_1$.

Next we define functions $f_i : \omega^\omega \to \omega^\omega$ for $i \in \omega$ by

$$f_i(x)(k) = \begin{cases} x(k) & \text{if } k < i, \\ \sigma_i(x(k)) & \text{otherwise.} \end{cases}$$

Clearly all functions f_i are continuous. Put $F(x) = \{f_i(x) : i \in \omega\}$ for $x \in \omega^\omega$.

LEMMA 2.1. *Suppose that $x_\alpha, y_\alpha \in \omega^\omega$ are such that there is no repetition in $\{x_\alpha, y_\alpha : \alpha \in \omega_1\}$. Then there exists $A \in [\omega_1]^{\omega_1}$ such that*

$$(\forall \alpha, \beta \in A)(\alpha < \beta \Rightarrow x_\alpha \notin F(y_\beta)).$$

P r o o f. For $\alpha < \omega_1$ let $n_\alpha = \min\{n : x_\alpha(n) \neq y_\alpha(n)\}$. We find $n \in \omega$ and $s, t \in \omega^n$ such that the set $A_0 = \{\alpha < \omega_1 : n = n_\alpha + 1 \ \& \ x_\alpha | n = s \ \& \ y_\alpha | n = t\}$ is stationary in ω_1. Clearly $s \neq t$ and $s|(n-1) = t|(n-1)$. Thus $\alpha, \beta \in A_0$ and $x_\alpha \in F(y_\beta)$ imply $x_\alpha \in \{f_i(y_\beta) : i \leq n\}$. Consequently, the set $\{\alpha \in A_0 \cap \beta : x_\alpha \in F(y_\beta)\}$ is finite for each $\beta \in A_0$.

We define the regressive function $\psi : A_0 \to \omega_1$ by $\psi(\beta) = \max\{\alpha \in A_0 \cap \beta : x_\alpha \in F(y_\beta)\}$ (with the convention that $\max \emptyset = 0$). By Fodor's lemma there are $\gamma < \omega_1$ and a stationary set $A_1 \subseteq A_0$ such that $\psi(\beta) = \gamma$ for all $\beta \in A_1$. Put $A = A_1 \setminus (\gamma + 1)$. Now, if $\alpha, \beta \in A$ and $\alpha < \beta$ then $\psi(\beta) < \alpha$ and hence $x_\alpha \notin F(y_\beta)$. ■

LEMMA 2.2. *Suppose that $\{W_\alpha : \alpha < \omega_1\}$ is a family of disjoint finite subsets of ω^ω. Then there exist $\beta < \omega_1$ and an infinite set $A \subseteq \beta$ such that*

$$(\forall \alpha \in A)(\forall x \in W_\alpha)(\forall y \in W_\beta)(x \notin F(y)).$$

P r o o f. We may assume that all sets W_α are of the same cardinality, say $|W| = n$ for $\alpha < \omega_1$. For $\alpha = \lambda + k$, where $\lambda < \omega_1$ is a limit ordinal and $k \in \omega$, we define $X_\alpha = W_{\lambda+2k}$ and $Y_\alpha = W_{\lambda+2k+1}$. Let $X_\alpha = \{x_i^\alpha : i < n\}$, $Y_\alpha = \{y_i^\alpha : i < n\}$. By induction on $l = l_1 n + l_2 < n^2$, $l_1, l_2 < n$, choose uncountable sets $A_l \subseteq \omega_1$ satisfying

- $A_{l+1} \subseteq A_l$ and
- if $l = l_1 n + l_2$, $l_1, l_2 < n$, $\alpha, \beta \in A_l$ and $\alpha < \beta$ then $x_{l_1}^\alpha \notin F(y_{l_2}^\beta)$.

Since there is no repetition in $\{x_{l_1}^\alpha, y_{l_2}^\alpha : \alpha \in A_{l-1}\}$ we may apply Lemma 2.1 to get A_l from A_{l-1}.

Consider A_{n^2-1}. Choose $\beta_0 \in A_{n^2-1}$ such that the set

$$A = \{\lambda + 2k < \beta_0 : \lambda + k \in A_{n^2-1} \ \& \ k \in \omega \ \& \ \lambda \text{ is a limit ordinal}\}$$

is infinite. Let $\beta_0 = \lambda_0 + k_0$ where $k_0 \in \omega$ and λ_0 is limit. Put $\beta = \lambda_0 + 2k_0 + 1$. Since $\beta_0 < \beta$ we have $A \subseteq \beta$. Suppose $\alpha = \lambda + 2k \in A$. Let $x \in W_\alpha$ and $y \in W_\beta$. Then $\lambda + k \in A_{n_1^2}$, $W_\alpha = X_{\lambda+k}$ and $W_\beta = Y_{\lambda_0+k_0} = Y_{\beta_0}$. Thus for some $l_1, l_2 < n$ we have $x = x_{l_1}^{\lambda+k}$ and $y = y_{l_2}^{\beta_0}$. Since $\lambda + k, \beta_0 \in A_{l_1 n + l_2}$ and $\lambda + k < \beta_0$ we get $x \notin F(y)$. The lemma is proved. ■

Let relations R_i on $\omega^{<\omega}$ be defined by

sR_it if and only if

$$i < |s| = |t|, \ s|i = t|i \text{ and } (\forall l \in [i, |s|))(s(l) = \sigma_i(t(l))).$$

Note that if $x, y \in \omega^\omega$ are such that $(\forall n > i)(x|nR_iy|n)$ then $x = f_i(y)$.

We define the following forcing notion $\mathbf{Q}$. A member q of $\mathbf{Q}$ is a finite function such that:

- $\operatorname{dom}(q) \in [\omega_1]^{<\omega}$, $\operatorname{rng}(q) \subseteq \omega^{<\omega}$,
- $(\forall \alpha, \beta \in \operatorname{dom}(q))(\alpha \neq \beta \Rightarrow q(\alpha) \neq q(\beta))$,
- there is $n(q) \in \omega$ such that $q(\alpha) \in \omega^{n(q)}$ for all $\alpha \in \operatorname{dom}(q)$.

The order is defined as follows: $q \leq p$ if and only if

- $\operatorname{dom}(q) \subseteq \operatorname{dom}(p)$,
- $(\forall \alpha \in \operatorname{dom}(q))(q(\alpha) \subseteq p(\alpha))$,
- if $\alpha, \beta \in \operatorname{dom}(q)$, $\alpha < \beta$, $i < n(q)$ and $q(\alpha) R_i q(\beta)$ then $p(\alpha) R_i p(\beta)$.

Lemma 2.3. $\mathbf{Q}$ *satisfies ccc.*

Proof. Suppose $\{q_\alpha : \alpha < \omega_1\} \subseteq \mathbf{Q}$. We find $\gamma < \omega_1$ and $A \in [\omega_1]^{\omega_1}$ such that for each $\alpha, \beta \in A$, $\alpha < \beta$, we have

- $n(q_\alpha) = n(q_\beta)$,
- $\operatorname{dom}(q_\alpha) \cap \gamma = \operatorname{dom}(q_\beta) \cap \gamma$, $(\operatorname{dom}(q_\alpha) \setminus \gamma) \cap (\operatorname{dom}(q_\beta) \setminus \gamma) = \emptyset$,
- $q_\alpha | (\operatorname{dom}(q_\alpha) \cap \gamma) = q_\beta | (\operatorname{dom}(q_\beta) \cap \gamma)$.

Suppose $\alpha, \beta \in A$. Clearly $\overline{q} = q_\alpha \cup q_\beta$ is a function. The only problem is that there may exist $\gamma_0 \in \operatorname{dom}(q_\alpha)$ and $\gamma_1 \in \operatorname{dom}(q_\beta)$ such that $q_\alpha(\gamma_0) = q_\beta(\gamma_1)$. Therefore to get a condition above both q_α and q_β we have to extend all $\overline{q}(\gamma)$. Let $\operatorname{dom}(\overline{q}) = \{\gamma_j : j < N\}$ be an increasing enumeration. For $i < n(q_\alpha)$ choose $\phi_i : N \to \omega$ such that

$$\begin{aligned} \text{if} \quad & j_1 < j_0 < N \text{ and } \overline{q}(\gamma_{j_1}) R_i \overline{q}(\gamma_{j_0}) \\ & \text{and either } \gamma_{j_0}, \gamma_{j_1} \in \operatorname{dom}(q_\alpha) \text{ or } \gamma_{j_0}, \gamma_{j_1} \in \operatorname{dom}(q_\beta) \\ \text{then } & \phi_i(j_0) = j_1 \,. \end{aligned}$$

Note that $\overline{q}(\gamma_{j_1}) R_i \overline{q}(\gamma_{j_0})$ and $\overline{q}(\gamma_{j_2}) R_i \overline{q}(\gamma_{j_0})$ imply $\overline{q}(\gamma_{j_1}) = \overline{q}(\gamma_{j_2})$. Hence if $j_2 < j_1 < j_0$ are as above then $\gamma_{j_2} \geq \gamma$ and consequently only one pair (j_1, j_0) or (j_2, j_0) will be considered in the definition of ϕ_i. Apply condition $(*)$ to find distinct $n_0, \ldots, n_{N-1}$ such that

$$(\forall i < n(q_\alpha))(\forall j_0, j_1 < N)(\phi_i(j_0) = j_1 \Rightarrow \sigma_i(n_{j_0}) = n_{j_1}) \,.$$

