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Bound quivers of three-separate stratified posets, their Galois coverings and socle projective representations

by

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Abstract. A class of stratified posets I_ϱ^* is investigated and their incidence algebras KI_ϱ^* are studied in connection with a class of non-shurian vector space categories. Under some assumptions on I_ϱ^* we associate with I_ϱ^* a bound quiver (Q,Ω) in such a way that $KI_\varrho^* \simeq K(Q,\Omega)$. We show that the fundamental group of (Q,Ω) is the free group with two free generators if I_ϱ^* is rib-convex. In this case the universal Galois covering of (Q,Ω) is described. If in addition I_ϱ is three-partite a fundamental domain $I^{*+\times}$ of this covering is constructed and a functorial connection between $\mathrm{mod}_{\mathrm{sp}}(KI_\varrho^{*+\times})$ and $\mathrm{mod}_{\mathrm{sp}}(KI_\varrho^*)$ is given.

1. Introduction. Socle projective representations of stratified posets introduced in [S1, S2] (see Definition 2.1 below) appear in a natural way in the study of vector space categories (see [S2], [S5, Chap. 17]) and lattices over orders (see [S5, Ch. 13], [S4]). The aim of this paper is to give some tools for studying these representations for a certain class of stratified posets.

Our main points of interest are the incidence algebra KI_ϱ^* over a field K of a three-separate stratified poset I_ϱ^* with a unique maximal element $*$ (see Definition 3.1) and the representation type of the category $\mathrm{mod}_{\mathrm{sp}}(KI_\varrho^*)$ of socle projective right KI_ϱ^*-modules. Following [S1, S2, S4] we associate with any such poset I_ϱ^* a bound quiver

$$(Q(I_\varrho^*),\Omega(I_\varrho^*))$$

in such a way that KI_ϱ^* is isomorphic to the bound quiver algebra $KQ(I_\varrho^*)/\Omega(I_\varrho^*)$. Under the assumption that I_ϱ^* is rib-convex (see Section 4) we show that the fundamental group $\Pi_1(Q(I_\varrho^*),\Omega(I_\varrho^*))$ is a free noncommutative group with two free generators and we give an explicit description of the universal covering $(\widetilde{Q},\widetilde{\Omega})$ of $(Q(I_\varrho^*),\Omega(I_\varrho^*))$. If in addition I_ϱ^* is three-partite we define, by means of $(\widetilde{Q},\widetilde{\Omega})$, a simply connected [AS]

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finite-dimensional three-peak algebra $KI_\varrho^{*+\times}$ and a functor

$$f_{+\times} : \mathrm{mod}_{\mathrm{sp}}(KI_\varrho^{*+\times}) \to \mathrm{mod}_{\mathrm{sp}}(KI_\varrho^{*})$$

preserving the representation type. In the case when the Auslander–Reiten quiver $\Gamma_{\mathrm{sp}}(KI_\varrho^{*+\times})$ of $\mathrm{mod}_{\mathrm{sp}}(KI_\varrho^{*+\times})$ has a preprojective component we get a simple criterion for the finite representation type of $\mathrm{mod}_{\mathrm{sp}}(KI_\varrho^{*+\times})$ (see Theorems 5.5, 5.6). In particular, we solve a problem stated in [S4, Remark 4.33].

I would like to thank Professor Daniel Simson for calling my interest to this subject and useful remarks concerning the paper.

2. Preliminaries and notation. We consider a poset I with partial order $\preccurlyeq$. We suppose that $I = \{1, \ldots, n\}$ and if $i \preccurlyeq j$ then $i \leq_{\mathbb{N}} j$ for $i, j \in I$. Define

$$\blacktriangle I := \{(i,j) : i, j \in I \text{ and } i \preccurlyeq j\},$$
$$\triangle I := \{(i,j) : i, j \in I \text{ and } i \prec j\}.$$

Given $(i,j) \in \blacktriangle I$ we put $[i,j] := \{s \in I : i \preccurlyeq s \preccurlyeq j\}$ and $\langle i,j\rangle := \{s \in I : i \prec s \prec j\}$. Throughout we identify (i,i) with i.

DEFINITION 2.1 [S2, S4]. A *stratification* of I is an equivalence relation ϱ on $\blacktriangle I$ such that if $(i,j)\varrho(p,q)$ then there exists a poset isomorphism $\sigma : [i,j] \to [p,q]$ such that $(i,t)\varrho(p,\sigma(t))$ and $(t,j)\varrho(\sigma(t),q)$ for any $t \in [i,j]$. A *stratified poset* is a pair

$$I_\varrho = (I, \varrho)$$

where I is a poset and ϱ is a stratification of I.

We denote by $r_\varrho(i,j)$ the cardinality of the ϱ-coset of (i,j), and call (i,j) a *rib* if $r_\varrho(i,j) > 1$ and $i \neq j$. The number $r_\varrho(i,j)$ is then the *rib rank* of the rib (i,j).

The full stratified subposet $\mathrm{rsk}(I_\varrho)$ of I_ϱ consisting of all beginnings and ends of ribs in I_ϱ is called the *rib skeleton* of I_ϱ. We fix a decomposition

$$\mathrm{rsk}(I_\varrho) = \Re_1 + \ldots + \Re_h$$

into rib-connected components with respect to the rib-equivalence relation generated by the following relation:

$$i — j \Leftrightarrow \text{either } (i,j) \text{ or } (j,i) \text{ is a rib}.$$

Fix a field K and a stratified poset I_ϱ. We recall from [S4] that the K-algebra

$$(2.2) \qquad KI_\varrho = \{b = (b_{pq}) \in \mathbb{M}_{n\times n}(K) : b_{pq} = 0 \text{ if } p \not\preccurlyeq q \text{ and } b_{ij} = b_{pq} \text{ if } (i,j)\varrho(p,q)\}$$

is called the *incidence algebra* of I_ϱ.

We denote by $I^* = I \cup \{*\}$ the enlargement of I by adjoining a unique maximal element $*$ (called the *peak*) and we extend trivially the relation ϱ from $\blacktriangle I$ to $\blacktriangle I^*$.

Thus we get a right peak algebra (see [S4]) of the form

$$KI_\varrho^* = \begin{pmatrix} KI_\varrho & M \\ 0 & K \end{pmatrix} \tag{2.3}$$

where

$$M = \left.\begin{pmatrix} K \\ \vdots \\ K \end{pmatrix}\right\}n$$

is a left KI_ϱ-module with respect to the usual matrix multiplication.

For a more detailed discussion of stratified posets, examples and applications the reader is referred to [S2] and [S5, Section 17.16].

In Section 3 below we will use the notion of the fundamental group of a quiver Q with a set of relations Ω ([Gr, MP]). For the convenience of the reader we briefly recall this concept. We follow [S4].

With a connected quiver Q we associate its fundamental group $\Pi_1(Q, q)$ computed as the group of homotopy classes $[\omega]$ of walks ω in Q starting and ending at the fixed point q. By a *walk* we mean a formal composition $\alpha_1 \ldots \alpha_r$ where α_p is an arrow of Q or its formal inverse and the sink of α_p is the source of α_{p+1}. Homotopy is the smallest equivalence relation $\approx$ (on the set of walks) such that:

(1) $1_x \approx 1_x^{-1}$ for each vertex x of Q,

(2) $\alpha\alpha^{-1} \approx 1_x$ and $\alpha^{-1}\alpha \approx 1_y$ for each arrow $\alpha : x \to y$,

(3) if $w \approx v$ then $uw \approx uv$ and $wu' \approx vu'$ whenever the walks involved are composable.

By the *fundamental group of a bound quiver* (Q, Ω) we mean the group

$$\Pi_1(Q, \Omega) = \Pi_1(Q, q)/N_\Omega\,, \tag{2.4}$$

where N_Ω is the normal subgroup generated by the conjugacy classes $C(u, v)$ of homotopy classes $[w^{-1}u^{-1}vw]$ in $\Pi_1(Q, q)$ where u, v are directed paths with a common sink and a common source, and there is a minimal relation

$$\omega = \lambda_1\omega_1 + \ldots + \lambda_t\omega_t \in (\Omega), \quad \lambda_i \in K^*\,,$$

with $t \geq 2$ and $u = \omega_1$, $v = \omega_2$. Let us recall from [MP] that a relation ω of the above form is a *minimal relation* if for every nonempty proper subset $J \subset \{1, \ldots, t\}$ we have

$$\sum_{j \in J} \lambda_j\omega_j \notin (\Omega)\,.$$

The following maximal tree lemma is a very useful method of computing the fundamental group. Before we formulate it we recall from [S4] that by an *Ω-contour* we mean a pair (u,v) of oriented paths with a common sink and a common source such that there is a minimal relation ω of the above form with $hug=\omega_1$ and $hvg=\omega_2$ for some oriented paths h,g such that the sink of h is the source of u and the source of g is the sink of u. We say that (u,v) is *defined with respect to the set* $\Omega'\subseteq(\Omega)$ if $\omega\in\Omega'$.

