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Order with successors is not interpretable in RCF

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Abstract. Using the monotonicity theorem of L. van den Dries for RCF-definable real functions, and a further result of that author about RCF-definable equivalence relations on $\mathbb{R}$, we show that the theory of order with successors is not interpretable in the theory RCF. This confirms a conjecture by J. Mycielski, P. Pudlák and A. Stern.

1. Let RCF be the theory of real closed fields. We may view RCF as the first order theory of the structure $\langle \mathbb{R}; +, \cdot, \leq, 0, 1\rangle$, where $\mathbb{R}$ is the set of real numbers and $+, \cdot, \leq, 0, 1$ have the usual meaning. It is conjectured in [6], (P_{23}), that the order theory of $\omega = \{0, 1, 2, \ldots\}$, i.e., $\mathrm{Th}(\langle \omega; \leq\rangle)$, cannot be interpreted in RCF:

$$|\mathrm{Th}(\langle \omega; \leq\rangle)| \not\leq |\mathrm{RCF}|\,. \tag{1.1}$$

Here $|T|$ denotes the *chapter of mathematics* containing a given theory T (i.e., the class of all theories which both interpret and are interpretable in T). $|T_1| \leq |T_2|$ means that T_1 is interpretable in T_2.

Our aim in this note is to prove (1.1). Familiarity with the definition of interpretability, as proposed by Jan Mycielski in [5], will be assumed. For further information, the reader is referred to the survey [6].

It is known that the order theories of ω and $\mathbb{Q}$ are not interpretable in each other: Both non-interpretability results

$$|\mathrm{Th}(\langle \omega; \leq\rangle)| \not\leq |\mathrm{Th}(\langle \mathbb{Q}; \leq\rangle)| \quad \text{and} \quad |\mathrm{Th}(\langle \mathbb{Q}; \leq\rangle)| \not\leq |\mathrm{Th}(\langle \omega; \leq\rangle)| \tag{1.2}$$

were obtained by J. Krajíček [4]. (The second part of (1.2) was proved independently by A. Stern; a proof which combines the methods of both authors is given in [8].) Obviously

$$|\mathrm{Th}(\langle \mathbb{Q}; \leq\rangle)| \leq |\mathrm{RCF}|\,, \tag{1.3}$$

so, using the second part of (1.2), we conclude that RCF is not interpretable in $\mathrm{Th}(\langle \omega; \leq\rangle)$. Hence, by (1.1), the chapters $|\mathrm{RCF}|$ and $|\mathrm{Th}(\langle \omega; \leq\rangle)|$ are incomparable. The first part of (1.2), easier to establish, clearly also follows from (1.1) and (1.3).

Since $|\mathrm{Th}(\langle \mathbb{Z}; \leq \rangle)| = |\mathrm{Th}(\langle \omega; \leq \rangle)|$, we may replace ω by $\mathbb{Z}$ in the above considerations. Another consequence of (1.1) is the well known fact that ω is not an RCF-definable subset of $\mathbb{R}$ (stemming from the undecidability of arithmetic).

To state our result quite exactly, let us specify first the axioms of a pre-ordering with successors. We shall call a binary relation $\ll$ a *pre-ordering* if $\ll$ is reflexive, transitive and there is universal comparability:

$$\forall x \forall y (x \ll y \vee y \ll x) .$$

For any such pre-ordering we abbreviate $(x \ll y) \wedge (y \ll x)$ as $x \approx y$. Clearly, $\approx$ is an equivalence relation; we call it the *equivalence relation associated with* $\ll$. We denote by $\mathrm{Succ}(x, y)$ the formula

$$x \ll y \wedge x \not\approx y \wedge \forall z (x \ll z \ll y \to z \approx x \vee z \approx y) ,$$

reading it: "y is an immediate successor of x". We say that $\ll$ is a *pre-ordering with successors* if every x has an immediate successor, i.e., if $\forall x \exists y \, \mathrm{Succ}(x, y)$.

It is clear that (1.1) will result if we prove:

Theorem 1.1. *Any sentence which proves the existence of a pre-ordering with successors is not interpretable in RCF.*

We take this opportunity to mention another recent result about RCF, and to pose a problem. A theory T is called *connected* if T interprets in the union $T_1 \dot{\cup} T_2$ of two theories with disjoint languages only if T interprets in T_1 or in T_2. The problem if RCF is connected was raised in [6], (P_2). A positive answer was found recently by A. Stern and the author [9].