Put $q(\gamma_j) = \overline{q}(\gamma_j)^\wedge n_j$ for $j < N$. Clearly $q \in \mathbf{Q}$. Suppose $\gamma_{j_1}, \gamma_{j_0} \in \operatorname{dom}(q_\alpha)$, $j_1 < j_0$ and $q_\alpha(\gamma_{j_1}) R_i q_\alpha(\gamma_{j_0})$ for some $i < n(q_\alpha)$. Then $\phi_i(j_0) = j_1$ and hence $\sigma_i(n_{j_0}) = n_{j_1}$. Thus $q(\gamma_{j_1}) R_i q(\gamma_{j_0})$. This shows that $q_\alpha \leq q$. Similarly $q_\beta \leq q$. Thus we have proved that $\mathbf{Q}$ satisfies the Knaster condition. ∎

Let $G \subseteq \mathbf{Q}$ be generic over $\mathbf{V}$. In $\mathbf{V}[G]$ we define $x^G_\alpha = \bigcup\{q(\alpha) : q \in G \ \&\ \alpha \in \operatorname{dom}(q)\}$ for $\alpha < \omega_1$. Obviously each x^G_α is a sequence of integers. As in the proof of Lemma 2.3 we can show that for each $q \in \mathbf{Q}$ there is $p \geq q$ such that $n(p) = n(q) + 1$. Consequently, $x^G_\alpha \in \omega^\omega$ for every $\alpha < \omega_1$.

Moreover, $x^G_\alpha \neq x^G_\beta$ for $\alpha < \beta < \omega_1$ (recall that $q(\alpha) \neq q(\beta)$ for distinct $\alpha, \beta \in \mathrm{dom}(q)$).

Note that if $\alpha < \beta$, $\alpha, \beta \in \mathrm{dom}(q)$, $q \in \mathbf{Q}$ and $i < n(q)$ then

$$\begin{aligned} q(\alpha) R_i q(\beta) \quad &\text{implies} \quad q \Vdash \dot{x}_\alpha = f_i(\dot{x}_\beta) \quad \text{and} \\ \neg q(\alpha) R_i q(\beta) \quad &\text{implies} \quad q \Vdash \dot{x}_\alpha \neq f_i(\dot{x}_\beta)\,. \end{aligned}$$

LEMMA 2.4. *Suppose $G \subseteq \mathbf{Q}$ is generic over $\mathbf{V}$. Then*

$$\mathbf{V}[G] \models (\forall A \in [\omega_1]^{\omega_1})(\exists \alpha, \beta \in A)(\alpha < \beta \ \&\ x^G_\alpha \in F(x^G_\beta))\,.$$

Proof. Let $\dot{A}$ be a $\mathbf{Q}$-name for an uncountable subset of ω_1. Given $p \in \mathbf{Q}$, find $A_0 \in [\omega_1]^{\omega_1}$ and $q_\alpha \geq p$ for $\alpha \in A_0$ such that $\alpha \in \mathrm{dom}(q_\alpha)$ and $q_\alpha \Vdash \alpha \in \dot{A}$. We may assume that for each $\alpha, \beta \in A_0$ we have $n = n(q_\alpha) = n(q_\beta)$ and $q_\alpha(\alpha) = q_\beta(\beta)$. Now we repeat the procedure of Lemma 2.3 with one small change. We choose suitable $A_1 \in [A_0]^{\omega_1}$ and $\gamma < \omega_1$, and we take $\alpha, \beta \in A_1$, $\alpha < \beta$. Defining integers $n_0, \ldots, n_{N-1}$ we consider functions $\phi_i : N \to \omega$ (for $i < n$) as in 2.3 and a function $\phi_n : N \to \omega$ such that $\phi_n(k) = l$, where $\alpha = \gamma_l$, $\beta = \gamma_k$. Then we get a condition $q \in \mathbf{Q}$ above both q_α and q_β and such that $\sigma_n(q(\beta)(n)) = q(\alpha)(n)$. Since $q(\beta)|n = q(\alpha)|n$ and $n(q) = n+1$ we have $q(\alpha) R_n q(\beta)$ and consequently $q \Vdash \dot{x}_\alpha = f_n(\dot{x}_\beta)$. Since $q \geq q_\alpha, q_\beta$ we get $q \Vdash$ "$\alpha, \beta \in \dot{A}\ \&\ \dot{x}_\alpha \in F(\dot{x}_\beta)$". ∎

Fix a Borel isomorphism $(\pi_0, \pi_1, \pi_2) : \omega^\omega \to (\omega^\omega)^\omega \times 2^{\omega \times \omega} \times \omega^\omega$. Thus if $x \in \omega^\omega$ then $\pi_1(x)$ is a relation on ω and $\pi_0(x)$ is a sequence of reals. Let Γ consist of all reals $x \in \omega^\omega$ such that

1. $(\forall n \neq m)(\pi_0(x)(n) \neq \pi_0(x)(m))$,
2. $\pi_1(x)$ is a linear order on ω,
3. $\pi_2(x) \in A_x = \{\pi_0(n) : n \in \omega\}$ and it is the $\pi_1(x)$-last element of A_x.

Note that in 3 we think of $\pi_1(x)$ as an order on A_x. We define relations $<_\Gamma$ and $\equiv_\Gamma$ on Γ by

$$\begin{aligned} &x <_\Gamma y \quad \text{if and only if} \\ &\qquad A_x \text{ is a proper } \pi_1(y)\text{-initial segment of } A_y \text{ and } \pi_1(y)|A_x = \pi_1(x), \\ &x \equiv_\Gamma y \quad \text{if and only if} \quad A_x = A_y \text{ and } \pi_1(y) = \pi_1(x) \end{aligned}$$

(we treat $\pi_1(x), \pi_1(y)$ as orders on A_x, A_y, respectively).

Clearly Γ is a Borel subset of ω^ω, $<_\Gamma$ is a Borel transitive relation on Γ and $\equiv_\Gamma$ is a Borel equivalence relation on Γ.

Now we define a forcing notion $\mathbf{P}_1$. Conditions in $\mathbf{P}_1$ are finite subsets p of Γ such that

$$\text{if} \quad x, y \in p \text{ and } x <_\Gamma y \quad \text{then} \quad \pi_2(x) \notin F(\pi_2(y))\,.$$

$\mathbf{P}_1$ is ordered by inclusion.

LEMMA 2.5. $\mathbf{P}_1$ *is a ccc Souslin forcing.*

Proof. $\mathbf{P}_1$ is Souslin since it can be easily coded as a Borel subset of ω^ω in such a way that the order is also Borel. We have to show that $\mathbf{P}_1$ satisfies the countable chain condition. First let us note some properties of the incompatibility in $\mathbf{P}_1$. Suppose $p, q \in \mathbf{P}_1$ are incompatible. Clearly $p \setminus q$ and $q \setminus p$ are incompatible. If $x \in p$ and $x \equiv_\Gamma x'$ then $(p \setminus \{x\}) \cup \{x'\}$ and q are incompatible.

Suppose now that $\{p_\alpha : \alpha < \omega_1\}$ is an antichain in $\mathbf{P}_1$. By the Δ-lemma and by the above remarks we may assume that

$$p_\alpha \cap p_\beta = \emptyset \quad \text{for } \alpha < \beta < \omega_1 , \tag{1}$$

$$\text{if} \quad x, x' \in \bigcup_{\alpha<\omega_1} p_\alpha \text{ and } x \neq x' \quad \text{then} \quad x \not\equiv_\Gamma x' . \tag{2}$$

Note that if $p \in \mathbf{P}_1$ then the set $\{[y]_{\equiv_\Gamma} : y \in \Gamma \ \& \ (\exists x \in p)(y <_\Gamma x)\}$ is countable. Hence, due to (2), we may assume that

$$(\forall \alpha < \beta < \omega_1)(\forall x \in p_\alpha)(\forall y \in p_\beta)(\neg y <_\Gamma x) . \tag{3}$$

Claim. *Let $d \in [\omega^\omega]^{<\omega}$. Then $d = \{\pi_2(x) : x \in p_\alpha\}$ for at most countably many $\alpha < \omega_1$.*

Indeed, assume not. Then we find $\beta < \omega_1$ such that $\{\pi_2(x) : x \in p_\beta\} = d$ and the set $B = \{\alpha < \beta : \{\pi_2(x) : x \in p_\alpha\} = d\}$ is infinite. Note that if $x', x'' <_\Gamma x$ and $\pi_2(x') = \pi_2(x'')$ then $x' \equiv_\Gamma x''$. Hence if $x \in p_\beta$ then for at most $|d|$ elements x' of $\bigcup_{\alpha \in A} p_\alpha$ we have $x' <_\Gamma x$. Thus we find $\alpha \in A$ such that $(\forall x' \in p_\alpha)(\forall x \in p_\beta)(\neg x' <_\Gamma x)$. It follows from (3) that

$$(\forall x \in p_\beta)(\forall x' \in p_\alpha)(\neg x <_\Gamma x')$$

and hence the conditions p_α and p_β are compatible—a contradiction.

