Bound quivers of three-separate stratified posets, their Galois coverings and socle projective representations

by

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Abstract. A class of stratified posets $I^*$ are investigated and their incidence algebras $KI^*$ are studied in connection with a class of non-shurian vector space categories. Under some assumptions on $I^*$ we associate with $I^*$ a bound quiver $(Q, \Omega)$ in such a way that $KI^* \simeq K(Q, \Omega)$. We show that the fundamental group of $(Q, \Omega)$ is the free group with two free generators if $I^*$ is rib-convex. In this case the universal Galois covering of $(Q, \Omega)$ is described. If in addition $I^*$ is three-partite a fundamental domain $I^{*+\times}$ of this covering is constructed and a functorial connection between $\text{mod}_{sp}(KI^{*+\times})$ and $\text{mod}_{sp}(KI^*)$ is given.

1. Introduction. Socle projective representations of stratified posets introduced in [S1, S2] (see Definition 2.1 below) appear in a natural way in the study of vector space categories (see [S2], [S5, Chap. 17]) and lattices over orders (see [S5, Ch. 13], [S4]). The aim of this paper is to give some tools for studying these representations for a certain class of stratified posets.

Our main points of interest are the incidence algebra $KI^*$ over a field $K$ of a three-separate stratified poset $I^*$ with a unique maximal element $*$ (see Definition 3.1) and the representation type of the category $\text{mod}_{sp}(KI^*)$ of socle projective right $KI^*$-modules. Following [S1, S2, S4] we associate with any such poset $I^*$ a bound quiver

$$(Q(I^*), \Omega(I^*))$$

in such a way that $KI^*$ is isomorphic to the bound quiver algebra $KQ(I^*)/\Omega(I^*)$. Under the assumption that $I^*$ is rib-convex (see Section 4) we show that the fundamental group $\Pi_1(Q(I^*), \Omega(I^*))$ is a free noncommutative group with two free generators and we give an explicit description of the universal covering $(\tilde{Q}, \tilde{\Omega})$ of $(Q(I^*), \Omega(I^*))$. If in addition $I^*$ is three-partite we define, by means of $(\tilde{Q}, \tilde{\Omega})$, a simply connected [AS]

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finite-dimensional three-peak algebra $KI^{++}_e$ and a functor

$$f_{++} : \text{mod}_{sp}(KI^{++}_e) \to \text{mod}_{sp}(KI^*_e)$$

preserving the representation type. In the case when the Auslander–Reiten quiver $I_{sp}(KI^{++}_e)$ of $\text{mod}_{sp}(KI^{++}_e)$ has a preprojective component we get a simple criterion for the finite representation type of $\text{mod}_{sp}(KI^*_e)$ (see Theorems 5.5, 5.6). In particular, we solve a problem stated in [S4, Remark 4.33].

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2. Preliminaries and notation. We consider a poset $I$ with partial order $\prec$. We suppose that $I = \{1, \ldots , n\}$ and if $i \prec j$ then $i \leq_N j$ for $i,j \in I$. Define

$$\Delta I := \{(i,j) : i,j \in I \text{ and } i \prec j\},$$

$$\triangle I := \{(i,j) : i,j \in I \text{ and } i \prec s \prec j\}.$$ Given $(i,j) \in \Delta I$ we put $[i,j] := \{s \in I : i \prec s \prec j\}$ and $\langle i,j \rangle := \{s \in I : i \prec s \prec j\}$. Throughout we identify $(i,i)$ with $i$.

Definition 2.1 [S2, S4]. A stratification of $I$ is an equivalence relation $\varrho$ on $\Delta I$ such that if $(i,j) \varrho (p,q)$ then there exists a poset isomorphism $\sigma : [i,j] \to [p,q]$ such that $(i,t) \varrho (p,\sigma(t))$ and $(t,j) \varrho (\sigma(t),q)$ for any $t \in [i,j]$. A stratified poset is a pair

$$I_{\varrho} = (I, \varrho)$$

where $I$ is a poset and $\varrho$ is a stratification of $I$.

We denote by $r_{\varrho}(i,j)$ the cardinality of the $\varrho$-coset of $(i,j)$, and call $(i,j)$ a rib if $r_{\varrho}(i,j) > 1$ and $i \neq j$. The number $r_{\varrho}(i,j)$ is then the rib rank of the rib $(i,j)$.

The full stratified subposet $\text{rsk}(I_{\varrho})$ of $I_{\varrho}$ consisting of all beginnings and ends of ribs in $I_{\varrho}$ is called the rib skeleton of $I_{\varrho}$. We fix a decomposition

$$\text{rsk}(I_{\varrho}) = \mathbb{R}_1 + \ldots + \mathbb{R}_h$$

into rib-connected components with respect to the rib-equivalence relation generated by the following relation:

$$i \sim j \iff \text{either } (i,j) \text{ or } (j,i) \text{ is a rib.}$$

Fix a field $K$ and a stratified poset $I_{\varrho}$. We recall from [S4] that the $K$-algebra

$$KI_{\varrho} = \{b = (b_{pq}) \in M_{n \times n}(K) : b_{pq} = 0 \text{ if } p \neq q \text{ and } b_{ij} = b_{pq} \text{ if } (i,j) \varrho (p,q)\}$$

is called the incidence algebra of $I_{\varrho}$. 
We denote by \( I^* = I \cup \{ \ast \} \) the enlargement of \( I \) by adjoining a unique maximal element \( \ast \) (called the peak) and we extend trivially the relation \( \bar{\rho} \) from \( \Delta I \) to \( \Delta I^* \).

