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The dimension of remainders of rim-compact spaces

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Abstract. Answering a question of Isbell we show that there exists a rim-compact space X such that every compactification Y of X has $\dim(Y \setminus X) \geq 1$.

It is known that a space X is rim-compact if and only if X has a compactification Y such that $Y \setminus X$ is zero-dimensionally embedded in Y, i.e. Y has a base for the open sets whose boundaries are disjoint from $Y \setminus X$. Note that this implies that $\operatorname{ind}(Y \setminus X) \leq 0$.

On the other hand, if there is a compactification Y of the space X such that $\operatorname{ind}(Y \setminus X) \leq 0$ then X need not be rim-compact; an example was given by Smirnov in [1958].

If X is Lindelöf at infinity, i.e. each compact subset of X is contained in a compact set with a countable base for its neighbourhoods, then X is rim-compact if and only if it has a compactification Y such that $\operatorname{ind}(Y \setminus X) \leq 0$; such a compactification then also has $\dim(Y \setminus X) \leq 0$. This suggests the question whether every rim-compact space X has a compactification Y such that $\dim(Y \setminus X) \leq 0$ (see Isbell [1964] and Aarts and Nishiura [1993]; see also Diamond, Hatzenbuhler and Mattson [1988] for related problems). We shall show that the answer is negative. We construct a strongly zero-dimensional space X such that for every compactification Y of X we have $\dim(Y \setminus X) \geq 1$. Our space X will be such that $\beta X \setminus X$ is metrizable, zero-dimensional but not strongly zero-dimensional. Originally, the remainder was Roy's space Δ; however, by using Kulesza's space we were able to obtain an example of the smallest possible weight ω_1.

Convention. We identify an ordinal with its set of predecessors so that for example $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$. All ordinals under consideration carry the order topology.

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The construction. Let K be Kulesza's space, i.e. K is a (completely) metrizable subspace of ω_1^ω of weight $w(K) = \omega_1$ which is dense in ω_1^ω and satisfies $\operatorname{ind} K = 0$ and $\operatorname{Ind} K = 1$ (see Kulesza [1990]).

Consider now

$$Z = (\omega_1 + 1)^{\omega+1} = (\omega_1 + 1)^\omega \times (\omega_1 + 1)$$

and let

$$X = Z \setminus (K \times \{\omega_1\}) .$$

Then $(\omega_1+1)^\omega \times \omega_1 \subseteq X \subseteq Z$. As $(\omega_1+1)^\omega$ is compact we may conclude that $Z = (\omega_1+1)^\omega \times (\omega_1+1)$ is the Čech–Stone compactification of $(\omega_1+1)^\omega \times \omega_1$ (see Engelking [1989, Problem 3.12.20(c)]). Hence $\beta X = Z$ as well. Note that βX is a product of compact zero-dimensional spaces hence βX is also compact and zero-dimensional and therefore strongly zero-dimensional. It follows that X itself is strongly zero-dimensional, hence zero-dimensional and a fortiori rim-compact. It is also easily seen that $w(X) = \omega_1$. It remains to show that $\dim(\alpha X \setminus X) \geq 1$ for every compactification αX of X.

Let αX be a compactification of X. Consider

$$f\colon \beta X \to \alpha X,$$

the extension of the natural embedding $\mathrm{id}_X\colon X \to \alpha X$ over βX. As βX and αX are compact, the mapping f is perfect. Now,

$$\beta X \setminus X = f^{-1}[\alpha X \setminus X]$$

so $f \restriction \beta X \setminus X$ is also perfect. But $\beta X \setminus X = K \times \{\omega_1\}$ and K is metrizable. To finish our argument we need the following theorem, due to Morita and Nagami (see Engelking [1989]).

Theorem. *If $f\colon X \to Y$ is a closed mapping of a metrizable space X to a metrizable space Y and for every $y \in Y$, $\operatorname{Ind} f^{-1}[\{y\}] \leq k$ for $k \geq 0$, then $\operatorname{Ind} X \leq \operatorname{Ind} Y + k$.*

Now, $K \times \{\omega_1\}$ and $\alpha X \setminus X$ are metrizable, the mapping $f\colon K \times \{\omega_1\} \to \alpha X \setminus X$ is perfect and for each $x \in \alpha X \setminus X$ the fiber $f^{-1}[\{x\}]$ is compact and zero-dimensional, hence strongly zero-dimensional. Note that this means that $\operatorname{Ind}(f^{-1}[\{x\}]) \leq 0$ for each $x \in \alpha X \setminus X$, so by the theorem

$$\operatorname{Ind}(K \times \{\omega_1\}) \leq \operatorname{Ind}(\alpha X \setminus X) + 0 ,$$

hence

$$\dim(\alpha X \setminus X) = \operatorname{Ind}(\alpha X \setminus X) \geq \operatorname{Ind}(K \times \{\omega_1\}) = 1 .$$

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