Let $d_\alpha = \{\pi_2(x) : x \in p_\alpha\}$. By the above claim we may assume that $d_\alpha \neq d_\beta$ for all $\alpha < \beta < \omega_1$. Applying the Δ-lemma we may assume that

$$\{d_\alpha : \alpha < \omega_1\} \text{ forms a } \Delta\text{-system with the root } d . \tag{4}$$

Since the set $\bigcup_{w \in d} F(w)$ is countable, without loss of generality

$$(\forall \alpha < \omega_1)(\forall v \in d_\alpha \setminus d)\Big(v \notin \bigcup_{w \in d} F(w)\Big) . \tag{5}$$

Apply Lemma 2.2 to the family $\{d_\alpha \setminus d : \alpha < \omega_1\}$ to get $\beta < \omega_1$ and an infinite set $A \subseteq \beta$ such that

$$(\forall \alpha \in A)(\forall v \in d_\alpha \setminus d)(\forall w \in d_\beta \setminus d)(v \notin F(w)) . \tag{6}$$

Let $y \in p_\beta$. As in the claim the set

$$\Big\{x \in \bigcup_{\alpha \in A} p_\alpha : \pi_2(x) \in d \ \& \ x <_\Gamma y\Big\}$$

is finite. Consequently, we find $\alpha \in A$ such that

$$(\forall x \in p_\alpha)(\forall y \in p_\beta)(\pi_2(x) \in d \Rightarrow \neg x <_\Gamma y)\,. \tag{7}$$

We claim that p_α and p_β are compatible. Let $x \in p_\alpha$ and $y \in p_\beta$. By (7), if $\pi_2(x) \in d$ then $\neg x <_\Gamma y$. If $\pi_2(x) \notin d$ and $\pi_2(y) \notin d$ then (6) applies and we get $\pi_2(x) \notin F(\pi_2(y))$. Finally, if $\pi_2(x) \notin d$ and $\pi_2(y) \in d$ then we use (5) to conclude that $\pi_2(x) \notin F(\pi_2(y))$. Hence $x \in p_\alpha$, $y \in p_\beta$ and $x <_\Gamma y$ imply $\pi_2(x) \notin F(\pi_2(y))$. Consequently, $p_\alpha \cup p_\beta \in \mathbf{P}_1$. ■

LEMMA 2.6. *Assume that there exists a sequence $\{x_\alpha : \alpha < \omega_1\}$ of elements of ω^ω such that*

$$(\forall A \in [\omega_1]^{\omega_1})(\exists \alpha, \beta \in A)(\alpha < \beta \ \&\ x_\alpha \in F(x_\beta))\,.$$

Then the forcing notion $\mathbf{P}_1$ does not satisfy the Knaster condition.

Proof. For $\alpha < \omega_1$ choose $y_\alpha \in \Gamma$ such that

- $A_{y_\alpha} = \{\pi_0(y_\alpha)(n) : n \in \omega\} = \{x_\gamma : \gamma \le \alpha\}$,
- $\pi_1(y_\alpha)$ is the natural order on A_{y_α}, $x_\gamma <_{\pi_1(y_\alpha)} x_\beta$ iff $\gamma < \beta$,
- $\pi_2(y_\alpha) = x_\alpha$.

Let $p_\alpha = \{y_\alpha\}$ for $\alpha < \omega_1$. Then $\{p_\alpha : \alpha < \omega_1\}$ does not have an uncountable subset of pairwise compatible elements. ■

Putting together Lemmas 2.4–2.6 we get

THEOREM 2.7. *It is consistent that there exists a ccc Souslin forcing notion which does not satisfy the Knaster condition.* ■

It is not difficult to see that this example does not satisfy the following requirement:

"The generic object is encoded by a real".

The next theorem says that we can also require such a condition. This answers a question of J. Bagaria.

THEOREM 2.8. *It is consistent that there exists a ccc Souslin forcing notion $\mathbf{Q}$ such that $\Vdash_{\mathbf{Q}} \mathbf{V}[G] = \mathbf{V}[\dot r]$ for some $\mathbf{Q}$-name $\dot r$ for a real and $\mathbf{Q}$ does not satisfy the Knaster condition.*

Proof. We follow the notation of the previous results. We work in the model of 2.7. Let $\mathbf{Q} = \{(p, w) : p \in \mathbf{P} \ \&\ w \in [\omega^{<\omega}]^{<\omega}\}$ be ordered by

$(p, w) \le (q, v)$ if and only if

$$p \le q,\ w \subseteq v \text{ and } x|n \notin v \backslash w \text{ for every } x \in p \text{ and } n \in \omega\,.$$

$\mathbf{Q}$ may be easily represented as a Souslin forcing notion (remember that "for each x in p" is a quantification on natural numbers). Note that if $p \le q$, $p, q \in \mathbf{P}$ and $w \in [\omega^{<\omega}]^{<\omega}$ then $(p, w) \le (q, w)$. Hence $\mathbf{Q}$ satisfies the countable chain condition. If $\mathbf{Q}$ satisfied the Knaster condition then $\mathbf{P}$ would have satisfied it.

We show that the $\mathbf{Q}$-generic object is encoded by a real. Let $\dot{r}$ be a $\mathbf{Q}$-name for a subset of $\omega^{<\omega}$ (a real) such that for any $\mathbf{Q}$-generic G we have $\dot{r}^G = \bigcup\{w : (\exists p)((p,w) \in G)\}$. Now in $\mathbf{V}[\dot{r}^G]$ define

$$H = \{(p,w) \in \mathbf{Q} : w \subseteq \dot{r}^G \ \& \ (\forall x \in p)(\forall n \in \omega)(x|n \in \dot{r}^G \Leftrightarrow x|n \in w)\}\,.$$

Note that H includes G since $x \in p$ and $x|n \notin w$ imply $(p,w) \Vdash x|n \notin \dot{r}$.

We show that H is a filter. Suppose $(p_0,w_0),(p_1,w_1) \in H$. For each $x \in p_0 \cup p_1$ we find $(p_x,w_x) \in G$ such that $(p_x,w_x) \Vdash (\forall n \geq N)(x|n \notin \dot{r})$ for some N. If $x \notin p_x$ we could take large n and add $x|n$ to w_x. Then we would have $(p_x,w_x) \leq (p_x, w_x \cup \{x|n\})$ and $(p_x, w_x \cup \{x|n\}) \Vdash x|n \in \dot{r}$. Thus $x \in p_x$ for all $x \in p_0 \cup p_1$. Let $p = \bigcup_{x \in p_0 \cup p_1} p_x$, $w = \bigcup_{x \in p_0 \cup p_1} w_x$. Then $(p,w) \in G \subseteq H$ and $(p_0,w_0),(p_1,w_1) \leq (p,w)$. Consequently, $H = G$ and the theorem is proved. ■

At the same time when the forcing notion $\mathbf{P}_1$ was constructed S. Todorčević found another example of this kind.

Let $\mathcal{F}$ be the family of all converging sequences s of real numbers such that $\lim s \notin s$. Todorčević's forcing notion $\mathbf{P}_1^*$ consists of finite subsets p of $\mathcal{F}$ with the property that

$$(\forall s,t \in p)(s \neq t \Rightarrow \lim s \notin t)\,.$$

Todorčević proved that $\mathbf{P}_1^*$ satisfies ccc and that if $\mathbf{b} = \omega_1$ then $\mathbf{P}_1^*$ does not have the Knaster property (see [To]).

3. A nonhomogeneous example. In this section we give an example of a ccc Souslin forcing notion which is very nonhomogeneous. Our forcing $\mathbf{P}_2$ will satisfy the following condition:

$$\text{there is no } p \in \mathbf{P}_2 \text{ such that } p \Vdash \text{“}\widehat{\mathbf{P}}_2|p \text{ is } \sigma\text{-centered”}.$$

Recall that if $\mathbf{Q}$ is the Amoeba Algebra for Measure or the Measure Algebra then $\Vdash_{\mathbf{Q}}$ “$\widehat{\mathbf{Q}}$ is σ-centered” (see [BJ]). The Todorčević example $\mathbf{P}_1^*$ also has this property.