Thus we get a right peak algebra (see [S4]) of the form

\[
(2.3) \quad KI_{\bar{\rho}}^* = \begin{pmatrix} KI_{\bar{\rho}} & M \\ 0 & K \end{pmatrix}
\]

where

\[
M = \begin{pmatrix} K \\ \vdots \\ K \end{pmatrix}_n
\]

is a left \( KI_{\bar{\rho}} \)-module with respect to the usual matrix multiplication.

For a more detailed discussion of stratified posets, examples and applications the reader is referred to [S2] and [S5, Section 17.16].

In Section 3 below we will use the notion of the fundamental group of a quiver \( Q \) with a set of relations \( \Omega \) ([Gr, MP]). For the convenience of the reader we briefly recall this concept. We follow [S4].

With a connected quiver \( Q \) we associate its fundamental group \( \Pi_1(Q, q) \) computed as the group of homotopy classes \( [\omega] \) of walks \( \omega \) in \( Q \) starting and ending at the fixed point \( q \). By a walk we mean a formal composition \( \alpha_1 \ldots \alpha_r \) where \( \alpha_p \) is an arrow of \( Q \) or its formal inverse and the sink of \( \alpha_p \) is the source of \( \alpha_{p+1} \). Homotopy is the smallest equivalence relation \( \approx \) (on the set of walks) such that:

1. \( 1_x \approx 1_x^{-1} \) for each vertex \( x \) of \( Q \),
2. \( \alpha \alpha^{-1} \approx 1_x \) and \( \alpha^{-1} \alpha \approx 1_y \) for each arrow \( \alpha : x \to y \),
3. if \( w \approx v \) then \( uw \approx uv \) and \( vu \approx vu \) whenever the walks involved are composable.

By the fundamental group of a bound quiver \( (Q, \Omega) \) we mean the group

\[
(2.4) \quad \Pi_1(Q, \Omega) = \Pi_1(Q, q)/N_{\Omega},
\]

where \( N_{\Omega} \) is the normal subgroup generated by the conjugacy classes \( C(u, v) \) of homotopy classes \( [w^{-1}u^{-1}vw] \) in \( \Pi_1(Q, q) \) where \( u, v \) are directed paths with a common sink and a common source, and there is a minimal relation

\[
\omega = \lambda_1\omega_1 + \ldots + \lambda_t\omega_t \in (\Omega), \quad \lambda_i \in K^*\,
\]

with \( t \geq 2 \) and \( u = \omega_1, v = \omega_2 \). Let us recall from [MP] that a relation \( \omega \) of the above form is a minimal relation if for every nonempty proper subset \( J \subset \{1, \ldots, t\} \) we have

\[
\sum_{j \in J} \lambda_j \omega_j \not\in (\Omega).
\]
The following maximal tree lemma is a very useful method of computing the fundamental group. Before we formulate it we recall from [S4] that by an $\Omega$-contour we mean a pair $(u, v)$ of oriented paths with a common sink and a common source such that there is a minimal relation $\omega$ of the above form with $hug = \omega_1$ and $hvg = \omega_2$ for some oriented paths $h, g$ such that the sink of $h$ is the source of $u$ and the source of $g$ is the sink of $u$. We say that $(u, v)$ is defined with respect to the set $\Omega' \subseteq (\Omega)$ if $\omega \in \Omega'$.

**Lemma 2.5** [S4, Remark 3.6, Lemma 3.7]. Suppose that $(Q, \Omega)$ is a bound quiver, let $T$ be a maximal tree in $Q$ and $q \in Q$.

(a) $N_\Omega$ is generated by the elements $C(u, v)$, where $(u, v)$ runs through all the $\Omega$-contours defined with respect to a fixed set of generators of the ideal $(\Omega)$.

(b) $\Pi_1(Q, q)$ is a free group generated by the elements $\hat{\beta} = [a\beta b]$ where $\beta \in Q_1 \setminus T$ and $a, b$ are walks in $T$ connecting $q$ with the sink and the source of $\beta$, respectively.

(c) If $(u, v)$ is an $\Omega$-contour and
\[ u = u_0\beta_1u_1\beta_2\ldots u_{s-1}\beta_su_s, \quad v = v_0\gamma_1v_1\gamma_2\ldots v_{r-1}\gamma rv_r, \]
where $\beta_i, \gamma_j \in Q_1 \setminus T$ and $u_i$ and $v_j$ are oriented paths in $T$ then
\[ \hat{\beta}_1\hat{\beta}_2\ldots\hat{\beta}_s \equiv \hat{\gamma}_1\hat{\gamma}_2\ldots\hat{\gamma}_r \mod N_\Omega. \]

If the fundamental group of $(Q, \Omega)$ is nontrivial we construct the universal Galois covering

\[ f: (\tilde{Q}, \tilde{\Omega}) \to (Q, \Omega) \]

of $(Q, \Omega)$ in the following way (see [MP, Corollary 1.5], [Gr]).

Fix $q \in Q$. Let $W$ be the topological universal cover of $Q$, i.e. a quiver $W$ whose vertices are the homotopy classes $[\omega]$ of walks $\omega$ in $Q$ starting at a fixed point $p ([Sp])$. There is an arrow $(\alpha, [\omega])$ from $[\omega]$ to $[\nu]$ in $W$ if $[\nu] = [\omega\alpha]$ for an arrow $\alpha$ in $Q$. $N_\Omega$ acts on $W$ in an obvious way. We take for $\tilde{Q}$ the orbit quiver $W/N_\Omega$ and for $\tilde{\Omega}$ the set of liftings of relations in $\Omega$ from $KQ$ to $K\tilde{Q}$. The bound quiver map $f$ is defined by

\[ f(N_\Omega(\alpha, [\omega])) = \alpha, \quad f(N_\Omega[\omega]) = \text{the sink of } \omega, \]

where $N_\Omega[\omega]$ (resp. $N_\Omega(\alpha, [\omega])$) denotes the orbit of $[\omega]$ (resp. $(\alpha, [\omega])$).