LEMMA 2.5 [S4, Remark 3.6, Lemma 3.7]. *Suppose that (Q,Ω) is a bound quiver, let T be a maximal tree in Q and $q\in Q$.*

(a) *N_Ω is generated by the elements $C(u,v)$, where (u,v) runs through all the Ω-contours defined with respect to a fixed set of generators of the ideal (Ω).*

(b) *$\Pi_1(Q,q)$ is a free group generated by the elements $\widehat{\beta}=[a\beta b]$ where $\beta\in Q_1\setminus T_1$ and a,b are walks in T connecting q with the sink and the source of β, respectively.*

(c) *If (u,v) is an Ω-contour and*

$$u=u_0\beta_1u_1\beta_2\ldots u_{s-1}\beta_su_s,\quad v=v_0\gamma_1v_1\gamma_2\ldots v_{r-1}\gamma_rv_r,$$

where $\beta_i,\gamma_j\in Q_1\backslash T_1$ and u_i and v_j are oriented paths in T then

$$\widehat{\beta}_1\widehat{\beta}_2\ldots\widehat{\beta}_s\equiv\widehat{\gamma}_1\widehat{\gamma}_2\ldots\widehat{\gamma}_r\ (\textit{modulo}\ N_\Omega).\ \blacksquare$$

If the fundamental group of (Q,Ω) is nontrivial we construct the *universal Galois covering*

$$f:(\widetilde{Q},\widetilde{\Omega})\to(Q,\Omega) \tag{2.6}$$

of (Q,Ω) in the following way (see [MP, Corollary 1.5], [Gr]).

Fix $q\in Q$. Let W be the topological universal cover of Q, i.e. a quiver W whose vertices are the homotopy classes $[\omega]$ of walks ω in Q starting at a fixed point p ([Sp]). There is an arrow $(\alpha,[\omega])$ from $[\omega]$ to $[\nu]$ in W if $[\nu]=[\omega\alpha]$ for an arrow α in Q. N_Ω acts on W in an obvious way. We take for $\widetilde{Q}$ the orbit quiver W/N_Ω and for $\widetilde{\Omega}$ the set of liftings of relations in Ω from KQ to $K\widetilde{Q}$. The bound quiver map f is defined by

$$f(N_\Omega(\alpha,[\omega]))=\alpha,\qquad f(N_\Omega[\omega])=\text{the sink of }\omega\,,$$

where $N_\Omega[\omega]$ (resp. $N_\Omega(\alpha,[\omega])$) denotes the orbit of $[\omega]$ (resp. $(\alpha,[\omega])$).

The group $\Pi_1(Q,\Omega)$ acts naturally on $(\widetilde{Q},\widetilde{\Omega})$ as a group of automorphisms. One can check that f is the universal Galois covering with group $\Pi_1(Q,\Omega)$ (see [Gr, MP]).

3. Three-separate stratified posets and the associated bound quivers. Let us start with our main definition which extends that given in [S1, S4].

DEFINITION 3.1. A *three-separate stratified poset* is a stratified poset I_ϱ such that I is the disjoint union of subsets $I^{(1)}, I^{(2)}, I^{(3)}$ and the following conditions hold:

(a) There is no relation $i \prec j$, where $i \in I^{(k)}$, $j \in I^{(l)}$ and $k > l$.

(b) $r_\varrho(i,j) \le 3$ for all $(i,j) \in \blacktriangle I$.

(c) If $(i,j)\varrho(s,t)$ and $(i,j) \ne (s,t)$ then there exist $k, l \le 3$ such that $k \ne l$, $i, j \in I^{(k)}$ and $s, t \in I^{(l)}$.

(d) If $r_\varrho(i,j) = 2$ then $i, j \notin I^{(1)}$.

We say that the decomposition $I = I^{(1)} + I^{(2)} + I^{(3)}$ is a *three-separation* of I_ϱ.

We call a rib of rank 3 a 3-*rib* and a rib of rank 2 a 2-*rib*. A pair $(i,j) \in \triangle I$ is called *short* if $\{i,j\} = [i,j]$. In this case we write β_{ij} instead of (i,j). A pair (i,j) is called 3-ϱ-*extremal* if it is not short, $r_\varrho(i,j) \le 2$ and $(i,s), (s,j)$ are 3-ribs for all s such that $i \prec s \prec j$. A pair (i,j) is called 2-ϱ-*extremal* if it is neither short nor 3-ϱ-extremal, $r_\varrho(i,j) = 1$ and $(i,s), (s,j)$ are ribs for all s such that $i \prec s \prec j$. We say that (i,j) is ϱ-*extremal* if it is either 2-ϱ-extremal or 3-ϱ-extremal.

EXAMPLE 3.2. Let I^* be the following poset:

$$\begin{array}{ccccc}
3 & & 1 & & \\
\downarrow & & \downarrow & & \\
6 & & 4 & \rightarrow & 2 \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
9 & & 7 & & 5 \\
 & & \downarrow & \nearrow & \downarrow \\
 & & 10 & & 8 \\
 & & & & \downarrow \\
 & & & & 11 \\
 & & & & \downarrow \\
 & & & & *
\end{array}$$

and ϱ be the relation given by

$$\begin{aligned}
&1\varrho 2\,,\\
&(3,6)\varrho(4,7)\varrho(5,8)\,,\\
&(6,9)\varrho(7,10)\varrho(8,11)\,,\\
&(4,10)\varrho(5,11)\,.
\end{aligned}$$

Then I^*_ϱ is a three-separate poset with three-separation $I = I^{(1)} + I^{(2)} + I^{(3)}$, where

$$I^{(1)} = \{3,6,9\}, \quad I^{(2)} = \{1,4,7,10\}, \quad I^{(3)} = \{2,5,8,11,*\}\,.$$

The pairs $(3,9)$, $(4,10)$ and $(5,11)$ are 3-ϱ-extremal.

We associate with I_ϱ the *bound quiver*

$$(Q(I_\varrho), \Omega(I_\varrho)) \tag{3.3}$$

as follows. The set $(Q(I_\varrho))_0$ of vertices of $Q(I_\varrho)$ is the set

$$I/\varrho = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$$

of the ϱ-cosets $\overline{q}$ of elements $q \in I$. We have the following arrows in $Q(I_\varrho)$.

(i) If (i,j) is short then the ϱ-coset $\overline{\beta}_{ij}$ of β_{ij} is a unique arrow from $\overline{i}$ to $\overline{j}$.

(ii) If $(i_k, j_k) \in \triangle I^{(k)}$ are 3-ϱ-extremal for $k = 1,2,3$, $i_1 \varrho i_2 \varrho i_3, j_1 \varrho j_2 \varrho j_3$ and $r_\varrho(i_k, j_k) = 1$ for $k = 1,2,3$ then we have exactly two arrows $\beta^*_{i_1 j_1}, \beta^*_{i_2 j_2} : \overline{i}_1 \to \overline{j}_1$.

If $(i_k, j_k) \in \triangle I^{(k)}$ and $(i_l, j_l) \in \triangle I^{(l)}$ are 3-ϱ-extremal, $i_k \varrho i_l \varrho i_m$, $j_k \varrho j_l \varrho j_m$, $(i_m, j_m) \in \triangle I^{(m)}$ is not 3-ϱ-extremal and (i_k, j_k) and (i_l, j_l) are unrelated then we have a unique arrow $\beta^*_{i_x j_x} : \overline{i}_1 \to \overline{j}_1$, where $x = \min(k,l)$.

If $(i_k, j_k) \in \triangle I^{(k)}$ are 3-ϱ-extremal for $k = 1,2,3$, $i_1 \varrho i_2 \varrho i_3, j_1 \varrho j_2 \varrho j_3$ and $(i_2, j_2)\varrho(i_3, j_3)$ then we have a unique arrow $\beta^*_{i_1 j_1} : \overline{i}_1 \to \overline{j}_1$.

If $(i_2, j_2) \in \triangle I^{(2)}$ and $(i_3, j_3) \in \triangle I^{(3)}$ are 2-ϱ-extremal, $i_2 \varrho i_3$ and $j_2 \varrho j_3$ then we have a unique arrow $\beta^*_{i_2 j_2} : \overline{i}_2 \to \overline{j}_2$.

A directed path ω in $Q(I_\varrho)$ is called a *rib path* if ω is a composition of arrows which are the ϱ-cosets of ribs in I_ϱ. It is called a 3-*rib path* if it is a composition of the ϱ-cosets of 3-ribs in I_ϱ. A path ω is called a 2-*rib path* if it is not a 3-rib path and it is a composition of ϱ-cosets of 3-ribs and 2-ribs in I_ϱ. A path ω is called a *nonrib path* if it is not a rib path. A nonrib path is called an $I^{(k)}$-*path* if it is a composition of arrows $\widetilde{\beta}_{ij}$ with $i,j \in I^{(k)}$, where $\widetilde{\beta}_{ij}$ denotes either $\overline{\beta}_{ij}$ or β^*_{ij}. An arrow $\overline{\beta}_{ij}$ is called 1-2-*skew* (resp. 2-3-*skew*, 1-3-*skew*) if $i \in I^{(1)}$ and $j \in I^{(2)}$ (resp. $i \in I^{(2)}$ and $j \in I^{(3)}$; $i \in I^{(1)}$ and $j \in I^{(3)}$). A directed path ω in Q is called 1-2-*skew* (resp. 2-3-*skew*; 1-3-*skew*) if ω contains a 1-2-skew arrow (resp. contains a 2-3-skew arrow; either contains a 1-3-skew arrow, or contains a 1-2-skew arrow and a 2-3-skew arrow).

We define the set of relations $\Omega = \Omega(I_\varrho)$ to consist of the following elements of the path algebra $KQ(I_\varrho)$:

(a) $\widetilde{\beta}_{i_1 j_1} \widetilde{\beta}_{i_2 j_2} \ldots \widetilde{\beta}_{i_r j_r}$ if there is no sequence $\beta_{t_0 t_1}, \beta_{t_1 t_2}, \ldots, \beta_{t_{r-1} t_r}$ such that $(i_k, j_k)\varrho(t_{k-1}, t_k)$ for $k = 1, \ldots, r$. (Recall that $\widetilde{\beta}_{ij}$ is either $\overline{\beta}_{ij}$ or β^*_{ij}.)