PROPOSITION 3.1. $\Vdash_{\mathbf{P}_1^*}$ *“$\widehat{\mathbf{P}}_1^*$ is σ-centered”.*

Proof. For a rational number $d \in Q$ let $\phi_d : \mathbf{P}_1^* \to \mathbf{P}_1^*$ be the translation by d. Thus $\phi_d(p) = \{s + d : s \in p\}$. Note that ϕ_d is an automorphism of $\mathbf{P}_1^*$. Moreover, if $p_1,p_2 \in \mathbf{P}_1^*$ then $\bigcup\{s - r : s \in p_1,\ r \in p_2\}$ is a nowhere dense set. Hence we find a rational d such that

$$-d \notin \{a - b : a \in s \cup \{\lim s\},\ b \in r \cup \{\lim r\},\ s \in p_1,\ r \in p_2\}\,.$$

Then the conditions $\phi_d(p_1)$ and p_2 are compatible. Thus we have proved that for each $p \in \mathbf{P}_1^*$ the set $\{\phi_d(p) : d \in Q\}$ is predense in $\mathbf{P}_1^*$. This implies that

$$\Vdash_{\mathbf{P}_1^*} \text{“}\widehat{\mathbf{P}}_1^* = \bigcup_{d\in Q} \phi_d[\dot{\Gamma}] \text{ and each } \phi_d[\dot{\Gamma}] \text{ is centered”},$$

where $\dot{\Gamma}$ is the canonical name for a generic filter. ∎

We do not know if

$$\Vdash_{\mathbf{P}_1} \text{“}\widehat{\mathbf{P}}_1 \text{ is } \sigma\text{-centered”}.$$

One can easily construct a ccc Souslin forcing $\mathbf{P}$ which does not force that $\widehat{\mathbf{P}}$ is σ-centered. An example of such a forcing notion is the disjoint union of the Cohen forcing and the measure algebra, $\mathbf{P} = (\{0\}\times\mathbf{C})\cup(\{1\}\times\mathbf{B})$. In this order we have $(0,\emptyset) \Vdash$ “$\{1\}\times\widehat{\mathbf{B}}$ is not σ-centered”. But in this example we can find a dense set of conditions $p\in\mathbf{P}$ such that

$$p \Vdash_{\mathbf{P}} \text{“}\widehat{\mathbf{P}}|p = \{q\in\widehat{\mathbf{P}} : q\geq p\} \text{ is } \sigma\text{-centered”}\,.$$

Define $T^*\subseteq\omega^{<\omega}$ and $f,g : T^*\to\omega$ in such a way that:

(α) T^* is a tree,

(β) if $\eta\in T^*$ then $\mathrm{succ}_{T^*}(\eta) = f(\eta)$,

(γ) if $\mathrm{lh}\,\eta < \mathrm{lh}\,\nu$ or $\mathrm{lh}\,\nu = \mathrm{lh}\,\eta$ but $(\exists k<\mathrm{lh}\,\eta)(\eta|k = \nu|k \ \&\ \eta(k)<\nu(k))$ then $f(\eta)<f(\nu)$,

(δ) $g(\eta) > |T^*\cap\omega^{\mathrm{lh}\,\eta}|\cdot\prod\{f(\nu) : f(\nu)<f(\eta)\}\cdot(100+\mathrm{lh}\,\eta)$,

(ε) $f(\eta) > g(\eta)\cdot\prod\{2^{f(\nu)} : f(\nu)<f(\eta)\}$.

For $\eta\in T^*$ and a set $A\subseteq\mathrm{succ}_{T^*}(\eta) = f(\eta)$ we define a norm of A:

$$\mathbf{nor}_\eta(A) = \frac{g(\eta)}{|f(\eta)\setminus A|}\,.$$

Lemma 3.2. *Suppose that $\eta\in T^*$ and $A_l\subseteq f(\eta)$ for $l<m$. Let $\zeta = \min\{\mathbf{nor}_\eta(A_l) : l<m\}$. Then*

(i) $\mathbf{nor}_\eta(\bigcap_{l<m} A_l)\geq\zeta/m$,

(ii) *if $\zeta\geq 1$ and $m\leq\prod\{2^{f(\nu)} : f(\nu)<f(\eta)\}$ then $\bigcap_{l<m} A_l\neq\emptyset$.*

Proof. (i) Note that

$$\Big|f(\eta)\setminus\bigcap_{l<m} A_l\Big| = \Big|\bigcup_{l<m} f(\eta)\setminus A_l\Big| \leq \sum_{l<m} |f(\eta)\setminus A_l| \leq m\cdot g(\eta)/\zeta\,.$$

Hence

$$\mathbf{nor}_\eta\Big(\bigcap_{l<m} A_l\Big) = \frac{g(\eta)}{|f(\eta)\setminus\bigcap_{l<m} A_l|} \geq \zeta/m\,.$$

(ii) Applying (i) we get $\mathbf{nor}_\eta(\bigcap_{l<m} A_l)\geq 1/m$. Hence

$$\Big|f(\eta)\setminus\bigcap_{l<m} A_l\Big| \leq g(\eta)\cdot m \leq g(\eta)\cdot\prod\{2^{f(\nu)} : f(\nu)<f(\eta)\} < f(\eta)$$

(the last inequality is guaranteed by condition (ε)). Consequently, the set $\bigcap_{l<m} A_l$ is nonempty. ∎

Let $\mathbf{P}_2$ consist of all trees $T \subseteq T^*$ such that

$$\lim_{n\to\infty} \min\{\mathbf{nor}_\eta(\mathrm{succ}_T(\eta)) : \eta \in T \cap \omega^n\} = \infty\,.$$

The order is by inclusion.

Recall that a forcing notion $\mathbf{Q}$ is *σ-k-linked* if there exist sets $R_n \subseteq \mathbf{Q}$ (for $n \in \omega$) such that $\bigcup_{n\in\omega} R_n = \mathbf{Q}$ and each R_n is k-linked (i.e. any k members of R_n have a common upper bound in $\mathbf{Q}$).

PROPOSITION 3.3. *For every $k < \omega$ the forcing notion $\mathbf{P}_2$ is σ-k-linked.*

Proof. Let $n \in \omega$ be such that for each $\eta \in T^* \cap \omega^n$,

$$k < \prod\{2^{f(\nu)} : f(\nu) < f(\eta)\}\,.$$

Note that the set

$$\{T \in \mathbf{P}_2 : \mathrm{lh}(\mathrm{root}\, T) \ge n \ \&\ (\forall \eta \in T)(\mathrm{root}\, T \subseteq \eta \Rightarrow \mathbf{nor}_\eta(\mathrm{succ}_T(\eta)) \ge 1)\}$$

is dense in $\mathbf{P}_2$. For $\eta \in T^*$ with $\mathrm{lh}\,\eta \ge n$, define

$$\mathcal{D}_\eta = \{T \in \mathbf{P}_2 : \mathrm{root}\ T = \eta \ \&\ (\forall \nu \in T)(\eta \subseteq \nu \Rightarrow \mathbf{nor}_\nu(\mathrm{succ}_T(\nu)) \ge 1)\}\,.$$

Since $\bigcup\{\mathcal{D}_\eta : \mathrm{lh}\,\eta \ge n\}$ is dense in $\mathbf{P}_2$ it is enough to show that each $\mathcal{D}_\eta$ is k-linked. Suppose $T_0, \ldots, T_{k-1} \in \mathcal{D}_\eta$. Since $k < \prod\{2^{f(\nu)} : f(\nu) < f(\eta)\}$ we may apply Lemma 3.2(ii) to conclude that if $\nu \in T = T_0 \cap \ldots \cap T_{k-1}$ and $\nu \supseteq \eta$ then $\mathrm{succ}_T(\nu) \ne \emptyset$. By 3.2 (i) we get $T \in \mathbf{P}_2$. ∎

For $\eta \in T^*$ we define a forcing notion $\mathbf{Q}_\eta$ by

$$\begin{aligned}\mathbf{Q}_\eta = \{t \subseteq T^* :\ & t \text{ is a finite tree of height } n \in \omega,\\ & \mathrm{root}\ t = \eta \text{ and}\\ & (\forall \nu \in t \cap \omega^{<n})(\eta \subseteq \nu \Rightarrow \mathbf{nor}_\nu(\mathrm{succ}_t(\nu)) \ge \mathrm{lh}\,\nu)\}\,.\end{aligned}$$

Since $\mathbf{Q}_\eta$ is countable and atomless it is isomorphic to the Cohen forcing $\mathbf{C}$. Let $\mathbf{P} = \prod\{\mathbf{Q}_{i,\eta} : i < \omega_1,\ \eta \in T^*\}$ be the finite support product such that each $\mathbf{Q}_{i,\eta}$ is a copy of $\mathbf{Q}_\eta$.