The group $\Pi_1(Q, \Omega)$ acts naturally on $(\tilde{Q}, \tilde{\Omega})$ as a group of automorphisms. One can check that $f$ is the universal Galois covering with group of automorphisms $\Pi_1(Q, \Omega)$ (see [Gr, MP]).

3. Three-separate stratified posets and the associated bound quivers. Let us start with our main definition which extends that given in [S1, S4].
Definition 3.1. A three-separate stratified poset is a stratified poset $I_\rho$ such that $I$ is the disjoint union of subsets $I^{(1)}, I^{(2)}, I^{(3)}$ and the following conditions hold:

(a) There is no relation $i \prec j$, where $i \in I^{(k)}, j \in I^{(l)}$ and $k > l$.

(b) $r_\rho(i,j) \leq 3$ for all $(i,j) \in \triangle I$.

(c) If $(i,j)\rho(s,t)$ and $(i,j) \neq (s,t)$ then there exist $k,l \leq 3$ such that $k \neq l$, $i,j \in I^{(k)}$ and $s,t \in I^{(l)}$.

(d) If $r_\rho(i,j) = 2$ then $i,j \notin I^{(1)}$.

We say that the decomposition $I = I^{(1)} + I^{(2)} + I^{(3)}$ is a three-separation of $I_\rho$.

We call a rib of rank 3 a 3-rib and a rib of rank 2 a 2-rib. A pair $(i,j) \in \triangle I$ is called short if $\{i,j\} = [i,j]$. In this case we write $\beta_{ij}$ instead of $(i,j)$. A pair $(i,j)$ is called 3-$\rho$-extremal if it is not short, $r_\rho(i,j) \leq 2$ and $(i,s),(s,j)$ are 3-ribs for all $s$ such that $i \prec s \prec j$. A pair $(i,j)$ is called 2-$\rho$-extremal if it is neither short nor 3-$\rho$-extremal, $r_\rho(i,j) = 1$ and $(i,s),(s,j)$ are ribs for all $s$ such that $i \prec s \prec j$. We say that $(i,j)$ is $\rho$-extremal if it is either 2-$\rho$-extremal or 3-$\rho$-extremal.

Example 3.2. Let $I^*$ be the following poset:

```
 3   1
\downarrow \downarrow
6   4 \rightarrow 2
\downarrow \searrow \downarrow
9   7   5
\downarrow \swarrow \downarrow
10  8
\downarrow
11
\downarrow
*
```

and $\rho$ be the relation given by

\begin{align*}
1&\rho 2, \\
(3,6)&\rho(4,7)\rho(5,8), \\
(6,9)&\rho(7,10)\rho(8,11), \\
(4,10)&\rho(5,11).
\end{align*}

Then $I_\rho^*$ is a three-separate poset with three-separation $I = I^{(1)} + I^{(2)} + I^{(3)}$, where

$I^{(1)} = \{3,6,9\}$, $I^{(2)} = \{1,4,7,10\}$, $I^{(3)} = \{2,5,8,11,*\}$.

The pairs $(3,9)$, $(4,10)$ and $(5,11)$ are 3-$\rho$-extremal.
We associate with $I_\varrho$ the bound quiver
\begin{equation}
(Q(I_\varrho), \Omega(I_\varrho))
\end{equation}
as follows. The set $(Q(I_\varrho))_0$ of vertices of $Q(I_\varrho)$ is the set
\[
I/\varrho = \{ I, \varrho, \ldots, \varrho^n \}
\]
of the $\varrho$-cosets of elements $q \in I$. We have the following arrows in $Q(I_\varrho)$.

(i) If $(i, j)$ is short then the $\varrho$-coset $\beta_{ij}$ of $\beta_{ij}$ is a unique arrow from $\overline{i}$ to $\overline{j}$.

(ii) If $(i_k, j_k) \in \Delta I^{(k)}$ are $3$-$\varrho$-extremal for $k = 1, 2, 3, i_1 g_2 g_3, j_1 g_2 j_3$ and $r_\varrho(i_k, j_k) = 1$ for $k = 1, 2, 3$ then we have exactly two arrows $\beta_{i_1j_1}^*, \beta_{i_2j_2}^*$:
\[
\overline{i_1} \rightarrow \overline{j_1}, \quad \overline{i_2} \rightarrow \overline{j_2}.
\]

A directed path $\omega$ in $Q(I_\varrho)$ is called a rib path if $\omega$ is a composition of arrows which are the $\varrho$-cosets of ribs in $I_\varrho$. It is called a $3$-rib path if it is a composition of the $\varrho$-cosets of 3-ribs in $I_\varrho$. A path $\omega$ is called a 2-rib path if it is not a 3-rib path and it is a composition of $\varrho$-cosets of 3-ribs and 2-ribs in $I_\varrho$. A path $\omega$ is called a nonrib path if it is not a rib path. A nonrib path is called an $I^{(k)}$-path if it is a composition of arrows $\beta_{ij}$ with $i, j \in I^{(k)}$, where $\beta_{ij}$ denotes either $\beta_{ij}^*$ or $\beta_{ij}^s$. An arrow $\beta_{ij}$ is called a 1-2-skew (resp. 2-3-skew, 1-3-skew) if $i \in I^{(1)}$ and $j \in I^{(2)}$ (resp. $i \in I^{(2)}$ and $j \in I^{(3)}$; $i \in I^{(1)}$ and $j \in I^{(3)}$). A directed path $\omega$ in $Q$ is called 1-2-skew (resp. 2-3-skew, 1-3-skew) if $\omega$ contains a 1-2-skew arrow (resp. contains a 2-3-skew arrow; either contains a 1-3-skew arrow, or contains a 1-2-skew arrow and a 2-3-skew arrow).