(b) $\widetilde{\beta}_{i_0 i_1} \widetilde{\beta}_{i_1 i_2} \ldots \widetilde{\beta}_{i_r i_{r+1}} - \widetilde{\beta}_{j_0 j_1} \widetilde{\beta}_{j_1 j_2} \ldots \widetilde{\beta}_{j_s j_{s+1}}$, where $i_0 = j_0$, $i_{r+1} = j_{s+1}$,

$$i_0 \prec i_1 \prec \ldots \prec i_r \prec i_{r+1}, \qquad j_0 \prec j_1 \prec \ldots \prec j_s \prec j_{s+1}$$

and there exist p, q such that (i_p, i_{p+1}) and (j_q, j_{q+1}) are not ribs.

(c) $w - u$ for all 3-rib paths (resp. 2-rib paths) w and u with a common sink and a common source.

(d) $w - w_1 - w_2 - w_3$, where w is a 3-rib path, w_k is an $I^{(k)}$-path for $k = 1, 2, 3$ and w, w_1, w_2, w_3 have a common sink and a common source.

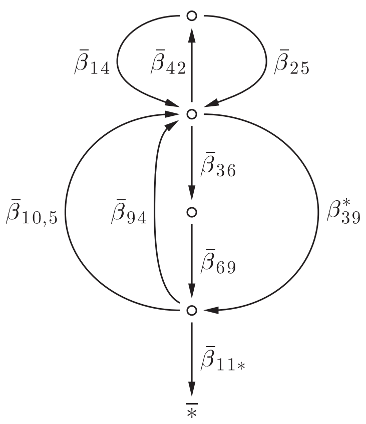
(e) $w - u$ for all $I^{(k)}$-paths w, u with a common sink and a common source for $k = 1, 2, 3$.

(f) $w - u_2 - u_3$, where w is a 2-rib path, u_k is an $I^{(k)}$-path for $k = 2, 3$ and w, u_2, u_3 have a common sink and a common source.

(g) $w - w' - u$ where w is a 3-rib path, w' is a 2-rib path, u is an $I^{(1)}$-path and w, w', u have a common sink and a common source.

In our example we have:

$Q(I_\varrho^*)$:



$$\begin{aligned}\Omega(I_\varrho^*) = \{&\overline{\beta}_{42}\overline{\beta}_{14}, \overline{\beta}_{25}\overline{\beta}_{42}, \overline{\beta}_{14}\beta_{39}^*, \overline{\beta}_{25}\beta_{39}^*, \overline{\beta}_{10,5}\overline{\beta}_{42}, \beta_{39}^*\overline{\beta}_{11*},\\ &\beta_{39}^*\overline{\beta}_{10,5}, \overline{\beta}_{94}\beta_{39}^*, \overline{\beta}_{10,5}\beta_{39}^*, \overline{\beta}_{36}\overline{\beta}_{69}\overline{\beta}_{10,5} - \overline{\beta}_{42}\overline{\beta}_{25},\\ &\beta_{39}^*\overline{\beta}_{94} - \overline{\beta}_{36}\overline{\beta}_{69}\overline{\beta}_{94}\}\,.\end{aligned}$$

Consider the K-algebra homomorphism

$$g : KQ(I_\varrho) \to KI_\varrho \tag{3.4}$$

defined by the formulas (compare with [S4]):

$$g(\overline{i}) = \begin{cases} e_{ii} & \text{if } r_\varrho(i) = 1\,,\\ e_{ii} + e_{i'i'} & \text{if } r_\varrho(i) = 2,\ i\varrho i',\ i \neq i'\,,\\ e_{ii} + e_{i'i'} + e_{i''i''} & \text{if } i\varrho i'\varrho i'',\ i \neq i' \neq i'' \neq i\,,\end{cases}$$

$$g(\overline{\beta}_{ij}) = \begin{cases} e_{ij} & \text{if } r_\varrho(i,j) = 1\,,\\ e_{ij} + e_{i'j'} & \text{if } r_\varrho(i,j) = 2, (i,j)\varrho(i',j') \\ & \quad\text{and } (i,j) \neq (i',j')\,,\\ e_{ij} + e_{i'j'} + e_{i''j''} & \text{if } (i,j)\varrho(i',j')\varrho(i'',j'') \text{ and}\\ & \quad (i,j) \neq (i',j') \neq (i'',j'') \neq (i,j)\,,\end{cases}$$

and

$$g(\beta^*_{ij}) = e_{ij}$$

where e_{ij} denotes the matrix with 1 in the (i, j)-entry and zeros elsewhere.

A connection between $(Q(I_\varrho), \Omega(I_\varrho))$ and I_ϱ is given by the following proposition (compare with [S4, Proposition 2.8]).

PROPOSITION 3.5. *Let I_ϱ be a three-separate stratified poset with a three-separation $I^{(1)} + I^{(2)} + I^{(3)}$. If $(Q(I_\varrho), \Omega(I_\varrho))$ is the bound quiver of I_ϱ* (see (3.3)) *then the homomorphism g of* (3.4) *induces a K-algebra isomorphism*

$$\overline{g} : K(Q(I_\varrho), \Omega(I_\varrho)) \to KI_\varrho ,$$

where $K(Q(I_\varrho), \Omega(I_\varrho)) = KQ(I_\varrho)/(\Omega(I_\varrho))$.

For the proof we will need the following technical lemma.

LEMMA 3.6. *Suppose $(s, t) \in \triangle I^{(k)}$, $(s', t') \in \triangle I^{(l)}$, $k \neq l$, $s\varrho s'$ and $t\varrho t'$.*

(a) *If (s', t') is not 3-ϱ-extremal and (s, t) is 3-ϱ-extremal then there exists a sequence $s_0 \prec s_1 \prec \ldots \prec s_r$, where $s_0 = s'$, $s_r = t'$, the pair (s_i, s_{i+1}) is short for any $i = 0, \ldots, r-1$, and there exists $i = 0, \ldots, r-1$ such that there is no relation $(s_i, s_{i+1})\varrho(u, v)$ with $(u, v) \in \triangle I^{(k)}$.*

(b) *If $k, l \neq 1$, (s', t') is not 2-ϱ-extremal and (s, t) is 2-ϱ-extremal then there exists a sequence $s_0 \prec s_1 \prec \ldots \prec s_r$, where $s_0 = s'$, $s_r = t'$, the pair (s_i, s_{i+1}) is short for any $i = 0, \ldots, r-1$, and there exists $i = 0, \ldots, r-1$ such that $r_\varrho(s_i, s_{i+1}) = 1$.*

Proof. We will prove (a); the proof of (b) is similar. Let

$$s_0 \prec s_1 \prec \ldots \prec s_r$$

be a sequence such that $s_0 = s'$, $s_r = t'$, the pair (s_i, s_{i+1}) is short for any $i = 0, \ldots, r-1$, and for some $i = 1, \ldots, r-1$ we have $r_\varrho(s', s_i) < 3$ or $r_\varrho(s_i, t') < 3$. The existence of such a sequence is obvious. Assume that for any $i = 0, \ldots, r-1$ there exist $(u, v) \in \triangle I^{(k)}$ such that $(s_i, s_{i+1})\varrho(u, v)$. Then it is easy to construct a sequence

$$s'_0 \prec s'_1 \prec \ldots \prec s'_r$$

such that $s'_0 = s$, $s'_r = t$ and for any $i = 0, \ldots, r$ we have $s'_i \varrho s_i$. But it follows from 3-ϱ-extremality of (s, t) that for any $i = 1, \ldots, r-1$ we have $r_\varrho(s, s'_i) = 3$ and $r_\varrho(s'_i, t) = 3$. This implies that for any $i = 1, \ldots, r-1$ we have $r_\varrho(s', s_i) = 3$ and $r_\varrho(s_i, t') = 3$, a contradiction. ■

Proof of Proposition 3.5. We set $(Q, \Omega) = (Q(I_\varrho), \Omega(I_\varrho))$ and $R = KI_\varrho$. Note that the idempotents $\widehat{e}_i := g(\overline{i})$, $i \in I^*$, form a complete set of primitive orthogonal idempotents of R. Moreover, the matrices $\widehat{e}_{ij}$,

$i \preccurlyeq j \preccurlyeq *$, defined as follows:

$$\widehat{e}_{ij} = \begin{cases} e_{ij} & \text{if } r_\varrho(i,j) = 1\,, \\ e_{ij} + e_{i'j'} & \text{if } r_\varrho(i,j) = 2,\ (i,j)\varrho(i',j') \\ & \quad \text{and } (i,j) \neq (i',j')\,, \\ e_{ij} + e_{i'j'} + e_{i''j''} & \text{if } (i,j)\varrho(i',j')\varrho(i'',j'') \text{ and} \\ & \quad (i,j) \neq (i',j') \neq (i'',j'') \neq (i,j) \end{cases}$$

form a K-basis of R. We shall show that $\widehat{e}_{st} \in \operatorname{Im}(g)$ for $(s,t) \in \blacktriangle I$. This is obvious if $s = t$. Assume that $s \neq t$. We proceed by induction on $m_{st} := |\langle s,t\rangle|$.

(1) If $m_{st} = 0$, i.e. (s,t) is short then $\widehat{e}_{st} = g(\overline{\beta}_{st}) \in \operatorname{Im}(g)$.