A theory T is called *compact* if there is a finitely axiomatizable theory T' such that $|T| = |T'|$, i.e., T and T' interpret each other.

Problem. Is RCF compact?

2. A subset S of $\mathbb{R}^d$, $d \geq 1$, will be called *definable* if S is definable in the language $\{+, \cdot, \leq, 0, 1\}$ of RCF, i.e., if there is a formula ϕ in that language and there are some $a_1, \ldots, a_k \in \mathbb{R}$ such that, for all $x_1, \ldots, x_d \in \mathbb{R}$,

$$(x_1, \ldots, x_d) \in S \leftrightarrow \phi(x_1, \ldots, x_d, a_1, \ldots, a_k) .$$

Functions and relations on Cartesian powers of $\mathbb{R}$ will be called definable if they have definable graphs. We shall show that for every $d \geq 1$, there does not exist on $\mathbb{R}^d$ a definable relation of pre-ordering with successors. Clearly, this will be enough to establish Theorem 1.1. The following known facts will be needed in the proof.

Theorem 2.1 ([1]). *Given a definable equivalence relation $\approx$ on $\mathbb{R}^d$ ($d \geq 1$), there is a definable function $f : \mathbb{R}^d \to \mathbb{R}^d$ which selects one representative* *from each* $(\approx)$*-equivalence class*, *i.e.*, *such that for any* d*-tuples* $\overline{x}, \overline{y} \in \mathbb{R}^d$,

$$\overline{x} \approx \overline{y} \leftrightarrow f(\overline{x}) = f(\overline{y}) \text{ and } \overline{x} \in f(\overline{x}) .$$

The next theorem is proved, in the general setting of O-minimal structures, in [7]:

THEOREM 2.2 (Monotonicity Theorem in [2], [3]). *For every definable function* $f : \mathbb{R} \to \mathbb{R}$ *there is a partition of* $\mathbb{R}$ *into finitely many points and open intervals such that on each of the intervals* f *is either constant or strictly monotone and continuous.*

So, for a definable $f : \mathbb{R} \to \mathbb{R}$ there are only finitely many $y \in \mathbb{R}$ such that the pre-image $f^{-1}(y)$ is infinite. Combining this observation with the case $d = 1$ of Theorem 2.1, we get:

LEMMA 2.3. *Every definable equivalence relation on* $\mathbb{R}$ *has only finitely many infinite equivalence classes.*

3. We need two more lemmas:

LEMMA 3.1. *Let* $L_0 \subset L_1 \subset L_2 \subset \ldots$ *be a strictly increasing sequence of subsets of* $\mathbb{R}$ *such that each boundary* ∂L_i *is a finite set. Suppose further that the number of elements* $\#(\partial L_i)$ *of any of these boundaries does not exceed a fixed constant* $K < \infty$. *Then the difference* $L_{i+1} \setminus L_i$ *is infinite for infinitely many* i.

Proof. Suppose there is an i such that $L_{j+1} \setminus L_j$ is finite for all $j \geq i$. Let $j > i$ and consider any $x \in L_j \setminus L_i$. If x is in the interior of L_j, then, as $x \notin L_i$, we must have $x \in \partial L_i$, for otherwise a neighbourhood of x would be disjoint from L_i, and $L_j \setminus L_i$ would be infinite. So

$$L_j \setminus L_i \subseteq \partial L_i \cup \partial L_j .$$

Since there are at least $j - i$ elements in $L_j \setminus L_i$, we get

$$j - i \leq \#(\partial L_i) + \#(\partial L_j) \leq 2K ,$$

which is obviously impossible for all $j \geq i$. ∎

LEMMA 3.2. *There does not exist a definable pre-ordering of* $\mathbb{R}$ *for which there is an infinite sequence* $x_0, x_1, x_2, \ldots$ *such that every* x_{i+1} *is an immediate successor of* x_i.