We define the set of relations $\Omega = \Omega(I_\varrho)$ to consist of the following elements of the path algebra $KQ(I_\varrho)$:

(a) $\beta_{i_1j_1} \beta_{i_2j_2} \ldots \beta_{i_rj_r}$ if there is no sequence $\beta_{i_0t_1}, \beta_{i_1t_2}, \ldots, \beta_{i_{r-1}t_r}$ such that $(i_k, j_k) g(t_{k-1}, t_k)$ for $k = 1, \ldots, r$. (Recall that $\beta_{ij}$ is either $\beta_{ij}^*$ or $\beta_{ij}^s$.)

(b) $\beta_{i_0t_1} \beta_{i_1t_2} \ldots \beta_{i_{r+1}j_{r+1}} - \beta_{j_0t_1} \beta_{j_1t_2} \ldots \beta_{j_{s+1}t_{r+1}}$, where $i_0 = j_0, i_{r+1} = j_{s+1}, i_0 < i_1 < \ldots < i_r < i_{r+1}, j_0 < j_1 < \ldots < j_s < j_{s+1}$ and there exist $p, q$ such that $(i_p, i_{p+1})$ and $(j_q, j_{q+1})$ are not ribs.
(c) \( w - u \) for all 3-rib paths (resp. 2-rib paths) \( w \) and \( u \) with a common sink and a common source.

(d) \( w - w_1 - w_2 - w_3 \), where \( w \) is a 3-rib path, \( w_k \) is an \( I^{(k)} \)-path for \( k = 1, 2, 3 \) and \( w, w_1, w_2, w_3 \) have a common sink and a common source.

(e) \( w - u \) for all \( I^{(k)} \)-paths \( w, u \) with a common sink and a common source for \( k = 1, 2, 3 \).

(f) \( w - u_2 - u_3 \), where \( w \) is a 2-rib path, \( u_k \) is an \( I^{(k)} \)-path for \( k = 2, 3 \) and \( w, u_2, u_3 \) have a common sink and a common source.

(g) \( w - w' - u \) where \( w \) is a 3-rib path, \( w' \) is a 2-rib path, \( u \) is an \( I^{(1)} \)-path and \( w, w', u \) have a common sink and a common source.

In our example we have:

\[
\Omega(I^*_o) = \{ \beta_{42} \beta_{14}, \beta_{25} \beta_{39}, \beta_{14} \beta_{39}, \beta_{25} \beta_{39}, \beta_{14} \beta_{39} \beta_{11}, \\
\beta_{39} \beta_{10,5}, \beta_{39} \beta_{94}, \beta_{39} \beta_{39} \beta_{10,5}, \beta_{39} \beta_{10,5} - \beta_{39} \beta_{25}, \\
\beta_{39} \beta_{39} \beta_{94} - \beta_{39} \beta_{69} \beta_{39} \}. \]

Consider the \( K \)-algebra homomorphism

\[
(3.4) \quad g : KQ(I_0) \rightarrow KI_0
\]

defined by the formulas (compare with [S4]):

\[
g(\bar{i}) = \begin{cases} 
 e_{ii} & \text{if } r_\bar{e}(i) = 1, \\
 e_{ii} + e_{i'i''} & \text{if } r_\bar{e}(i) = 2, \ i \neq i' \neq i'' \neq i, \\
 e_{ii} + e_{i'i''} + e_{i'i''} & \text{if } i_\bar{e} i' i'' \neq i \neq i' \neq i'' \neq i.
\end{cases}
\]

\[
g(\bar{\beta}_{ij}) = \begin{cases} 
 e_{ij} & \text{if } r_\bar{e}(i, j) = 1, \\
 e_{ij} + e_{i'j'} & \text{if } r_\bar{e}(i, j) = 2, \ (i, j) \neq (i', j') \text{ and } (i, j) \neq (i', j'), \\
 e_{ij} + e_{i'j'} + e_{i'j''} & \text{if } (i, j) \neq (i', j') \neq (i'', j'') \neq (i, j) \text{ and } (i, j) \neq (i', j') \neq (i'', j'') \neq (i, j).
\end{cases}
\]
and

\[ g(\beta^*_i) = e_{ij} \]

where \( e_{ij} \) denotes the matrix with 1 in the \((i,j)\)-entry and zeros elsewhere.

A connection between \((Q(I_\emptyset), \Omega(I_\emptyset))\) and \(I_\emptyset\) is given by the following proposition (compare with [S4, Proposition 2.8]).

**Proposition 3.5.** Let \( I_\emptyset \) be a three-separate stratified poset with a three-separation \( I^{(1)} + I^{(2)} + I^{(3)} \). If \((Q(I_\emptyset), \Omega(I_\emptyset))\) is the bound quiver of \( I_\emptyset \) (see (3.3)) then the homomorphism \( g \) of (3.4) induces a \( K \)-algebra isomorphism

\[ \mathfrak{g}: K(Q(I_\emptyset), \Omega(I_\emptyset)) \rightarrow KI_\emptyset, \]

where \( K(Q(I_\emptyset), \Omega(I_\emptyset)) = KQ(I_\emptyset)/\Omega(I_\emptyset) \).

For the proof we will need the following technical lemma.