Assume that $m > 0$ and $\widehat{e}_{st} \in \operatorname{Im}(g)$ for $(s,t) \in \vartriangle I$ such that $m_{st} < m$. Suppose that $m_{st} = m$.

(2) If (s,t) is not ϱ-extremal then there exists $p \in \langle s,t\rangle$ such that $r_\varrho(s,p) = r_\varrho(s,t)$ or $r_\varrho(p,t) = r_\varrho(s,t)$. Then $\widehat{e}_{st} = \widehat{e}_{sp}\widehat{e}_{pt}$ and since by the induction hypothesis $\widehat{e}_{sp}, \widehat{e}_{pt} \in \operatorname{Im}(g)$ we get $\widehat{e}_{st} \in \operatorname{Im}(g)$.

(3) Suppose that $r_\varrho(s,t) = 2$ and (s,t) is 3-ϱ-extremal. Then there exist $s', t' \in I^{(1)}$ such that $s'\varrho s$ and $t'\varrho t$. It is easy to see that $s' \prec t'$. If (s',t') is not 3-ϱ-extremal then it follows from Lemma 3.6 and (1) that $\widehat{e}_{s't'} \in \operatorname{Im}(g)$. Indeed, we take a sequence $s_0 \prec s_1 \prec \ldots \prec s_r$ such that $s_0 = s$, $s_r = t$, the pairs (s_j, s_{j+1}) are short for $j = 0,\ldots,r-1$ and there is no relation $(s_i,s_{i+1})\varrho(u,v)$ with $u, v \in I^{(2)} \cup I^{(3)}$, for some $i = 0,\ldots,r-1$. Since $s', t' \in I^{(1)}$ we get $r_\varrho(s_i, s_{i+1}) = 1$ for some $i = 0,\ldots,r-1$. Then

$$\widehat{e}_{s't'} = \widehat{e}_{s_0s_1}\widehat{e}_{s_1s_2}\ldots\widehat{e}_{s_{r-1}s_r}\,.$$

The right side of this equality belongs to $\operatorname{Im}(g)$ by (1). Thus $\widehat{e}_{s't'} \in \operatorname{Im}(g)$.

If (s',t') is 3-ϱ-extremal then $\widehat{e}_{s't'} = g(\beta^*_{s't'}) \in \operatorname{Im}(g)$ as well. Since by the induction hypothesis we have $\widehat{e}_{sp}\widehat{e}_{pt} \in \operatorname{Im}(g)$, where $p \in \langle s,t\rangle$, we conclude that

$$\widehat{e}_{st} = \widehat{e}_{sp}\widehat{e}_{pt} - \widehat{e}_{s't'} \in \operatorname{Im}(g)\,.$$

(4) Suppose that $r_\varrho(s,t) = 1$ and (s,t) is 3-ϱ-extremal. Let $s\varrho s'\varrho s''$ and $t\varrho t'\varrho t''$, where $s,t \in I^{(k)}$, $s',t' \in I^{(l)}$, $s'',t'' \in I^{(n)}$, and k,l,n are pairwise different. It is easy to check that $s' \prec t'$ and $s'' \prec t''$. Consider the following cases.

(a) If both (s',t') and (s'',t'') are 3-ϱ-extremal and $k \neq 3$ then $\widehat{e}_{st} = g(\beta^*_{st}) \in \operatorname{Im}(g)$. If $k = 3$ then by the same argument (since $l, n \neq 3$) we get $\widehat{e}_{s't'}, \widehat{e}_{s''t''} \in \operatorname{Im}(g)$. By the induction hypothesis for any $p \in \langle s,t\rangle$ we have

$$\widehat{e}_{st} + \widehat{e}_{s't'} + \widehat{e}_{s''t''} = \widehat{e}_{sp}\widehat{e}_{pt} \in \operatorname{Im}(g)$$

and hence we conclude that $\widehat{e}_{st} \in \operatorname{Im}(g)$.

(b) Suppose that (s',t') is 3-ϱ-extremal but (s'',t'') is not. If $k<l$ then $\widehat{e}_{st}=g(\beta^*_{st})\in \mathrm{Im}(g)$. If $k>l$ then by the same reason $\widehat{e}_{s't'}\in \mathrm{Im}(g)$. Moreover, using Lemma 3.6 and arguments similar to those used in (3) we prove that $\widehat{e}_{s''t''}\in \mathrm{Im}(g)$. Then as in (a) we conclude that $\widehat{e}_{st}\in \mathrm{Im}(g)$.

(c) Suppose that (s',t'), (s'',t'') are not 3-ϱ-extremal. Then using Lemma 3.6 one can show that $e_{s't'}+e_{s''t''}\in \mathrm{Im}(g)$. Then as above we get

$$\widehat{e}_{st}=\widehat{e}_{sp}\widehat{e}_{pt}-e_{s't'}-e_{s''t''}\in \mathrm{Im}(g)$$

if $p\in\langle s,t\rangle$.

(5) Suppose that $r_\varrho(s,t)=1$ and (s,t) is 2-ϱ-extremal. Let $s\varrho s'$ and $t\varrho t'$, where $s,t\in I^{(k)}$, $s',t'\in I^{(l)}$, and $\{k,l\}=\{1,2\}$. Then $s'\prec t'$ and $r_\varrho(s',t')=1$. It is easy to check that (s',t') is not 3-ϱ-extremal. If (s',t') is 2-ϱ-extremal and $k<l$ then $\widehat{e}_{st}=g(\beta^*_{st})\in \mathrm{Im}(g)$. If $k>l$ then by the same reason $\widehat{e}_{s't'}\in \mathrm{Im}(g)$. Taking $p\in\langle s,t\rangle$ such that $r_\varrho(s,p)=2$ or $r_\varrho(p,t)=2$ we obtain

$$\widehat{e}_{st}+\widehat{e}_{s't'}=\widehat{e}_{sp}\widehat{e}_{pt}\in \mathrm{Im}(g)$$

by the induction hypothesis and hence $\widehat{e}_{st}\in \mathrm{Im}(g)$.

If (s',t') is not 2-ϱ-extremal then using Lemma 3.6 we prove that $\widehat{e}_{s't'}\in \mathrm{Im}(g)$. Thus again we see that

$$\widehat{e}_{st}=\widehat{e}_{sp}\widehat{e}_{pt}-\widehat{e}_{s't'}\in \mathrm{Im}(g)\,.$$

We have shown that g is an epimorphism. It is easy to check that $g(\Omega)=0$. Thus g induces a K-algebra epimorphism

$$\overline{g}:K(Q,\Omega)=KQ/(\Omega)\to R\,.$$

Now we show that $\overline{g}$ is injective. It is enough to prove that for all $i,j\in I$ we have

$$\dim_K e(i)(KQ/\Omega)e(j)\le \dim_K \widehat{e}_{ii}R\widehat{e}_{jj}\,,$$

where $e(i)$ denotes the idempotent corresponding to the trivial path at $\overline{i}$. As an example consider the case when $r_\varrho(i)=2$, $r_\varrho(j)=1$. Then $\overline{i}$ can be joined to $\overline{j}$ by paths of the following kinds:

(1) $I^{(2)}$-paths,
(2) 2-3-skew paths,
(3) $I^{(3)}$-paths.

Paths of the same kind are equal modulo Ω. Thus $e(i)K(Q,\Omega)e(j)$ has a basis $\mathfrak{B}$ consisting of paths of pairwise different kinds. Moreover, all the kinds (1)–(3) cannot appear in $\mathfrak{B}$ simultaneously. One can check that $g(\mathfrak{B})$ is a linearly independent set and the required inequality holds. The proof in the remaining cases is analogous. ∎

4. A covering for $(Q(I_\varrho^*), \Omega(I_\varrho^*))$. Suppose that I_ϱ is a three-separate stratified poset and I_ϱ^* is its one-peak enlargement (see Section 2). Let

$$I^* = I^{(1)} + I^{(2)} + I^{(3)}$$

be a three-separation of I^*. Note that $* \in I^{(3)}$.

Let $(Q, \Omega) = (Q(I_\varrho^*), \Omega(I_\varrho^*))$ be the bound quiver associated with I_ϱ^* (see (3.3)). Let

$$\begin{aligned} a_i &= \overline{\beta}_{p_i q_i} : \overline{p}_i \to \overline{q}_i, \quad & i &= 1, \ldots, k_1, \\ b_i &= \overline{\beta}_{r_i s_i} : \overline{r}_i \to \overline{s}_i, \quad & i &= 1, \ldots, k_2, \\ d_i &= \overline{\beta}_{t_i u_i} : \overline{t}_i \to \overline{u}_i, \quad & i &= 1, \ldots, k_3, \end{aligned}$$

be all the 1-2-skew, 2-3-skew and 1-3-skew arrows respectively, where $p_i \in I^{(1)}$, $q_j \in I^{(2)}$, $r_i \in I^{(2)}$, $s_j \in I^{(3)}$, $t_i \in I^{(1)}$, $u_j \in I^{(3)}$. Denote by Q^- the quiver obtained from Q by removing all arrows a_i, b_i, d_i, and by Ω^- the set of relations in Ω which do not involve skew arrows.

Let $G = \mathbb{Z}\alpha * \mathbb{Z}\beta$ be the free noncommutative group with two free generators α, β. Following [S1, S4] we define a Galois covering

$$f : (\widetilde{Q}, \widetilde{\Omega}) \to (Q, \Omega) \tag{4.1}$$

with group G as follows.