THEOREM 3.4. *Let $G \subseteq \mathbf{P}$ be a generic filter over $\mathbf{V}$. Then, in $\mathbf{V}[G]$, there is no $S \in \mathbf{P}_2$ such that*

$$S \Vdash_{\mathbf{P}_2} \text{“}\widehat{\mathbf{P}}_2|S \textit{ is } \sigma\textit{-centered”}.$$

Proof. We work in $\mathbf{V}[G]$. Assume $S \Vdash$ “$\widehat{\mathbf{P}}_2|S$ is σ-centered”. Let $\dot{R}_n$ ($n \in \omega$) be $\mathbf{P}_2$-names for subsets of $\mathbf{P}_2$ such that

$$S \Vdash_{\mathbf{P}_2} \text{“}\widehat{\mathbf{P}}_2|S \subseteq \bigcup_{n\in\omega} \dot{R}_n \ \&\ \text{each } \dot{R}_n \text{ is directed”}\,.$$

Take $n \in \omega$ such that

$$(\forall \eta \in S)(n \le \operatorname{lh} \eta \Rightarrow \mathbf{nor}_\eta(\operatorname{succ}_S(\eta)) \ge 1)\,.$$

Fix any $\eta \in S \cap \omega^n$ and choose $l, m \in \operatorname{succ}_S(\eta)$, $l < m$. For $i < \omega_1$ put

$$T_i = \bigcup \{t : \{((i, \eta^\frown m), t)\} \in G\}\,.$$

Each T_i is the tree added by $G \cap \mathbf{Q}_{i,\eta^\frown m}$ and it is an element of $\mathbf{P}_2$. Moreover, root $T_i = \eta^\frown m$ and for each $\nu \in T_i$ if $\eta^\frown m \subseteq \nu$ then $\mathbf{nor}_\nu(\operatorname{succ}_{T_i}(\nu)) \ge \operatorname{lh} \nu$. Hence, by Lemma 3.2, $T_i \cap S \in \mathbf{P}_2$ for each $i \in \omega_1$.

Now we work in $\mathbf{V}$. We find $p_i, \dot{S}_i, \eta_i, n_i$ such that for each $i \in \omega_1$:

- $p_i \in \mathbf{P}$, $n_i \in \omega, \eta_i \in T^*$ and $\dot{S}_i$ is a $\mathbf{P}$-name for a member of $\mathbf{P}_2$,
- $\Vdash_{\mathbf{P}} (\forall \nu \in \dot{S}_i)(\text{root } \dot{S}_i \subseteq \nu \Rightarrow \mathbf{nor}_\nu(\operatorname{succ}_{\dot{S}_i}(\nu)) \ge 1)$,
- $p_i \Vdash_{\mathbf{P}}$ "$\eta^\frown l \subseteq \eta_i = \text{root } \dot{S}_i$ & $\dot{S}_i \Vdash_{\mathbf{P}_2} \dot{T}_i \in \dot{R}_{n_i}$",
- $(i, \eta^\frown m) \in \operatorname{dom}(p_i)$.

Next we find a set $I \in [\omega_1]^{\omega_1}$ such that $\{\operatorname{dom}(p_i) : i \in I\}$ forms a Δ-system with the root d, and for each $i \in I$:

- $\eta_i = \eta^*$ and $n_i = n^*$,
- $p_i | d = p^*$,
- $p_i(i, \eta^\frown m) = t$, $(i, \eta^\frown m) \notin d$.

Let $n^\#$ be the height of the tree t. Clearly we may assume that $n^\# > \operatorname{lh} \eta^*$. Fix an enumeration $\{\varrho_k : k < k^\#\}$ of $t \cap \omega^{n^\#}$. Put

$$H = \{(a_k : k < k^\#) : a_k \subseteq f(\varrho_k) \ \&\ \mathbf{nor}_{\varrho_k}(a_k) \ge n^\#\}\,.$$

Choose distinct $i_{\bar{\mathbf{a}}} \in I$ for $\bar{\mathbf{a}} \in H$. We define a condition $q \in \mathbf{P}$ extending all $p_{i_{\bar{\mathbf{a}}}}$ ($\bar{\mathbf{a}} \in H$) by

$$\operatorname{dom}(q) = \bigcup \{\operatorname{dom}(p_{i_{\bar{\mathbf{a}}}}) : \bar{\mathbf{a}} \in H\}\,;$$

$$\text{if} \quad (i, \nu) \in \operatorname{dom}(p_{i_{\bar{\mathbf{a}}}}) \text{ and } (i, \nu) \ne (i_{\bar{\mathbf{a}}}, \eta^\frown m) \quad \text{then} \quad q(i, \nu) = p_{i_{\bar{\mathbf{a}}}}(i, \nu)\,;$$
$$q(i_{\bar{\mathbf{a}}}, \eta^\frown m) = t \cup \{\varrho_k{}^\frown c : k < k^\#,\ c \in \bar{\mathbf{a}}(k)\}\,.$$

Now we take $r \ge q$ such that r decides all $\dot{S}_{i_{\bar{\mathbf{a}}}} | (n^\# + 1)$. Thus we have finite trees $s_{\bar{\mathbf{a}}}$ (for $\bar{\mathbf{a}} \in H$) such that $r \Vdash_{\mathbf{P}} \dot{S}_{i_{\bar{\mathbf{a}}}} | (n^\# + 1) = s_{\bar{\mathbf{a}}}$.

Claim. *There exists $H' \subseteq H$ such that*

(i) $\bigcap_{\bar{\mathbf{a}} \in H'} (s_{\bar{\mathbf{a}}} \cap \omega^{n^\# + 1}) \ne \emptyset$ *and*
(ii) *for each $k < k^\#$ the set $\bigcap \{\bar{\mathbf{a}}(k) : \bar{\mathbf{a}} \in H'\}$ is empty.*

Indeed, let $H_\varrho = \{\bar{\mathbf{a}} \in H : \varrho \in s_{\bar{\mathbf{a}}}\}$ for $\varrho \in T^* \cap \omega^{n^\# + 1}$ with $\eta^\frown l \subseteq \varrho$. Clearly $\varrho \in \bigcap_{\bar{\mathbf{a}} \in H_\varrho} s_{\bar{\mathbf{a}}}$, so it is enough to show that for some ϱ the family H_ϱ satisfies (ii). Suppose that for each $\varrho \in T^* \cap \omega^{n^\# + 1}$ with $\varrho \supseteq \eta^\frown l$ we can find $k_\varrho < k^\#$ and c_ϱ such that $c_\varrho \in \bigcap\{\overline{\mathbf{a}}(k_\varrho) : \overline{\mathbf{a}} \in H_\varrho\}$. Put

$$\overline{\mathbf{a}}^*(k) = f(\varrho_k)\setminus\{c_\varrho : \varrho \in T^* \cap \omega^{n^\#+1} \ \&\ \eta^\frown l \subseteq \varrho\}\,.$$

Let $\varrho^+ \in T^* \cap \omega^{n^\#}$ be such that

$$f(\varrho^+) = \max\{f(\varrho) : \varrho \in T^* \cap \omega^{n^\#},\ \eta^\frown l \subseteq \varrho\}\,.$$

By condition (γ) we get

$$|\{\varrho \in T^* \cap \omega^{n^\#+1} : \eta^\frown l \subseteq \varrho\}| \le \prod\{f(\nu) : f(\nu) \le f(\varrho^+)\}\,.$$

Now, for each $k < k^\#$ we have $f(\varrho^+) < f(\varrho_k)$ (recall that $\eta^\frown l \subseteq \varrho^+$, $\eta^\frown m \subseteq \varrho_k$ and $l < m$ so condition (γ) works). Hence

$$\begin{aligned}\mathbf{nor}_{\varrho_k}(\overline{\mathbf{a}}^*(k)) &\ge \frac{g(\varrho_k)}{\prod\{f(\nu) : f(\nu) \le f(\varrho^+)\}}\\ &\ge \frac{g(\varrho_k)}{\prod\{f(\nu) : f(\nu) < f(\varrho_k)\}} > n^\#\,.\end{aligned}$$

Thus $\overline{\mathbf{a}}^* \in H$. Since $c_\varrho \notin \overline{\mathbf{a}}^*(k_\varrho)$ we have $\overline{\mathbf{a}}^* \notin H_\varrho$ for every $\varrho \in T^* \cap \omega^{n^\#+1}$ with $\eta^\frown l \subseteq \varrho$. Since $\bigcup\{H_\varrho : \varrho \in T^* \cap \omega^{n^\#+1},\ \eta^\frown l \subseteq \varrho\} = H$ we get a contradiction. The claim is proved.

Now let $H' \subseteq H$ be a family given by the claim. Condition (ii) implies that

$$r \Vdash_{\mathbf{P}} \text{“the family } \{T_{i_{\overline{\mathbf{a}}}} : \overline{\mathbf{a}} \in H'\} \text{ has no upper bound in } \mathbf{P}_2\text{”.}$$

Since $|H| \le \prod\{2^{f(\varrho_k)} : k < k^\#\}$ we see that for each $\varrho \in T^* \cap \omega^{n^\#+1}$,

$$|H'| \le \prod\{2^{f(\nu)} : f(\nu) < f(\varrho)\}\,.$$

Hence we may apply 3.2(ii) to conclude that for every $\varrho \supseteq \eta^\frown l$ with $\operatorname{lh}\varrho \ge n^\# + 1$,

$$r \Vdash_{\mathbf{P}} \text{“if } \varrho \in \bigcap_{\overline{\mathbf{a}} \in H'} \dot{S}_{i_{\overline{\mathbf{a}}}} \text{ then } \bigcap_{\overline{\mathbf{a}} \in H'} \operatorname{succ}_{\dot{S}_{i_{\overline{\mathbf{a}}}}}(\varrho) \ne \emptyset\text{”.}$$

Thus

$$r \Vdash_{\mathbf{P}} \text{“the family } \{\dot{S}_{i_{\overline{\mathbf{a}}}} : \overline{\mathbf{a}} \in H'\} \text{ has an upper bound”.}$$

Since $r \Vdash_{\mathbf{P}}$ “$\dot{S}_{i_{\overline{\mathbf{a}}}} \Vdash_{\mathbf{P}_2} \dot{T}_{i_{\overline{\mathbf{a}}}} \in \dot{R}_{n^*}$” we get a contradiction. ∎

Remark. 1) In the above theorem we worked in the model $\mathbf{V}[G]$ for technical reasons only. The assertion of the theorem can be proved in ZFC.