Proof. Suppose that $\ll$ is a definable pre-ordering of $\mathbb{R}$ and $(x_i)_{i<\omega}$ is a sequence such that $\mathrm{Succ}(x_i, x_{i+1})$ holds for each i. By the Tarski–Seidenberg quantifier elimination theorem for RCF, the formula $x \ll y$ is RCF-equivalent to a Boolean combination of atomic formulas:

$$x \ll y \leftrightarrow \bigvee_{j=1}^{r} \Big[(p_j(x,y) = 0) \wedge \bigwedge_{k=1}^{n} (q_k^{(j)}(x,y) > 0) \Big] ,$$

where $p_j(x,y)$, $q_k^{(j)}(x,y)$ are real polynomials in two variables. Let us check that if $L_i = \{x : x \ll x_i\}$, $i = 0,1,2,\ldots$, then all assumptions of Lemma 3.1 are satisfied. Putting $y = x_i$ in the above Boolean combination, we conclude that each L_i is a finite union of intervals, so ∂L_i is a finite set. Let $a \in \partial L_i$. Then, for some j, there are numbers x, arbitrarily close to a, satisfying

$$(p_j(x,x_i) = 0) \wedge \bigwedge_{k=1}^{n} (q_k^{(j)}(x,y) > 0)\,.$$

Clearly $p_j(a,x_j) = 0$. So, if $p_j(x,x_i)$ does not vanish identically as a polynomial in x, then the number of possible values for $a \in \partial L_i$ does not exceed the degree with respect to x of the polynomial $p_j(x,y)$. If $p_j(x,x_i) \equiv 0$ as a polynomial in x, then $a \in \partial L_i$ implies that $q_k^{(j)}(a,x_i) = 0$ for at least one k. We conclude that the number of elements of any ∂L_i is not greater than the sum of the degrees with respect to x of all the polynomials $p_j(x,y)$, $q_k^{(j)}(x,y)$, i.e., there is a common finite bound K for the number of elements of each boundary ∂L_i.

If $\approx$ is the equivalence relation associated with $\ll$ then it is clear that the $(\approx)$-equivalence class $[x_{i+1}]$ of x_{i+1} equals $L_{i+1} \backslash L_i$. Thus each inclusion $L_i \subset L_{i+1}$ is proper and we can apply Lemma 3.1, which tells us that infinitely many equivalence classes $[x_i]$ are infinite. But this contradicts Lemma 2.3. ∎

4. Proof of Theorem 1.1. Suppose that α is a sentence which proves the existence of a pre-ordering with successors and α can be d-dimensionally interpreted in RCF. Then there exists a definable pre-ordering $\ll$ of $\mathbb{R}^d$ with successors. If $d = 1$, we get a contradiction with Lemma 3.2, and we are done. Otherwise, consider the equivalence relation on $\mathbb{R}^d$ associated with $\ll$ and a definable map $f : \mathbb{R}^d \to \mathbb{R}^d$ which selects one representative from each equivalence class, as in Theorem 2.1. We put $E = f(\mathbb{R}^d)$ and denote by $\ll_E$ the restriction of $\ll$ to E. Clearly $\ll_E$ is an ordering of E where each element has a unique immediate successor. Next we choose $e_0 \in E$, and denote by $(e_i)_{i<\omega}$ the sequence of elements of E such that each e_{i+1} is the immediate successor of e_i.

By Baire's Category Theorem, there is a straight line in $\mathbb{R}^d$ such that all e_i, $0 \le i < \omega$, have different orthogonal projections on that line. In other words, there is a $w \in \mathbb{R}^d$ such that, if we denote for each $x \in \mathbb{R}$ by H_x the hyperplane $\{v : v \cdot w = x\}$ in $\mathbb{R}^d$, we never get two distinct e_i, e_j in one such hyperplane. We put now $E^+ = \{e \in E : e_0 \ll_E e\}$ and define a pre-ordering $\le$ of $\mathbb{R}$ by

$$x \le y \leftrightarrow (H_x \cap E^+ = \emptyset) \vee \forall (u \in H_y \cap E^+) \exists (v \in H_x \cap E^+)(v \ll_E u)\,.$$

This gives a definable pre-ordering of $\mathbb{R}$. Denoting by x_i the unique $x \in \mathbb{R}$ for which $e_i \in H_x$, we check that, for every $0 \le i < \omega$, x_{i+1} is the immediate successor of x_i. We have again reached a contradiction with Lemma 3.2, and this shows that a d-dimensional interpretation of α in RCF cannot exist. ■

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