Let $\widetilde{Q}^{(x)} = Q^- \times \{x\}$ for $x \in G$. We put $j^{(x)} = (j, x)$ and $\gamma_{ij}^{(x)} = (\gamma_{ij}, x)$ where j is a vertex of Q^- and γ_{ij} is an arrow in Q^-. We define $\widetilde{Q}$ to be the disjoint union of $\widetilde{Q}^{(x)}$ over all $x \in G$ connected by the edges

$$\begin{aligned} a_i^{(x)} &: \overline{p}_i^{(x)} \to \overline{q}_i^{(\alpha x)}, \qquad & i &= 1, \ldots, k_1, \\ b_i^{(x)} &: \overline{r}_i^{(x)} \to \overline{s}_i^{(\beta x)}, \qquad & i &= 1, \ldots, k_2, \\ d_i^{(x)} &: \overline{t}_i^{(x)} \to \overline{u}_i^{(\beta\alpha x)}, \qquad & i &= 1, \ldots, k_3 \end{aligned}$$

(see Fig. 4.2). We define f by setting $f(j^{(x)}) = j$ and $f(\gamma_{ij}^{(x)}) = \gamma_{ij}$. We take for $\widetilde{\Omega}$ the natural lift of Ω along f. The group G acts on $\widetilde{Q}$ in the following way:

$$y * j^{(x)} = j^{(yx)}, \quad y * \gamma_{ij}^{(x)} = \gamma_{ij}^{(yx)} \quad \text{for } y \in G\,.$$

We note that f induces a bound quiver isomorphism

$$(\widetilde{Q}/G, \widetilde{\Omega}/G) \simeq (Q, \Omega)\,.$$

In general I_ϱ admits many different three-separations. However, it is easy to see that the isomorphism class of the covering (4.1) does not depend on the choice of the three-separation.

We are especially interested in the case when the covering (4.1) is the universal cover of (Q, Ω). For this purpose we need the following definition.

$\widetilde{Q}$:

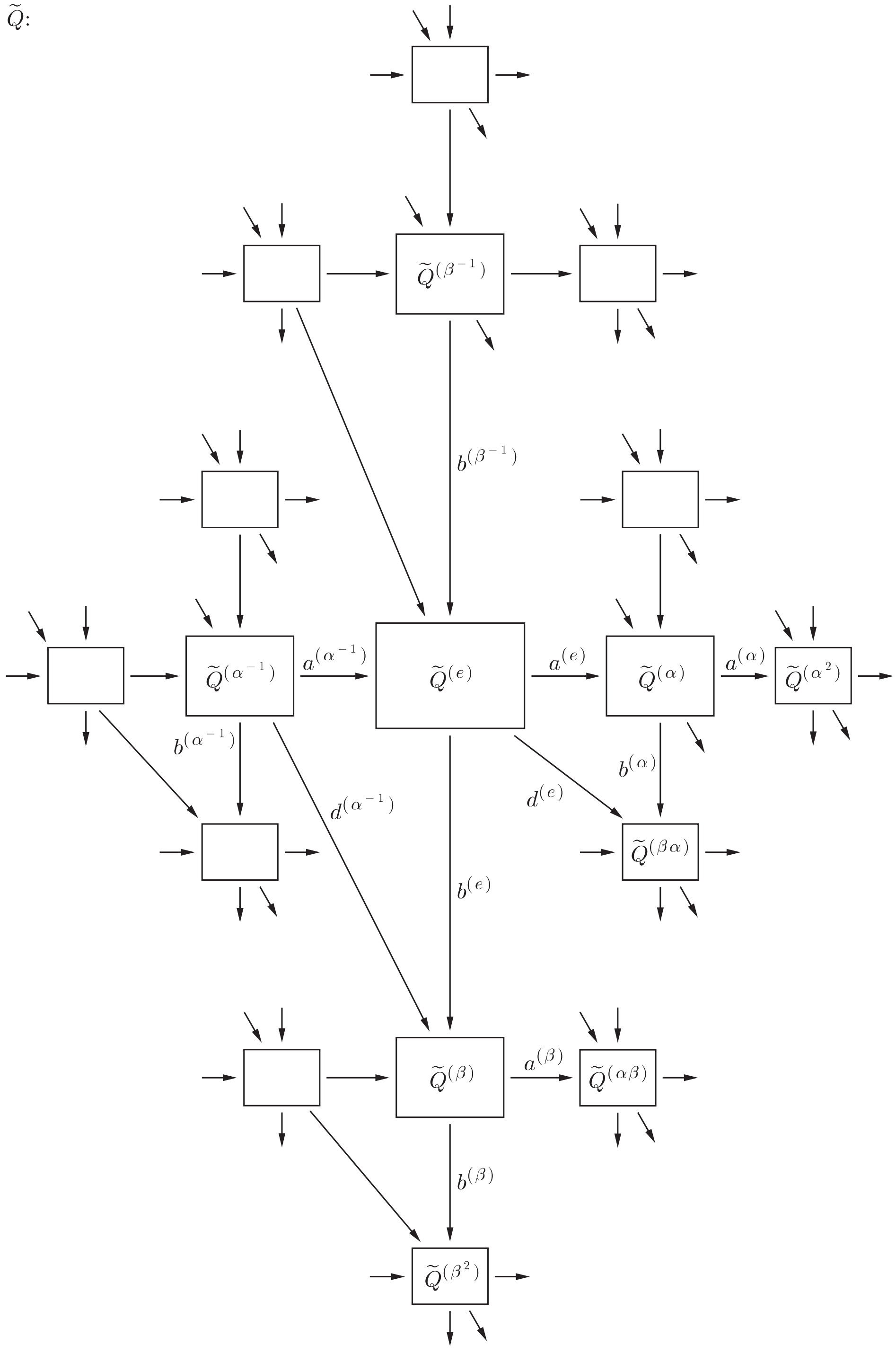


Fig. 4.2

DEFINITION 4.3. We call a three-separate poset I_ϱ a *rib convex poset* if the following hold.

(1) The rib skeleton $\operatorname{rsk}(I_\varrho)$ of I_ϱ has exactly three rib-connected components $\Re_1$, $\Re_2$, $\Re_3$; we assume that $\Re_i \subseteq I^{(i)}$ for $i = 1, 2, 3$.

(2) If $r_\varrho(i) > 1$ then $i \in \operatorname{rsk}(I_\varrho)$.

(3) For any $(i, j) \in \triangle\Re_k$ for some k there exists a rib path from $\overline{i}$ to $\overline{j}$.

PROPOSITION 4.4 (compare [S4, Proposition 3.8]). *Let I_ϱ be a rib convex three-separate poset and $(Q, \Omega) = (Q(I_\varrho^*), \Omega(I_\varrho^*))$ be the bound quiver associated with I_ϱ^** (see (3.3)).

(a) *The fundamental group $\Pi_1(Q, \Omega)$ of (Q, Ω) is a free group with two free generators.*

(b) *The covering $f : (\widetilde{Q}, \widetilde{\Omega}) \to (Q, \Omega)$ defined in* (4.1) *is the universal Galois covering of (Q, Ω).*

Proof. (a) Note that we can assume that the three-separation $I^{(1)} + I^{(2)} + I^{(3)}$ of I_ϱ^* is such that

$$I^{(1)} = \{i \in I : i \preccurlyeq x \text{ for some } x \in \Re_1\},$$
$$I^{(2)} = \{i \in I \setminus I^{(1)} : i \preccurlyeq x \text{ for some } x \in \Re_2\},$$
$$I^{(3)} = I \setminus (I^{(1)} \cup I^{(2)}) \quad \text{and} \quad * \in I^{(3)}.$$

We keep the notation of skew arrows introduced above. Note that the quiver Q^- obtained from Q by removing all the skew arrows has no oriented cycles and has the following property:

$(*_{Q^-})$ *for each vertex $\overline{i} \in Q^-$ there exists an oriented path $\omega : \overline{i} \to *$ in Q^-.*

We denote by Q'' the full subquiver of Q^- consisting of the vertices $\overline{i}$ for $i \in I^{(3)}$, and by Q' the full subquiver of Q^- consisting of the vertices $\overline{i}$ for $i \in I^{(2)} \cup I^{(3)}$. We have quiver embeddings $Q'' \subseteq Q' \subseteq Q^- \subseteq Q$. Note that Q' and Q'' have the property $(*_{Q'})$ and $(*_{Q''})$ respectively and they are closed under taking successors in Q^-.

First we construct a maximal tree $T'' \subseteq Q''$ with the property $(*_{T''})$ by induction on $|Q''_0|$.

If $|Q''_0| = 2$ then we take $T'' = Q''$.

Suppose that if $|Q''_0| < m$ then there exists T'' with the required properties. Let $|Q''_0| = m$ and $\overline{a}$ be a minimal element in Q'' (i.e. a source in Q''). Let T''_+ be the maximal tree in the quiver obtained from Q'' by removing the vertex $\overline{a}$. Let $\overline{\beta}_{at}$ be an arrow in Q'' from $\overline{a}$ to some $\overline{t} \in T''_+$. Then $T'' = T''_+ \cup \{\overline{a}\} \cup \{\overline{\beta}_{at}\}$ is a tree with the required property.

Next, just as above, by induction on $|Q'_0 \setminus Q''_0|$ we construct a maximal tree T' in Q' with the property $(*_{T'})$ and such that $T' \cap Q'' = T''$. Finally, applying an induction on $|Q_0^- \setminus Q_0'|$ we extend T' to a maximal tree T in Q^- having the property $(*_T)$. Note that T is a maximal tree in Q.