2) The forcing notion $\mathbf{P}_2$ is a special case of the forcing studied in [Sh1].

Problem 3.5. *Does there exist a ccc Souslin forcing* $\mathbf{P}$ *such that*

(i) $\mathbf{P}$ *is homogeneous* (*i.e. for each* $p \in \mathbf{P}$, $\Vdash_{\mathbf{P}}$ *“there exists a generic filter* G *over* $\mathbf{V}$ *such that* $p \in G$”), *and*

(ii) $\nVdash_{\mathbf{P}}$ “$\widehat{\mathbf{P}}$ *is* σ*-centered*”?

4. On "small subsets of P are σ-centered". Our next example is connected with the following, still open, question:

PROBLEM 4.1. *Assume that for each ccc Souslin forcing* $\mathbf{P}$ *every set* $Q \in [\mathbf{P}]^{\omega_1}$ *is* σ*-centered* (*in* $\mathbf{P}$). *Does* $\mathbf{MA}_{\omega_1}$(*Souslin*) *hold true*?

As an illustration of this subject let us recall a property of the Random (Solovay) Algebra $\mathbf{B}$ (see [BaJ]):

if every $B \in [\mathbf{B}]^{\omega_1}$ is σ-centered
then the real line cannot be covered by ω_1 null sets
and consequently $\mathbf{MA}_{\omega_1}(\mathbf{B})$ holds true.

Our example shows that the above property of the algebra $\mathbf{B}$ does not extend to other forcing notions. Let

$$\mathbf{P}_3 = \{(n,T) : n \in \omega \ \&\ T \subseteq 2^{<\omega} \text{ is a tree } \ \&\ (\forall t \in T \cap 2^n)(\mu([T_t]) > 0)\}\,.$$

The order is defined by

$$(n_1,T_1) \le (n_2,T_2) \quad \text{if and only if} \quad n_1 \le n_2,\ T_2 \subseteq T_1 \text{ and } T_1|n_1 = T_2|n_1\,.$$

LEMMA 4.2. $\mathbf{P}_3$ *is a* σ*-linked Souslin forcing which is not* σ*-centered.*

Proof. Note that $\Vdash_{\mathbf{P}_3}$"there exists a perfect set of random reals over $\mathbf{V}$". Hence $\mathbf{P}_3$ is not σ-centered. To show that it is σ-linked define sets $U(W,n,m)$ for $n < m < \omega$ and finite trees $W \subseteq 2^{\le m}$:

$$\begin{aligned} &U(W,n,m)\\ &\quad = \{(n,T) \in \mathbf{P}_3 : T|m = W \ \&\ (\forall t \in T \cap 2^n)(\mu([T_t]) > W(t)/2^{m+1})\} \end{aligned}$$

where $W(t) = |\{s \in W \cap 2^m : t \subseteq s\}|$ (for $t \in W \cap 2^n$). Clearly each set $U(W,n,m)$ is linked (i.e. any two members of it are compatible in $\mathbf{P}_3$) and $\mathbf{P}_3 = \bigcup\{U(W,n,m) : n < m < \omega \ \&\ W \subseteq 2^{\le m}\}$. Since obviously $\mathbf{P}_3$ is Souslin we are done. ∎

Let $\mathbf{B}(\kappa)$ stand for the Random Algebra for adding κ many random reals. This is the measure algebra of the space 2^κ.

THEOREM 4.3. *Assume* $\mathbf{V} \vDash \mathbf{CH}$. *Let* $G \subseteq \mathbf{B}(\omega_2)$ *be a generic set over* $\mathbf{V}$. *Then*, *in* $\mathbf{V}[G]$,

(i) *the Martin axiom fails for* $\mathbf{P}_3$ *but*
(ii) *each* $Q \in [\mathbf{P}_3]^{\omega_1}$ *is* σ*-centered* (*in* $\mathbf{P}_3$).

Proof. Cichoń proved that one random real does not produce a perfect set of random reals (see [BaJ]). Hence in $\mathbf{V}[G]$ there is no perfect set of random reals over $\mathbf{V}$. Consequently, the first assertion is satisfied in $\mathbf{V}[G]$. Since $\mathbf{V}[G] \vDash$ "each $B \in [\mathbf{B}]^{\omega_1}$ is σ-centered in $\mathbf{B}$" (compare Section 3) it is enough to show the following:

CLAIM. *Suppose that each $B \in [\mathbf{B}]^{\omega_1}$ is σ-centered. Then every set $Q \in [\mathbf{P}_3]^{\omega_1}$ is σ-centered.*

Indeed, let $Q \in [\mathbf{P}_3]^{\omega_1}$. For $n \in \omega$ and $t \in 2^n$ put

$$B(t,n) = \{[T_t] : (n,T) \in Q \ \&\ t \in T\}\,.$$

By our assumption we find sets $B(t,n,k)$ for $k,n \in \omega$ and $t \in 2^n$ such that $B(t,n) = \bigcup_{k\in\omega} B(t,n,k)$ and for all $A_1, A_2 \in B(t,n,k)$ the set $A_1 \cap A_2$ is of positive measure. Now define sets $Q(n,W,\sigma)$ for $n \in \omega$, a finite tree $W \subseteq 2^{\le n}$ and a function $\sigma : W \cap 2^n \to \omega$ by

$$Q(n,W,\sigma) = \{(n,T) \in Q : T|n = W \ \&\ (\forall t \in T \cap 2^n)([T_t] \in B(t,n,\sigma(t)))\}\,.$$

Note that if $(n,T_1),(n,T_2) \in Q(n,W,\sigma)$ then for each $t \in W \cap 2^n$ the set $[(T_1)_t] \cap [(T_2)_t]$ is of positive measure. Consequently, each $Q(n,W,\sigma)$ is linked and we are done. ■

5. A σ-centered example. In this section we define a very simple σ-centered Souslin forcing notion. Next we show that in any generic extension of some model of **CH** via finite support iteration of the Dominating (Hechler) Algebra, the Martin Axiom fails for this forcing notion. Consequently, we get the consistency of the following sentence:

any union of less than continuum meager sets is meager + $\neg$**CH** + **MA** *fails for some σ-centered Souslin forcing.*

Our example $\mathbf{P}_4$ consists of all pairs (n,F) such that $n \in \omega$, $F \in [2^\omega]^{<\omega}$ and all elements of the list $\{x|n : x \in F\}$ are distinct. $\mathbf{P}_4$ is ordered by

$$(n,F) \le (n',F') \quad \text{if and only if} \quad n \le n',\ F \subseteq F' \text{ and } \{x|n : x \in F\} = \{x|n : x \in F'\}\,.$$

LEMMA 5.1. *$\mathbf{P}_4$ is a σ-centered Souslin forcing.*

Proof. Clearly $\mathbf{P}_4$ is Souslin (even Borel). To show that $\mathbf{P}_4$ is σ-centered note that if $\{x|n : x \in F_0\} = \ldots = \{x|n : x \in F_k\}$ then the conditions $(n,F_0),\ldots,(n,F_k)$ are compatible (if m is large enough then $(m, F_0 \cup \ldots \cup F_k)$ is a witness for this). ■

Now we want to define the model we will start with. At the beginning we work in **L**. Applying the technique of [Sh] we can construct a sequence $(\mathbf{P}_\xi : \xi \le \omega_1)$ of forcing notions such that for all $\alpha, \beta < \omega_1$ and $\xi \le \omega_1$:

(1) if $\alpha < \beta$ then $\mathbf{P}_\alpha$ is a complete suborder of $\mathbf{P}_\beta$,

(2) there is $\gamma > \beta$ such that $\mathbf{P}_{\gamma+1} = \mathbf{P}_\gamma * \dot{\mathbf{D}}_\alpha$, where $\dot{\mathbf{D}}_\alpha$ is the $\mathbf{P}_\gamma$-name for finite support, length α iteration of the Hechler forcing,

(3) $\mathbf{P}_\xi$ satisfies ccc,

(4) if ξ is limit then $\mathbf{P}_\xi = \overrightarrow{\lim}_{\zeta<\xi}\mathbf{P}_\zeta$,
(5) $\mathbf{P}_{\omega_1} \Vdash$ "every projective set of reals has the Baire property"