**Lemma 3.6.** Suppose \((s, t) \in \Delta I^{(k)}, (s', t') \in \Delta I^{(l)}, k \neq l, sgs' \text{ and } tgt' \).

(a) If \((s', t')\) is not 3-\(q\)-extremal and \((s, t)\) is 3-\(q\)-extremal then there exists a sequence \(s_0 < s_1 < \ldots < s_r\), where \(s_0 = s', s_r = t'\), the pair \((s_i, s_{i+1})\) is short for any \(i = 0, \ldots, r - 1\), and there exists \(i = 0, \ldots, r - 1\) such that there is no relation \((s_i, s_{i+1})\rho(u, v)\) with \((u, v) \in \Delta I^{(k)}\).

(b) If \(k, l \neq 1\), \((s', t')\) is not 2-\(q\)-extremal and \((s, t)\) is 2-\(q\)-extremal then there exists a sequence \(s_0 < s_1 < \ldots < s_r\), where \(s_0 = s', s_r = t'\), the pair \((s_i, s_{i+1})\) is short for any \(i = 0, \ldots, r - 1\), and there exists \(i = 0, \ldots, r - 1\) such that \(r_\emptyset(s_i, s_{i+1}) = 1\).

**Proof.** We will prove (a); the proof of (b) is similar. Let

\[ s_0 < s_1 < \ldots < s_r \]

be a sequence such that \(s_0 = s', s_r = t'\), the pair \((s_i, s_{i+1})\) is short for any \(i = 0, \ldots, r - 1\), and for some \(i = 1, \ldots, r - 1\) we have \(r_\emptyset(s', s_i) < 3\) or \(r_\emptyset(s_i, t') < 3\). The existence of such a sequence is obvious. Assume that for any \(i = 0, \ldots, r - 1\) there exist \((u, v) \in \Delta I^{(k)}\) such that \((s_i, s_{i+1})\rho(u, v)\). Then it is easy to construct a sequence

\[ s'_0 < s'_1 < \ldots < s'_r \]

such that \(s'_0 = s, s'_r = t\) and for any \(i = 0, \ldots, r\) we have \(s'_i \rho s_i\). But it follows from 3-\(q\)-extremality of \((s, t)\) that for any \(i = 1, \ldots, r - 1\) we have \(r_\emptyset(s, s'_i) = 3\) and \(r_\emptyset(s'_i, t) = 3\). This implies that for any \(i = 1, \ldots, r - 1\) we have \(r_\emptyset(s'_i, s_i) = 3\) and \(r_\emptyset(s_i, t') = 3\), a contradiction. \(\blacksquare\)

**Proof of Proposition 3.5.** We set \((Q, \Omega) = (Q(I_\emptyset), \Omega(I_\emptyset))\) and \(R = KI_\emptyset\). Note that the idempotents \(\hat{e}_i := g(\hat{e})\), \(i \in I^s\), form a complete set of primitive orthogonal idempotents of \(R\). Moreover, the matrices \(\hat{e}_{ij},\)
\[
i \preceq j \preceq \ast, \text{ defined as follows:}
\]
\[
\hat{e}_{ij} = \begin{cases} 
e_{ij} & \text{if } r_\beta(i, j) = 1, \\
e_{ij} + e_{i'j'} & \text{if } r_\beta(i, j) = 2, (i, j) \not\in (i', j'), \\
e_{ij} + e_{i'j'} + e_{i''j''} + e_{i''j'} & \text{if } (i, j) \not\in (i', j') \text{ and } (i', j') \not\in (i'', j''), \\
e_{ij} + e_{i'j'} + e_{i''j''} & \text{if } (i, j) \not\in (i', j') \text{ and } (i', j') \not\in (i'', j'') \not\in (i, j).
\end{cases}
\]
form a \(K\)-basis of \(R\). We shall show that \(\hat{e}_{st} \in \text{Im}(g)\) for \((s, t) \in \Delta I\). This is obvious if \(s = t\). Assume that \(s \neq t\). We proceed by induction on \(m_{st} := |\{s, t\}|\).

(1) If \(m_{st} = 0\), i.e. \((s, t)\) is short then \(\hat{e}_{st} = g([s, t]) \in \text{Im}(g)\).

Assume that \(m > 0\) and \(\hat{e}_{st} \in \text{Im}(g)\) for \((s, t) \in \Delta I\) such that \(m_{st} < m\). Suppose that \(m_{st} = m\).

(2) If \((s, t)\) is not \(g\)-extremal then there exists \(p \in \langle s, t \rangle\) such that \(r_\beta(s, p) = r_\beta(s, t)\) or \(r_\beta(p, t) = r_\beta(s, t)\). Then \(\hat{e}_{st} = \hat{e}_{sp} \hat{e}_{pt}\) and since by the induction hypothesis \(\hat{e}_{sp}, \hat{e}_{pt} \in \text{Im}(g)\) we get \(\hat{e}_{st} \in \text{Im}(g)\).

(3) Suppose that \(r_\beta(s, t) = 2\) and \((s, t)\) is 3-\(g\)-extremal. Then there exist \(s', t' \in I^{(1)}\) such that \(s' \prec s\) and \(t' \prec t\). It is easy to see that \(s' \prec t'\). If \((s', t')\) is not 3-\(g\)-extremal then it follows from Lemma 3.6 and (1) that \(\hat{e}_{s't'} \in \text{Im}(g)\).

Indeed, we take a sequence \(s_0 \prec s_1 \prec \ldots \prec s_r\) such that \(s_0 = s, s_r = t\), the pairs \((s_j, s_{j+1})\) are short for \(j = 0, \ldots, r - 1\) and there is no relation \((s_i, s_{i+1}) \not\in \text{Im}(g)\) with \(u, v \in I^{(2)} \cup I^{(3)}\), for some \(i = 0, \ldots, r - 1\). Since \(s', t' \in I^{(1)}\) we get \(r_\beta(s_i, s_{i+1}) = 1\) for some \(i = 0, \ldots, r - 1\). Then
\[
\hat{e}_{s't'} = \hat{e}_{s_0s_1} \hat{e}_{s_1s_2} \ldots \hat{e}_{s_{r-1}s_r}.
\]
The right side of this equality belongs to \(\text{Im}(g)\) by (1). Thus \(\hat{e}_{s't'} \in \text{Im}(g)\).