Suppose that $(Q^-)_0$ consists of the elements i_k, $k = 0, \ldots, m$, where $i_0 = \overline{*}$. Since Q^- has no oriented cycle, without loss of generality we can suppose that if there exists a directed path from i_k to i_j in Q^- then $k > j$.

(1) We show by induction on k that if $b = \overline{\beta}_{st} \in Q''$ is an arrow beginning at $i_k = \overline{s}$ then $\widehat{b} \in N_\Omega$ (we keep the notation of Lemma 2.5). For $k = 0$ this is obvious. Suppose that for $k < m$ the statement is proved. Let $k = m$ and $s, t \in I^{(3)}$. If $b \in T$ then $\widehat{b} \in N_\Omega$. Suppose that $b \notin T$. Then there exists an arrow $\overline{\beta}_{ss_1}$ in T such that $s_1 \in I^{(3)}$. Consider two sequences $\beta_{ss_1}, \beta_{s_1s_2}, \ldots, \beta_{s_m*}$ and $\beta_{tt_1}, \beta_{t_1t_2}, \ldots, \beta_{t_l*}$ of short pairs in $I^{(3)}$. Then

$$(b\overline{\beta}_{tt_1}\overline{\beta}_{t_1t_2}\ldots\overline{\beta}_{t_l*}, \overline{\beta}_{ss_1}\overline{\beta}_{s_1s_2}, \ldots, \overline{\beta}_{s_m*})$$

is an Ω-contour.

Since $\overline{\beta}_{ss_1} \in T$ and by the induction hypothesis $\widehat{\overline{\beta}}_{t_it_{i+1}}, \widehat{\overline{\beta}}_{s_js_{j+1}} \in N_\Omega$ for $i = 0, \ldots, l$ and $j = 1, \ldots, m$, we get $\widehat{b} \in N_\Omega$. (Here we put $t_{l+1} = s_{m+1} = *$ and $t_0 = t$.)

In particular, we have shown that $\widehat{b} \in N_\Omega$ if b is the ϱ-coset of a rib.

(2) Now we are going to prove that for skew arrows b_p, b_q with $r_p, r_q \in \Re_2$ we have $\widehat{b}_p \equiv \widehat{b}_q$ (modulo N_Ω). By our assumptions on I_ϱ there exist points $x_1, \ldots, x_l \in \Re_2$ and rib paths u_i, v_i for $i = 1, \ldots, l$ as in the figure:

$$\begin{array}{ccccccccccc} & & \overline{x}_1 & & & & \overline{x}_2 & & & \overline{x}_l & & \\ & \swarrow^{u_1} & & \searrow^{v_1} & & \swarrow^{u_2} & & \searrow & \swarrow & & \searrow^{v_l} & \\ \overline{r}_{p_0} & & & & \overline{r}_{p_1} & & & \ldots & & & & \overline{r}_{p_l} \end{array}$$

where $\overline{r}_{p_i}$ is the source of the 2-3-skew arrow $b_{p_i}, p_0 = p$ and $p_l = q$. Denote by w_i an $I^{(3)}$-path from the sink $\overline{s}_{p_i}$ of b_{p_i} to $\overline{*}$ for $i = 1, \ldots, l$. Then $(u_ib_{i-1}w_{i-1}, v_ib_iw_i)$ is an Ω-contour for $i = 1, \ldots, l$. By (1) above we get $\widehat{v}_i, \widehat{u}_i, \widehat{w}_i \in N_\Omega$, hence $\widehat{b}_p\widehat{b}_q^{-1} \in N_\Omega$.

(3) By induction on k we shall show that if $b = \overline{\beta}_{st} \in Q'$ is not a skew arrow and it begins at $i_k = \overline{s}$ then $\widehat{b} \in N_\Omega$. For $k = 0$ this is obvious. Suppose that for $k < m$ the statement is proved. Let $k = m$. Suppose that $b \notin T$. Let $\overline{\beta}_{s's_1}$ be an arrow in $T' \subseteq T$ beginning at i_k and $s', s_1 \in I^{(2)} \cup I^{(3)}$.

If $s = s' \in I^{(3)}$ then $s_1, t \in I^{(3)}$ and we prove the statement as in (1).

Suppose that $s = s' \in I^{(2)}$. Then $s_1, t \in I^{(2)}$. Let

$$\beta_{s_1s_2}, \ldots, \beta_{s_mr_p} \quad \text{and} \quad \beta_{tt_1}, \ldots, \beta_{t_lr_q}$$

be sequences of short pairs in $I^{(2)}$ ending at $r_p, r_q \in \Re_2$ whose ϱ-cosets are the sources of the arrows b_p, b_q respectively. The sinks of these arrows are $\overline{s}_p, \overline{s}_q$. Let u_p and u_q be paths composed of the ϱ-cosets of short pairs in

$I^{(3)}$ connecting $\overline{s}_p$ and $\overline{s}_q$ with $\overline{*}$ respectively. Then

$$(b\overline{\beta}_{tt_1},\dots,\overline{\beta}_{t_l r_q} b_q u_q, \overline{\beta}_{s' s_1}\overline{\beta}_{s_1 s_2},\dots,\overline{\beta}_{s_m r_p} b_p u_p)$$

is an Ω-contour and since $\overline{\beta}_{s' s_1} \in T$, $\widehat{b}_p \equiv \widehat{b}_q$ (modulo N_Ω) and by the induction hypothesis we get $\widehat{b} \in N_\Omega$.

Suppose now that $s \neq s'$. Then $r_\varrho(s) > 1$, hence $s \in \operatorname{rsk}(I_\varrho)$. If $s \in \Re_2$ then, since $b \in Q'$, $t \in I^{(2)}$. We have a path u_p composed of the ϱ-cosets of short pairs from $I^{(2)}$ connecting $\overline{t}$ with the source $\overline{r}_p$ of a skew arrow b_p such that $r_p \in \Re_2$. From the rib convexity of $\Re_2$ we get the existence of a rib path v from $\overline{s}$ to $\overline{r}_p$. Then

$$(vb_p, bu_pb_p)$$

is an Ω-contour and since $\widehat{v} \in N_\Omega$ and by the induction hypothesis $\widehat{u}_p \in N_\Omega$, we get $\widehat{b} \in N_\Omega$.

If $s \in \Re_3$ then $s, t \in I^{(3)}$ and we prove the statement as in (1).

(4) We show that $\widehat{b}_p \equiv \widehat{b}_q$ (modulo N_Ω) for any p, q. Note that it is enough to show that $\widehat{b}_p \equiv \widehat{b}_q$ (modulo N_Ω) if $r_p \notin \Re_2, r_q \in \Re_2$ and $r_p \prec r_q$. Let v be a path composed of the ϱ-cosets of short pairs in $I^{(2)}$ from $\overline{r}_p$ to $\overline{r}_q$, and u_p, u_q be the paths in Q'' composed of the ϱ-cosets of short pairs connecting $\overline{s}_p$ and $\overline{s}_q$ with $\overline{*}$ respectively. Then (b_pu_p, vb_qu_q) is an Ω-contour and by (1) and (3) we get $\widehat{b}_p\widehat{b}_q^{-1} \in N_\Omega$.

(5) We show as in (2) that $\widehat{d}_p \equiv \widehat{d}_r$ (modulo N_Ω) for all 1-3-skew arrows d_p, d_r whose sources are the ϱ-cosets of elements of $\Re_1$; $\widehat{d}_p \equiv \widehat{a}_r\widehat{b}_q$ (modulo N_Ω) for any 1-3-skew arrow d_p, 1-2-skew arrow a_r and 2-3-skew arrow b_q such that the sources of d_p and a_r are the ϱ-cosets of elements of $\Re_1$, and $\widehat{a}_p \equiv \widehat{a}_r$ (modulo N_Ω) for any 1-2-skew paths $\widehat{a}_p, \widehat{a}_r$ whose sources are the ϱ-cosets of elements of $\Re_1$.

(6) We show as in (3) that if $b = \overline{\beta}_{st}$ is an arrow in Q^- then $\widehat{b} \in N_\Omega$.

(7) We show as in (4) that

$$\widehat{d}_p \equiv \widehat{d}_r, \quad \widehat{a}_q \equiv \widehat{a}_r, \quad \widehat{d}_p \equiv \widehat{a}_r\widehat{b}_q \text{ (modulo } N_\Omega)$$

for arbitrary p, q, r. Note that there exists at least one 2-3-skew arrow and at least one 1-3-skew arrow or 1-2-skew arrow.

(8) We show that $\widehat{\beta}^*_{ij} \in N_\Omega$ for any i, j such that the arrow β^*_{ij} exists. There is a rib path u from $\overline{i}$ to $\overline{j}$ and a nonzero path v from $\overline{j}$ to $\overline{*}$ composed of the ϱ-cosets of short pairs in I^*. Then $(uv, \beta^*_{ij}v)$ is an Ω-contour and since $\widehat{u}, \widehat{v} \in N_\Omega$ we get $\widehat{\beta}^*_{ij} \in N_\Omega$ as well.

We have shown that $\widehat{a}_1 N_\Omega$, $\widehat{b}_1 N_\Omega$ generate the group $\Pi_1(Q,\Omega)$ if there exists a 1-2-skew arrow a_1, and $\widehat{d}_1 N_\Omega$, $\widehat{b}_1 N_\Omega$ generate $\Pi_1(Q,\Omega)$ if there exists a 1-3-skew arrow d_1.