(for details see also [JR]). Recall that the Hechler forcing $\mathbf{D}$ consists of all pairs (n,f) such that $n\in\omega$ and $f\in\omega^\omega$. These pairs are ordered by

$$(n,f)\le(n',f') \quad \text{if and only if} \quad n\le n',\ f|n=f'|n' \text{ and } f(k)\le f'(k) \text{ for all } k\in\omega\,.$$

Suppose $G\subseteq\mathbf{P}_{\omega_1}$ is a generic set over $\mathbf{L}$. We work in $\mathbf{L}[G]$. For distinct $x,y\in 2^\omega$ we define $h(x,y)=\min\{n : x(n)\ne y(n)\}$. Easy calculations show the following:

LEMMA 5.2. *Let $b\subseteq\omega$. Then the following conditions are equivalent*:

(i) *there exists a* Borel *equivalence relation R on 2^ω with countably many equivalence classes such that $\{h(x,y): x,y\in 2^\omega\cap\mathbf{L}\ \&\ x\ne y\ \&\ R(x,y)\}\subseteq b$*,

(ii) *there exists an equivalence relation R on 2^ω with countably many equivalence classes such that $\{h(x,y): x,y\in 2^\omega\cap\mathbf{L}\ \&\ x\ne y\ \&\ R(x,y)\}\subseteq b$*,

(iii) *there exist sets $Y_n\subseteq 2^\omega$ (for $n\in\omega$) such that $\mathbf{L}\cap 2^\omega\subseteq\bigcup_{n\in\omega}Y_n$ and $\bigcup_{n\in\omega}\{h(x,y): x\ne y\ \&\ x,y\in Y_n\}\subseteq b$*,

(iv) $(\exists f: 2^{<\omega}\to 2)(\forall x\in 2^\omega\cap\mathbf{L})(\exists m\in\omega)(\forall n>m)(n\notin b\Rightarrow f(x|n)=x(n))$.

The Raisonnier filter $\mathcal{F}$ consists of all sets $b\subseteq\omega$ satisfying one of the conditions of 5.2 (cf. [Ra]). $\mathcal{F}$ is a proper filter on ω. Directly from (iv) of 5.2 one can see that $\mathcal{F}$ is a Σ^1_3-subset of 2^ω. Consequently, it has the Baire property (recall that we are in $\mathbf{L}[G]$).

THEOREM 5.3 (Talagrand, [Ta]). *For any proper filter F on ω the following conditions are equivalent*:

(i) *F does not have the Baire property*,

(ii) *for every increasing sequence $(n_k : k\in\omega)$ of integers there exists $b\in F$ such that $(\exists^\infty_k)(b\cap[n_k,n_{k+1})=\emptyset)$.* ■

Applying the above theorem we can find an increasing function $r\in\omega^\omega\cap\mathbf{L}[G]$ such that (in $\mathbf{L}[G]$)

$$(\forall b\in\mathcal{F})(\forall^\infty_k)(b\cap[r(k),r(k+1))\ne\emptyset)\,.$$

Let $\dot r$ be the $\mathbf{P}_{\omega_1}$-name for r and let $\alpha_0<\omega_1$ be such that $\dot r$ is a $\mathbf{P}_{\alpha_0}$-name.
Our basic model will be $\mathbf{L}[r]$.

THEOREM 5.4. *Let κ be a regular cardinal. Let $\mathbf{D}_\kappa$ be the finite support iteration of the Hechler forcing of length κ. Suppose $H\subseteq\mathbf{D}_\kappa$ is a generic set over $\mathbf{L}[r]$. Then*

$$\mathbf{L}[r][H]\models \text{"there is no } \mathbf{P}_4\text{-generic object over } \mathbf{L}[r]\text{"}.$$

Proof. Assume not. Let $H^* \in \mathbf{L}[r][H]$ be a $\mathbf{P}_4$-generic object over $\mathbf{L}[r]$. Put $T = \bigcup\{F : (\exists n \in \omega)((n, F) \in H^*)\}$. Then in $\mathbf{L}[r][H^*]$ we have:

(6) T is a closed subset of 2^ω,

(7) $(\exists^\infty_k)(\forall x, y \in T)(x \neq y \Rightarrow h(x, y) \notin [r(k), r(k+1)))$ and

(8) $(\forall x \in 2^\omega \cap \mathbf{L})(\exists q \in Q)(q + x \in T)$

(Q stands for the set of all sequences which are eventually 0, and $+$ denotes the addition modulo 2). Since both (7) and (8) are absolute (Π^1_2) sentences they are also satisfied in $\mathbf{L}[r][H]$. Let $\dot{T} \in \mathbf{L}[r]$ be a $\mathbf{P}_\kappa$-name for T. Since T is a closed subset of 2^ω we can think of $\dot{T}$ as a name for a real.

Now we work in $\mathbf{L}[r]$. Let $p \in \mathbf{D}_\kappa \cap \mathbf{L}[r]$ be such that

$$p \Vdash \text{“}\dot{T} \text{ satisfies (6)–(8)”}.$$

By the properties of the Souslin forcing (see §1 of [JS1]) we find a (closed) countable set $S \subseteq \kappa$ such that:

(9) $\dot{T}$ is a $\mathbf{D}_\kappa|S$-name, $p \in \mathbf{D}_\kappa|S$ and

(10) $\mathbf{D}_\kappa|S$ is a complete suborder of $\mathbf{D}_\kappa$.

Since (6)–(8) are absolute we get

(11) $p \Vdash_{\mathbf{D}_\kappa|S}$ “$\dot{T}$ satisfies (6)–(8)”.

But $\mathbf{D}_\kappa|S$ is isomorphic to the finite support iteration of the Hechler forcing of a countable length α ($\alpha < \omega_1$). Thus we can treat $\dot{T}$ as a $\mathbf{D}_\alpha$-name and p as a condition in $\mathbf{D}_\alpha$. Then, in $\mathbf{L}[r]$,

(12) $p \Vdash_{\mathbf{D}_\alpha|S}$ “$\dot{T}$ satisfies (6)–(8)”.

By (2) we find $\gamma > \alpha_0$ such that

(13) $\mathbf{P}_{\gamma+1} = \mathbf{P}_\gamma * \dot{\mathbf{D}}_\alpha$ and

(14) $p \equiv (\mathbf{1}, p)$ interpreted as a member of $\mathbf{P}_{\gamma+1}$ belongs to G.

By the properties of the Souslin forcing (12) holds true in $\mathbf{L}[G \cap \mathbf{P}_\gamma]$ and hence

(15) $\mathbf{L}[G \cap \mathbf{P}_{\gamma+1}] \models$ “$\dot{T}^G$ satisfies (6)–(8)”

(we treat here $\dot{T}$ as a $\mathbf{P}_{\gamma+1}$-name). Let $b = \{h(x, y) : x \neq y \ \&\ x, y \in \dot{T}^G\} \in [\omega]^{<\omega} \cap \mathbf{L}[G]$. By (15) and by Shoenfield absoluteness we have

(16) $\mathbf{L}[G] \models$ “$\dot{T}^G$ satisfies (6)–(8)”.

Since $\{h(x, y) : x \neq y \ \&\ x, y \in \dot{T}^G\} = \{h(x, y) : x \neq y \ \&\ x, y \in \dot{T}^G + q\}$ we conclude that

(17) $\mathbf{L}[G] \models$ “the sets $\dot{T}^G + q$ (for $q \in Q$) witness that $b \in \mathcal{F}$” and

(18) $\quad \mathbf{L}[G] \models (\exists^\infty_k)(b \cap [r(k), r(k+1)) = \emptyset)$.

The last condition contradicts our choice of r. ■

Since $\Vdash_{\mathbf{D}_\kappa}$ "any union of less than κ meager sets is meager" we get

COROLLARY 5.5. *The following theory is consistent*: $\mathbf{ZFC} + \neg\mathbf{CH} +$ *"Martin Axiom fails for some σ-centered Souslin forcing" + "any union of less than continuum meager sets is meager".* ■

6. On "Souslin not ccc". In this section we will give a negative answer to the following question of Woodin:

Suppose $\mathbf{P}$ is a Souslin forcing notion which is not ccc. Does there exist a perfect set $T \subseteq \mathbf{P}$ such that any distinct $t_1, t_2 \in T$ are incompatible?

Recall that in the case of non-ccc partial orders we do not require Souslin forcings to satisfy the condition: "the set $\{(p,q) : p$ is incompatible with $q\}$ is Σ^1_1".

Thus a forcing notion $\mathbf{P}$ is *Souslin not ccc* if both $\mathbf{P}$ and $\le_{\mathbf{P}}$ are analytic sets. The reason for this is that we want to cover in our definition various standard forcing notions with simple definitions for which incompatibility is not analytic (e.g. the Laver forcing).