If \((s', t')\) is 3-\(g\)-extremal then \(\hat{e}_{s't'} = g(\beta_{s't'}) \in \text{Im}(g)\) as well. Since by the induction hypothesis we have \(\hat{e}_{sp} \hat{e}_{pt} \in \text{Im}(g)\), where \(p \in \langle s, t \rangle\), we conclude that
\[
\hat{e}_{st} = \hat{e}_{sp} \hat{e}_{pt} - \hat{e}_{s't'} \in \text{Im}(g).
\]

(4) Suppose that \(r_\beta(s, t) = 1\) and \((s, t)\) is 3-\(g\)-extremal. Let \(s\)gs'gs'' and \(t\)gt'gt'', where \(s, t \in I^{(k)}, s', t' \in I^{(l)}, s'', t'' \in I^{(n)}\), and \(k, l, n\) are pairwise different. It is easy to check that \(s' \prec t'\) and \(s'' \prec t''\). Consider the following cases.

(a) If both \((s', t')\) and \((s'', t'')\) are 3-\(g\)-extremal and \(k \neq 3\) then \(\hat{e}_{st} = g(\beta_{st}) \in \text{Im}(g)\). If \(k = 3\) then by the same argument (since \(l, n \neq 3\)) we get \(\hat{e}_{s't'}, \hat{e}_{s''t''} \in \text{Im}(g)\). By the induction hypothesis for any \(p \in \langle s, t \rangle\) we have
\[
\hat{e}_{st} + \hat{e}_{s't'} + \hat{e}_{s''t''} = \hat{e}_{sp} \hat{e}_{pt} \in \text{Im}(g)
\]
and hence we conclude that \(\hat{e}_{st} \in \text{Im}(g)\).
(b) Suppose that \((s', t')\) is 3-\(\rho\)-extremal but \((s'', t'')\) is not. If \(k < l\) then 
\[ \hat{e}_{st} = g(\beta^*_{st}) \in \text{Im}(g). \]
If \(k > l\) then by the same reason \(\hat{e}_{s't'} \in \text{Im}(g)\). Moreover, using Lemma 3.6 and arguments similar to those used in (3) we prove that \(\hat{e}_{s't''} \in \text{Im}(g)\). Then as in (a) we conclude that \(\hat{e}_{st} \in \text{Im}(g)\).

(c) Suppose that \((s', t'), (s'', t'')\) are not 3-\(\rho\)-extremal. Then using Lemma 3.6 one can show that \(e_{s't'} + e_{s''t''} \in \text{Im}(g)\). Then as above we get 
\[ \hat{e}_{st} = \hat{e}_{sp}\hat{e}_{pt} - e_{s't'} - e_{s''t''} \in \text{Im}(g) \]
if \(p \in (s, t)\).

(5) Suppose that \(r_\rho(s, t) = 1\) and \((s, t)\) is 2-\(\rho\)-extremal. Let \(s' s'\) and \(t' t'\), where \(s, t \in I^{(k)}\), \(s', t' \in I^{(l)}\), and \(\{k, l\} = \{1, 2\}\). Then \(s' \prec t'\) and \(r_\rho(s', t') = 1\). It is easy to check that \((s', t')\) is not 3-\(\rho\)-extremal. If \((s', t')\) is 2-\(\rho\)-extremal and \(k < l\) then \(\hat{e}_{st} = g(\beta^*_{st}) \in \text{Im}(g)\). If \(k > l\) then by the same reason \(\hat{e}_{s't'} \in \text{Im}(g)\). Taking \(p \in (s, t)\) such that \(r_\rho(s, p) = 2\) or \(r_\rho(p, t) = 2\) we obtain
\[ \hat{e}_{st} + \hat{e}_{s't'} = \hat{e}_{sp}\hat{e}_{pt} \in \text{Im}(g) \]
by the induction hypothesis and hence \(\hat{e}_{st} \in \text{Im}(g)\).

We have shown that \(g\) is an epimorphism. It is easy to check that \(g(\Omega) = 0\). Thus \(g\) induces a \(K\)-algebra epimorphism
\[ \overline{g} : K(Q, \Omega) = KQ/\Omega \to R. \]

Now we show that \(\overline{g}\) is injective. It is enough to prove that for all \(i, j \in I\) we have
\[ \dim_K e(i)(KQ/\Omega)e(j) \leq \dim_K \hat{e}_{ii}R\hat{e}_{jj}, \]
where \(e(i)\) denotes the idempotent corresponding to the trivial path at \(i\).

As an example consider the case when \(r_\rho(i) = 2\), \(r_\rho(j) = 1\). Then \(\overline{i}\) can be joined to \(\overline{j}\) by paths of the following kinds:

1. \(I^{(2)}\)-paths,
2. 2-3-skew paths,
3. \(I^{(3)}\)-paths.

Paths of the same kind are equal modulo \(\Omega\). Thus \(e(i)K(Q, \Omega)e(j)\) has a basis \(\mathcal{B}\) consisting of paths of pairwise different kinds. Moreover, all the kinds (1)–(3) cannot appear in \(\mathcal{B}\) simultaneously. One can check that \(g(\mathcal{B})\) is a linearly independent set and the required inequality holds. The proof in the remaining cases is analogous. ■
4. A covering for \((Q(I^*_q), \Omega(I^*_q))\). Suppose that \(I_q\) is a three-separate stratified poset and \(I^*_q\) is its one-peak enlargement (see Section 2). Let

\[ I^* = I^{(1)} + I^{(2)} + I^{(3)} \]

be a three-separation of \(I^*_q\). Note that \(\ast \in I^{(3)}\).