(9) Now we prove that $\{\widehat{a}_1 N_\Omega, \widehat{b}_1 N_\Omega\}$ (or $\{\widehat{d}_1 N_\Omega, \widehat{b}_1 N_\Omega\}$) is a set of free generators of $\Pi_1(Q,\Omega)$.

Suppose that a_1 exists. We have to show that no word of the form

$$\kappa = \widehat{a}_1^{s_1} \widehat{b}_1^{t_1} \ldots \widehat{a}_1^{s_l} \widehat{b}_1^{t_l}$$

such that $s_{i+1} \neq 0 \neq t_i$ for $i = 1, \ldots, l-1$ or $s_1 \neq 0$ or $t_l \neq 0$ belongs to N_Ω.

Suppose that $\omega = \lambda_1\omega_1 + \ldots + \lambda_m\omega_m$ is a minimal relation in Ω such that $m \geq 2$. Then all the ω_i have a common sink and a common source. Moreover, ω is a sum of elements of the form $a_1 b a_2$ where $a_1, a_2 \in KQ$ and b is a relation of type (a), (b), (c), (d), (e), (f) or (g) (see (3.3)). Since ω is minimal and $m \geq 2$ we have $\omega = a_1 b a_2$ where $a_1, a_2 \in KQ$ are paths in Q and b is a relation of one of the above types. Thus the following types of Ω-contours are possible:

- (γ_1,γ_2), $(\gamma_1 b_i \gamma_2, \gamma_3 b_j \gamma_4)$, $(\gamma_1 a_i \gamma_2, \gamma_3 a_j \gamma_4)$, $(\gamma_1 d_i \gamma_2, \gamma_3 d_j \gamma_4)$, $(\gamma_1 d_i \gamma_2, \gamma_3 a_j \gamma_4 b_k \gamma_5)$

(induced by relations of type (b)), and

- (γ_i, γ_j),

(induced by relations of types (c) to (g)), where the γ_s denote paths in Q which do not contain skew arrows.

Hence we get the following types of generators of N_Ω:

$$\widehat{\gamma},\quad \widehat{\gamma}_1 \widehat{b}_i^{-1} \widehat{\gamma}_2 \widehat{b}_j \widehat{\gamma}_3,\quad \widehat{\gamma}_1 \widehat{a}_i^{-1} \widehat{\gamma}_2 \widehat{a}_j \widehat{\gamma}_3,\quad \widehat{\gamma}_1 \widehat{d}_i^{-1} \widehat{\gamma}_2 \widehat{d}_j \widehat{\gamma}_3,\quad \widehat{\gamma}_1 \widehat{d}_i \widehat{\gamma}_2 \widehat{b}_j^{-1} \widehat{\gamma}_3 \widehat{a}_k^{-1} \widehat{\gamma}_4\,,$$

where the $\widehat{\gamma}_i$ are elements of the free group $\Pi_1(Q)$ which are words without the letters $\widehat{a}_i$, $\widehat{b}_i$, $\widehat{d}_i$.

Consider the group homomorphism

$$h : \Pi_1(Q) \to \mathbb{Z}a * \mathbb{Z}b$$

defined by $h(\widehat{\gamma}_i) = 1$, $h(\widehat{a}_i) = a$, $h(\widehat{b}_i) = b$, $h(\widehat{d}_i) = ab$. Note that all the generators of N_Ω listed above are contained in $\mathrm{Ker}(h)$. Hence $N_\Omega \subseteq \mathrm{Ker}(h)$. If κ is as above then $h(\kappa) \neq 1$, so $\kappa \notin N_\Omega$.

If there is no 1-2-skew arrow a_i in Q then we prove in a similar way that $\{\widehat{d}_1 N_\Omega, \widehat{b}_1 N_\Omega\}$ freely generates $\Pi_1(Q,\Omega)$. This finishes the proof of (a).

The statement (b) follows from the above considerations and from the construction of the universal cover described in (2.6). Since $\Pi_1(Q,\Omega) = \mathbb{Z}\alpha * \mathbb{Z}\beta$ it is easy to see that the construction in our case coincides with the construction (4.1) applied to $G = \Pi_1(Q,\Omega)$. ∎

5. Three-partite posets and the associated three-peak bound quivers. In this section we discuss some special case of three-separate posets, namely the three-partite posets in the sense of Definition 5.1 below.

If $I_1, I_2 \subseteq I$ are subposets then we write $I_1 < I_2$ if for all $i_1 \in I_1$ and $i_2 \in I_2$ we have $i_1 \prec i_2$. We say that I_1 is *connected* if it is connected with respect to the equivalence relation generated by the following relation:

$$i \prec\succ j \;\Leftrightarrow\; \text{either } i \prec j \text{ or } j \prec i \text{ is a minimal relation in } I\,.$$

DEFINITION 5.1 (compare with [S4, Def. 4.1]). A three-separate poset I^*_ϱ with a three-separation $I^{(1)} + I^{(2)} + I^{(3)}$ and a unique maximal element $*$ is called *three-partite* if

(a) $I^{(k)}$ is the disjoint union of subposets $C^{(k)}$ and $J^{(k)}$ such that $C^{(k)}$ is either empty or it is a chain

$$C^{(k)} : c_1^{(k)} \to c_2^{(k)} \to \ldots \to c_{m_k}^{(k)}$$

for $k = 2, 3$, $I^{(1)} < J^{(2)} < J^{(3)}$ and $C^{(2)} < C^{(3)}$.

(b) The stratified poset I_ϱ is rib-convex.

(c) There exist connected subposets $I_0^{(1)} \subseteq I^{(1)}, I_0^{(2)} \subseteq J_0^{(2)} \subseteq J^{(2)}$, $I_0^{(3)} \subseteq J_0^{(3)} \subseteq J^{(3)}$ and poset isomorphisms $\sigma_1 : I_0^{(1)} \to I_0^{(2)}$, $\sigma_2 : I_0^{(2)} \to I_0^{(3)}$ and $\sigma_3 : J_0^{(2)} \to J_0^{(3)}$ satisfying the following conditions:

(i) σ_2 is the restriction of σ_3 to $I_0^{(2)}$,

(ii) $r_\varrho(i) = 3$ if and only if i belongs to $I_0^{(k)}$ for some $k = 1, 2, 3$, and $r_\varrho(i) = 2$ if and only if i belongs to $J_0^{(k)} \setminus I_0^{(k)}$ for some $k = 2, 3$,

(iii) $(i,j)\varrho(\sigma_1(i), \sigma_1(j))\varrho(\sigma_2\sigma_1(i), \sigma_2\sigma_1(j))$ provided $i \preccurlyeq j$, $i, j \in I_0^{(1)}$, $[i,j] = \{i,j\}$, and $(i,j)\varrho(\sigma_3(i), \sigma_3(j))$ provided $i \preccurlyeq j$, $i,j \in J_0^{(2)}$, $[i,j] = \{i,j\}$.

We visualize this notion in Fig. 5.2.

Following an idea in [S4] we associate with any three-partite stratified poset I^*_ϱ a three-peak bound quiver

$$I_\varrho^{*+\times} = (Q^{+\times}, \Omega^{+\times}) \tag{5.3}$$

defined as follows:

For the quiver $Q^{+\times}$ we take the disjoint union of Q^- (see (4.1)) and two chains:

$$C^+ : c_1^{(3)+} \to c_2^{(3)+} \to \ldots \to c_{m_3}^{(3)+} \to +,$$
$$C^\times : c_1^{(2)\times} \to \ldots \to c_{m_2}^{(2)\times} \to c_1^{(3)\times} \to \ldots \to c_{m_3}^{(3)\times} \to \times,$$

connected by the following arrows (we use the notation from Section 4):

(i) $\beta_{r_i+} : \overline{r}_i \to +$ if r_i and $c_{m_3}^{(3)}$ are unrelated,