Let $\mathbf{Q}$ be the following partially ordered set: $W \in \mathbf{Q}$ if W is a finite set of pairs (α, β) with $\alpha \le \beta < \omega_1$ such that if $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ are in W, then $\beta_1 < \alpha_2$ or $\beta_2 < \alpha_1$. $\mathbf{Q}$ is ordered by inclusion. It follows from [Je1] that $\mathbf{Q}$ is proper. Clearly $|\mathbf{Q}| = \omega_1$.

Next we define a forcing notion $\mathbf{P}_5$. It consists of all $r \in \omega^\omega$ such that r codes a pair (E^r, w^r) where

• E^r is a relation on ω such that $(\omega, E^r) \models \mathbf{ZFC}^-$ and E^r encodes all elements of $\omega \cup \{\omega\}$,
• $w^r \in \omega$ and $E^r \models$ "$w^r \in \mathbf{Q}$".

We say that a one-to-one function $f \in \omega^\omega$ *interprets E^{r_1} in E^{r_2}* if there exists $n \in \omega$ such that $\mathrm{rng}(f) = \{k \in \omega : E^{r_2}(k,n)\}$ and $E^{r_1}(l,k) \equiv E^{r_2}(f(l), f(k))$.

If f interprets E^{r_1} in E^{r_2} then E^{r_2} may "discover" that some of the ordinals of E^{r_1} are not ordinals (i.e. are not well-founded). Let $w(r_1, r_2, f) = w^{r_1} \cap \{(\alpha, \beta) : \alpha \le \beta$ are ordinals in $E^{r_2}\}$. Then, in E^{r_2}, $w(r_1, r_2, f)$ is an initial segment of w^{r_1} and it is in $\mathbf{Q}^{E^{r_2}}$.

Now we can define an order $\le$ on $\mathbf{P}_5$:

$r_1 \le r_2$ if and only if $r_1 = r_2$ or *there exists* $f \in \omega^\omega$ which interprets E^{r_1} in E^{r_2} and such that $(\omega, E^{r_2}) \models w(r_1, r_2, f) \subseteq w^{r_2}$.

Obviously both $\mathbf{P}_5$ and the order $\le$ are Σ^1_1-sets.

For $r \in \mathbf{P}_5$ we define $W(r)$ as $w^r \cap \{(\alpha, \beta) : \alpha \le \beta$ are well-founded$\}$. Note that $W(r_1) = W(r_2)$ implies r_1 and r_2 are equivalent in $\mathbf{P}_5$ (i.e. they have the same compatible elements of $\mathbf{P}_5$). Consequently, $\mathbf{Q}$ may be densely embedded in the complete Boolean algebra determined by $\mathbf{P}_5$. It follows from [Je1] that $\mathbf{P}_5$ is proper, Souslin and does not satisfy the countable chain condition. Moreover, if $\omega_1 < 2^\omega$ then $\mathbf{P}_5$ does not contain a perfect set of pairwise incompatible elements (recall $|\mathbf{Q}| = \omega_1$). ■

An interesting question appears here:

Suppose $\mathbf{P}$ is ω-proper and Souslin. Does there exist a perfect set of pairwise incompatible elements of $\mathbf{P}$?

The negative answer to this question is given by the following result.

THEOREM 6.1. *Assume $\omega_1 < \mathrm{cf}(2^\omega)$. There exists an ω-proper Souslin non-ccc forcing notion $\mathbf{P}^*_5$ with no perfect set of pairwise incompatible elements.*

Proof. Let $\delta \le \omega_1$ be additively indecomposable. Let $\mathbf{Q}^*$ be the order defined by $W \in \mathbf{Q}^*$ if and only if W is a countable set of pairs (α, β) with $\alpha \le \beta < \omega_1$ such that:

- $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in W \Rightarrow \beta_1 < \alpha_2$ or $\beta_2 < \alpha_1$,
- $\{(\alpha, \beta) \in W : \alpha \ne \beta\}$ is finite,
- the order type of the set $\{\alpha : (\exists \beta)((\alpha, \beta) \in W)\}$ is less than δ.

$\mathbf{Q}^*$ is ordered by inclusion.

It follows from Chapter XVII, §3 of [Sh2] that $\mathbf{Q}^*$ is α-proper for each $\alpha < \omega_1$.

Now we can repeat the coding procedure that we applied to define the forcing notion $\mathbf{P}_5$. Thus we get a Souslin forcing notion $\mathbf{P}^*_5$ such that $\mathbf{Q}^*$ can be densely embedded in the Boolean algebra determined by $\mathbf{P}^*_5$.

For $W \in \mathbf{Q}^*$ let $\mathrm{heart}(W) = \{(\alpha, \beta) \in W : \alpha \ne \beta\}$.

Assume that $\{(E^{r_\eta}, w^{r_\eta}) : \eta \in 2^\omega\} \subseteq \mathbf{P}^*_5$ is a perfect set of pairwise incompatible elements. Let W_η be the well-founded part of w^{r_η}. Since without loss of generality we can assume that $\sup\{\beta : (\exists \alpha)((\alpha, \beta) \in W_\eta)\}$ is constant and $\mathrm{heart}(W_\eta)$ is constant we easily get a contradiction. ■

7. On ccc Σ^1_2. Souslin ccc notions of forcing are indestructible ccc (see [JS1]):

Suppose $\mathbf{P}$ is a ccc Souslin notion of forcing. Let $\mathbf{Q}$ be a ccc forcing notion. Then $\Vdash_{\mathbf{Q}}$ "$\widehat{\mathbf{P}}$ is ccc".

The above property does not hold true for more complicated forcing notions. In this section we show that there may exist two ccc Σ^1_2-notions of forcing $\mathbf{P}_6$ and $\mathbf{P}^*_6$ such that $\mathbf{P}_6 \times \mathbf{P}^*_6$ does not satisfy ccc.

We start with $\mathbf{V} = \mathbf{L}$. Let $\mathbf{Q}$ be a ccc notion of forcing such that

$$\Vdash_{\mathbf{Q}} \ \mathbf{MA} + \neg\mathbf{CH}\,.$$

Let $G \subseteq \mathbf{Q}$ be a generic set over $\mathbf{L}$ and let r be a random real over $\mathbf{L}[G]$. Recall that by a theorem of Roitman (cf. [Ro]) we have $\mathbf{L}[G][r] \models \mathbf{MA}(\sigma\text{-centered})$.

Fix a sequence $(f_\alpha : \alpha < \omega_1) \in \mathbf{L}$ of one-to-one functions $f_\alpha : \alpha \xrightarrow{1\text{-}1} \omega$ and define in $\mathbf{L}[r]$ sets E_1, E_2 by

$$E_i = \{\{\alpha,\beta\} \in [\omega_1]^2 : \beta < \alpha \ \&\ r(f_\alpha(\beta)) = i\} \quad \text{for } i = 0, 1\,.$$

We define forcing notions $\mathbf{P}_6, \mathbf{P}^*_6$:

$$\mathbf{P}_6 = \{H \in [\omega_1]^2 : [H]^2 \subseteq E_0\}\,, \qquad \mathbf{P}^*_6 = \{H \in [\omega_1]^2 : [H]^2 \subseteq E_1\}\,.$$

Orders are inclusions.

Both $\mathbf{P}_6$ and $\mathbf{P}^*_6$ are elements of $\mathbf{L}[r]$. Moreover, they can be thought of as subsets of $\mathbf{L}[r] \cap 2^\omega$. Applying $\mathbf{MA}(\sigma\text{-centered})$ we find that (cf. [Je])

$$\mathbf{L}[G][r] \models \text{“any subset of } \mathbf{L}[r] \cap 2^\omega \text{ is a relative } \Sigma^0_2\text{-set”}.$$

Consequently,

$$\mathbf{L}[G][r] \models \text{“any subset of } \mathbf{L}[r] \cap 2^\omega \text{ is } \Sigma^1_2\text{”}.$$

Thus $\mathbf{P}_6$ and $\mathbf{P}^*_6$ are Σ^1_2-notions of forcing in $\mathbf{L}[G][r]$ (i.e. both $\mathbf{P}_6$, $\mathbf{P}^*_6$, and the orders and incompatibility relations are Σ^1_2-sets). Roitman proved the following:

THEOREM 7.1 (Roitman, Prop. 4.6 of [Ro]). *In* $\mathbf{L}[G][r]$ *both* $\mathbf{P}_6$ *and* $\mathbf{P}^*_6$ *satisfy ccc and* $\mathbf{P}_6 \times \mathbf{P}^*_6$ *does not satisfy ccc.* ∎

COROLLARY 7.2. *The following theory is consistent*: $\mathbf{ZFC} + \mathbf{MA}(\sigma$-*centered*$) + \neg\mathbf{CH} +$ *“there exist ccc* Σ^1_2*-notions of forcing* $\mathbf{P}_6$, $\mathbf{P}^*_6$ *such that* $\mathbf{P}^*_6 \Vdash$ «$\widehat{\mathbf{P}}_6$ *is not ccc*»*”.* ∎

PROBLEM 7.3. *Is there a ccc Souslin forcing notion* $\mathbf{P}$ *such that* $\mathbf{MA}(\mathbf{P})$ *always fails after adding a random real*?

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