Let \((Q, \Omega) = (Q(I^*_q), \Omega(I^*_q))\) be the bound quiver associated with \(I^*_q\) (see (3.3)). Let \(a_i = \beta p_i q_i: p_i \to q_i, i = 1, \ldots, k_1\), \(b_i = \beta r_i s_i: r_i \to s_i, i = 1, \ldots, k_2\), \(d_i = \beta t_i u_i: t_i \to u_i, i = 1, \ldots, k_3\) be all the 1-2-skew, 2-3-skew and 1-3-skew arrows respectively, where \(p_i \in I^{(1)}, q_j \in I^{(2)}, r_i \in I^{(2)}, s_j \in I^{(3)}, t_i \in I^{(1)}, u_j \in I^{(3)}\). Denote by \(Q^-\) the quiver obtained from \(Q\) by removing all arrows \(a_i, b_i, d_i\), and by \(\Omega^-\) the set of relations in \(\Omega\) which do not involve skew arrows.

Let \(G = \mathbb{Z}\alpha * \mathbb{Z}\beta\) be the free noncommutative group with two free generators \(\alpha, \beta\). Following [S1, S4] we define a Galois covering

\[(4.1) \quad f: (\tilde{Q}, \tilde{\Omega}) \to (Q, \Omega)\]

with group \(G\) as follows.

Let \(\tilde{Q}^{(x)} = Q^- \times \{x\}\) for \(x \in G\). We put \(j^{(x)} = (j, x)\) and \(\gamma^{(x)}_{ij} = (\gamma_{ij}, x)\) where \(j\) is a vertex of \(Q^-\) and \(\gamma_{ij}\) is an arrow in \(Q^-\). We define \(\tilde{Q}\) to be the disjoint union of \(\tilde{Q}^{(x)}\) over all \(x \in G\) connected by the edges

\[ a_i^{(x)}: \tilde{p}_i^{(x)} \to \tilde{q}_i^{(ax)}, \quad i = 1, \ldots, k_1, \]
\[ b_i^{(x)}: \tilde{r}_i^{(x)} \to \tilde{s}_i^{(bx)}, \quad i = 1, \ldots, k_2, \]
\[ d_i^{(x)}: \tilde{t}_i^{(x)} \to \tilde{u}_i^{(bx)}, \quad i = 1, \ldots, k_3 \]

(see Fig. 4.2). We define \(f\) by setting \(f(j^{(x)}) = j\) and \(f(\gamma^{(x)}_{ij}) = \gamma_{ij}\). We take for \(\tilde{\Omega}\) the natural lift of \(\Omega\) along \(f\). The group \(G\) acts on \(\tilde{Q}\) in the following way:

\[ y \ast j^{(x)} = j^{(yx)}, \quad y \ast \gamma^{(x)}_{ij} = \gamma^{(yx)}_{ij} \quad \text{for} \ y \in G. \]

We note that \(f\) induces a bound quiver isomorphism

\[(\tilde{Q}/G, \tilde{\Omega}/G) \simeq (Q, \Omega).\]

In general \(I_q\) admits many different three-separations. However, it is easy to see that the isomorphism class of the covering (4.1) does not depend on the choice of the three-separation.

We are especially interested in the case when the covering (4.1) is the universal cover of \((Q, \Omega)\). For this purpose we need the following definition.
Fig. 4.2
Definition 4.3. We call a three-separate poset $I_\varrho$ a rib convex poset if the following hold.

1. The rib skeleton $\text{rsk}(I_\varrho)$ of $I_\varrho$ has exactly three rib-connected components $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$; we assume that $\mathbb{R}_i \subseteq I^{(i)}$ for $i = 1, 2, 3$.
2. If $r_\varrho(i) > 1$ then $i \in \text{rsk}(I_\varrho)$.
3. For any $(i, j) \in \Delta \mathbb{R}_k$ for some $k$ there exists a rib path from $\overline{i}$ to $\overline{j}$.

Proposition 4.4 (compare [S4, Proposition 3.8]). Let $I_\varrho$ be a rib convex three-separate poset and $(Q, \Omega) = (Q(I^*_\varrho), \Omega(I^*_\varrho))$ be the bound quiver associated with $I^*_\varrho$ (see (3.3)).

(a) The fundamental group $\Pi_1(Q, \Omega)$ of $(Q, \Omega)$ is a free group with two free generators.
(b) The covering $f: (\overline{Q}, \overline{\Omega}) \to (Q, \Omega)$ defined in (4.1) is the universal Galois covering of $(Q, \Omega)$.

Proof. (a) Note that we can assume that the three-separation $I^{(1)} + I^{(2)} + I^{(3)}$ of $I^*_\varrho$ is such that

$$I^{(1)} = \{ i \in I : i \preceq x \text{ for some } x \in \mathbb{R}_1 \},$$
$$I^{(2)} = \{ i \in I \setminus I^{(1)} : i \preceq x \text{ for some } x \in \mathbb{R}_2 \},$$
$$I^{(3)} = I \setminus (I^{(1)} \cup I^{(2)}) \text{ and } * \in I^{(3)}.$$ 

We keep the notation of skew arrows introduced above. Note that the quiver $Q^-$ obtained from $Q$ by removing all the skew arrows has no oriented cycles and has the following property:

(*$_Q^-$) for each vertex $\overline{i} \in Q^-$ there exists an oriented path $\omega : \overline{i} \to *$ in $Q^-$. 