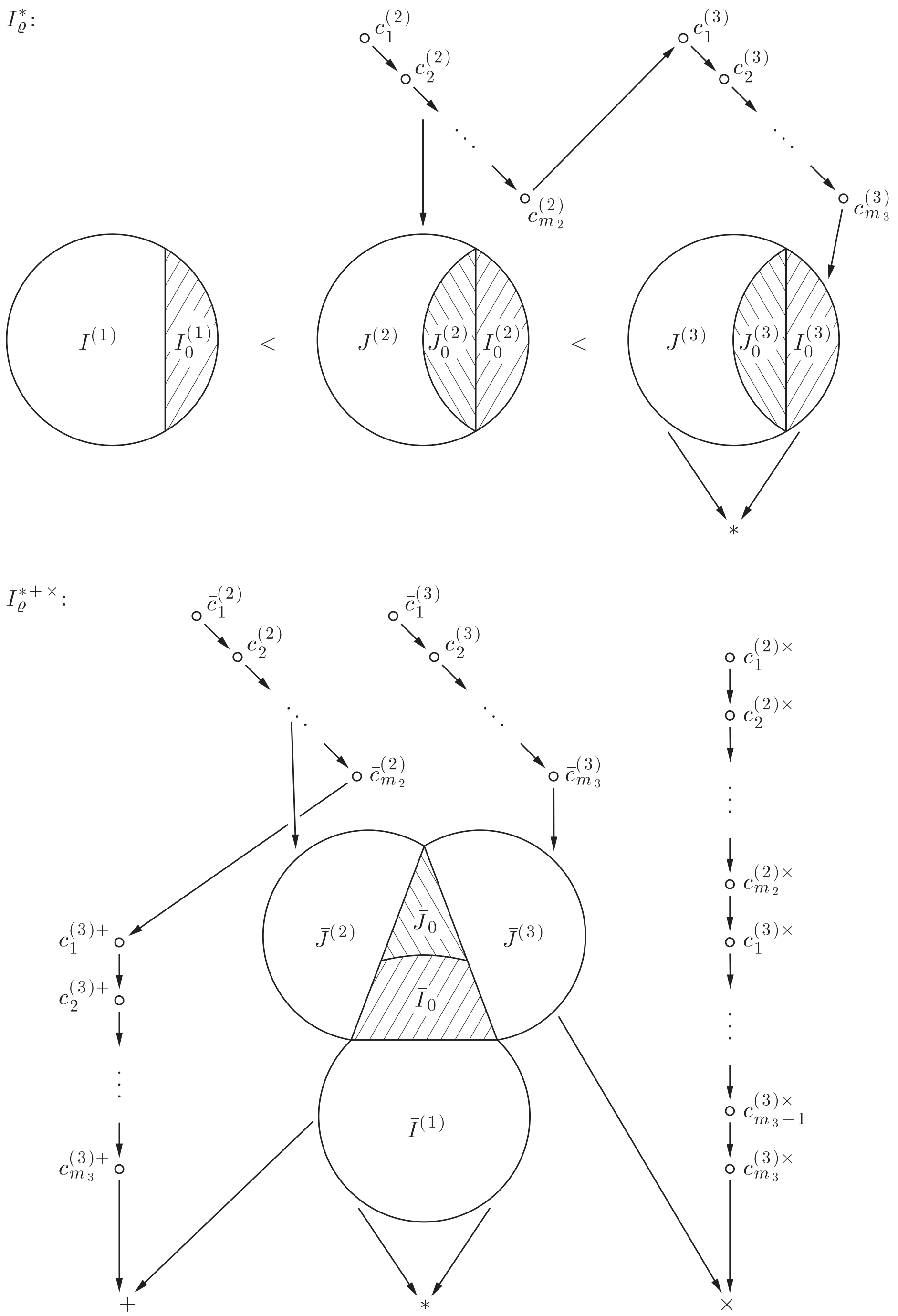

I_ϱ^*:
$c_1^{(2)}$
$c_2^{(2)}$
$c_{m_2}^{(2)}$
$c_1^{(3)}$
$c_2^{(3)}$
$c_{m_3}^{(3)}$
$I^{(1)}$
$I_0^{(1)}$
<
$J^{(2)}$
$J_0^{(2)}$
$I_0^{(2)}$
<
$J^{(3)}$
$J_0^{(3)}$
$I_0^{(3)}$
*
$I_\varrho^{*+\times}$:
$\bar{c}_1^{(2)}$
$\bar{c}_2^{(2)}$
$\bar{c}_{m_2}^{(2)}$
$\bar{c}_1^{(3)}$
$\bar{c}_2^{(3)}$
$\bar{c}_{m_3}^{(3)}$
$c_1^{(2)\times}$
$c_2^{(2)\times}$
$c_{m_2}^{(2)\times}$
$c_1^{(3)\times}$
$c_{m_3-1}^{(3)\times}$
$c_{m_3}^{(3)\times}$
$c_1^{(3)+}$
$c_2^{(3)+}$
$c_{m_3}^{(3)+}$
$\bar{J}^{(2)}$
$\bar{J}_0$
$\bar{J}^{(3)}$
$\bar{I}_0$
$\bar{I}^{(1)}$
+
*
×

Fig. 5.2

(ii) $\beta_{p_i\times}: \overline{p}_i \to \times$ if p_i and $c^{(3)}_{m_3}$ are unrelated and
$\beta_{t_i\times}: \overline{t}_i \to \times$ if t_i and $c^{(3)}_{m_3}$ are unrelated,

(iii) $a^{\times}_i : \overline{p}_i \to \overline{q}^{(\times)}_i$ if $q_i \in C^{(2)}$,

(iv) $b^{+}_i : \overline{r}_i \to \overline{s}^{(+)}_i$ if $s_i \in C^{(3)}$,

(v) $d^{\times}_i : \overline{t}_i \to \overline{u}^{(\times)}_i$ if $u_i \in C^{(3)}$

(see Fig. 5.2).

For $\Omega^{+\times}$ we take the set Ω^{-} with the following additional relations:

(1) $w\beta_{r_i+}$ and wb^{+}_j if w is neither a 3-rib path nor a 2-rib path nor an $I^{(2)}$-path,

(2) $w\beta_{t_i\times}$, $w\beta_{p_i\times}$, $wa^{\times}_i$ and $wd^{\times}_i$ if w is neither a 3-rib path nor an $I^{(1)}$-path,

(3) $wu - vu'$, where $w : \overline{p} \to \overline{r}_i$ is a 3-rib path, a 2-rib path or an $I^{(2)}$-path, and where $v : \overline{p} \to \overline{r}_j$ and u and u' have a common sink in C^{+}.

(4) $wu - vu'$, where $w : \overline{p} \to \overline{p}_i$ is a 3-rib path or an $I^{(1)}$-path, and where $v : \overline{p} \to \overline{p}_j$, u and u' have a common sink in $C^{\times}$.

Analogously to the bipartite case [S4] there is an algebra isomorphism

$$KI^{*+\times}_{\varrho} \simeq \xi\widetilde{R}\xi\,,$$

where $\widetilde{R} = K(\widetilde{Q}, \widetilde{\Omega})$ and

$$\xi = \sum_{t\in Q^{(e)}_0} e_t + \sum_{t\in C^{(2)}} e_{(t,\alpha)} + \sum_{t\in C^{(3)}} (e_{(t,\beta)} + e_{(t,\beta\alpha)}) + e_{(*,\beta)} + e_{(*,\beta\alpha)}$$

and we consider the diagram

$$\text{(5.4)}\qquad \begin{array}{ccccc} \operatorname{mod}_{\rm sp}(\xi\widetilde{R}\xi) & \xrightarrow{L_\xi} & \operatorname{mod}_{\rm sp}(\nu\widetilde{R}\nu) & \xrightarrow{T_\nu} & \operatorname{mod}_{\rm sp}(\widetilde{R}) \\ \uparrow{\scriptstyle\iota} & & & & \downarrow{\scriptstyle f_{\rm sp}} \\ \operatorname{mod}_{\rm sp}(KI^{*+\times}_{\varrho}) & & \xrightarrow[f_{+\times}]{\qquad\qquad} & & \operatorname{mod}_{\rm sp}(R) \end{array}$$

where ι is the natural equivalence, $f_{\rm sp}$ is the covering functor (see [Ga], [S4, 4.20]), $\nu = \sum_t e_t$ where t runs over the set of vertices of the union of all quivers $Q^{(\omega)}_0$ for ω of the form $\omega = \alpha^{s_1}\beta^{t_1}\ldots\alpha^{s_m}\beta^{t_m}$, where $s_i, t_i \ge 0$ for $i = 1,\ldots,m$, L_ξ and T_ν are the lower and upper induction functors respectively (see [S4, S5]), $f_{+\times}$ is the composed functor $f_{\rm sp}\circ T_\nu \circ L_\xi \circ \iota$.

By the Splitting Theorem of [S3], Proposition 4.3 above, Theorem 4.19 and Remark 4.21 of [S4] we get the following.

THEOREM 5.5. *If I^{*}_{ϱ} is a three-partite stratified poset then*:

(a) *The functor* $f_{+\times} : \operatorname{mod}_{\rm sp}(KI^{*+\times}_{\varrho}) \to \operatorname{mod}_{\rm sp}(R)$ *is exact*, *faithful*, *dense and preserves indecomposability.*

(b) *The category* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *is of finite representation type if and only if so is the category* $\operatorname{mod}_{\mathrm{sp}}(R)$.

(c) *If* K *is an algebraically closed field then* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *is of tame* (*resp. wild*) *representation type if and only if* $\operatorname{mod}_{\mathrm{sp}}(R)$ *is of tame* (*resp. wild*) *representation type.* ■

Applying arguments similar to those used in [S4, Proposition 4.9] and Proposition 4.3 above one can prove the following.

THEOREM 5.6. *Let* I_{ϱ}^{*} *be a three-partite poset and let* $I_{\varrho}^{*+\times}$ *be the associated three-peak bound quiver* (5.3).

(a) *The fundamental group* $\Pi_1(I_{\varrho}^{*+\times})$ *is trivial. If in addition every vertex of* $I_{\varrho}^{*+\times}$ *is separating then the Auslander–Reiten quiver* $\Gamma_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *of* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *has a preprojective component.*

(b) *If the Auslander–Reiten quiver* $\Gamma_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *of* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*+\times})$ *has a preprojective component then* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*})$ *is of finite representation type if and only if* $I_{\varrho}^{*+\times}$ *contains no Weichert's critical forms* (see [W]). ■

Let us finish with a simple corollary from the above considerations.

COROLLARY 5.7. *If* I_{ϱ}^{*} *is a three-partite stratified poset and* $\operatorname{mod}_{\mathrm{sp}}(KI_{\varrho}^{*})$ *is of finite representation type then* I_{ϱ} *does not contain any rib.*

Proof. It is easy to check that if I_{ϱ} contains a rib then $I_{\varrho}^{*+\times}$ contains a subquiver of type $\widetilde{\mathbb{D}}_4$ which is of infinite representation type. Thus the statement follows from Theorem 5.6 above. ■

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