We denote by $Q''$ the full subquiver of $Q^-$ consisting of the vertices $\overline{i}$ for $i \in I^{(3)}$, and by $Q'$ the full subquiver of $Q^-$ consisting of the vertices $\overline{i}$ for $i \in I^{(2)} \cup I^{(3)}$. We have quiver embeddings $Q'' \subseteq Q' \subseteq Q^- \subseteq Q$. Note that $Q'$ and $Q''$ have the property (*$_{Q'}$) and (*$_{Q''}$) respectively and they are closed under taking successors in $Q^-$. 

First we construct a maximal tree $T'' \subseteq Q''$ with the property (*$_{T''}$) by induction on $|Q''_0|$. If $|Q''_0| = 2$ then we take $T'' = Q''$. Suppose that if $|Q''_0| < m$ then there exists $T''$ with the required properties. Let $|Q''_0| = m$ and $\overline{a}$ be a minimal element in $Q''$ (i.e. a source in $Q''$). Let $T''_0$ be the maximal tree in the quiver obtained from $Q''$ by removing the vertex $\overline{a}$. Let $\overline{a}_{at}$ be an arrow in $Q''$ from $\overline{a}$ to some $\overline{i} \in T''_0$. Then $T'' = T''_0 \cup \{ \overline{a} \} \cup \{ \overline{a}_{at} \}$ is a tree with the required property. 

Next, just as above, by induction on $|Q'_0 \setminus Q''_0|$ we construct a maximal tree $T'$ in $Q'$ with the property (*$_{T'}$) and such that $T' \cap Q'' = T''$. Finally,
applying an induction on $|Q_0^− \setminus Q_0^0|$ we extend $T'$ to a maximal tree $T$ in $Q^−$ having the property $(+T)$. Note that $T$ is a maximal tree in $Q$.

Suppose that $(Q^−)_0$ consists of the elements $i_k$, $k = 0, \ldots, m$, where $i_0 = \frak{s}$. Since $Q^−$ has no oriented cycle, without loss of generality we can suppose that if there exists a directed path from $i_k$ to $i_j$ in $Q^−$ then $k > j$.

(1) We show by induction on $k$ that if $b = \beta_{st} \in Q''$ is an arrow beginning at $i_k = \frak{s}$ then $\hat{b} \in N_Q$ (we keep the notation of Lemma 2.5). For $k = 0$ this is obvious. Suppose that for $k < m$ the statement is proved. Let $k = m$ and $s, t \in I^{(3)}$. If $b \in T$ then $\hat{b} \in N_Q$. Suppose that $b \not\in T$. Then there exists an arrow $\beta_{ss_1}$ in $T$ such that $s_1 \in I^{(3)}$. Consider two sequences $\beta_{ss_1}, \beta_{ss_1s_2}, \ldots, \beta_{sm*}$ and $\beta_{tt_1}, \beta_{tt_1t_2}, \ldots, \beta_{tt_k}$ of short pairs in $I^{(3)}$. Then

$$(b_0, \beta_{tt_1}, \beta_{tt_1t_2}, \ldots, \hat{p}, \beta_{ss_1}, \beta_{ss_1s_2}, \ldots, \beta_{sm*})$$

is an $\Omega$-contour.

Since $\beta_{ss_1} \in T$ and by the induction hypothesis $\hat{\beta}_{t_i, t_{i+1}}, \beta_{ss_1s_{i+1}} \in \Omega$ for $i = 0, \ldots, l$ and $j = 1, \ldots, m$, we get $\hat{b} \in N_Q$. (Here we put $t_l = t, s_l = s$.)

In particular, we have shown that $\hat{b} \in N_Q$ if $b$ is the $\varrho$-coset of a rib.

(2) Now we are going to prove that for skew arrows $b_p, b_q$ with $r_p, r_q \in \mathbb{R}_2$ we have $b_p \equiv b_q$ (modulo $N_Q$). By our assumptions on $I_\varrho$ there exist points $x_1, \ldots, x_l \in \mathbb{R}_2$ and rib paths $u_i, v_i$ for $i = 1, \ldots, l$ as in the figure:

$$
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & u_1 \ar[dl] & \ar[dr] & \ar[dr] & \ar[dr] & \ar[dr] & \ar[dr] & \ar[dr] & v_1 \ar[dl]\ \mathfrak{t}_{p_0} & \mathfrak{t}_{p_1} & \cdots & \mathfrak{t}_{p_l} & \mathfrak{t}_{p_{l+1}} \}
\end{array}
\end{array}
$$

where $\mathfrak{t}_{p_i}$ is the source of the 2-3-skew arrow $b_{p_i}, p_0 = p$ and $p_l = q$. Denote by $u_i$ an $I^{(3)}$-path from the sink $\mathfrak{t}_{p_i}$ of $b_{p_i}$ to $\frak{s}$ for $i = 1, \ldots, l$. Then $(u_i, b_{i-1}, v_{i-1}, u_i, b_i, v_i)$ is an $\Omega$-contour for $i = 1, \ldots, l$. By (1) above we get $\hat{u}_i, \hat{v}_i, \hat{w}_i \in N_Q$, hence $b_p \hat{b}_q^{-1} \in N_Q$.

(3) By induction on $k$ we shall show that if $b = \beta_{st} \in Q'$ is not a skew arrow and it begins at $i_k = \frak{s}$ then $\hat{b} \in N_Q$. For $k = 0$ this is obvious. Suppose that for $k < m$ the statement is proved. Let $k = m$. Suppose that $b \not\in T$. Let $\beta_{s's_1}$ be an arrow in $T' \subseteq T$ beginning at $i_k$ and $s', s_1 \in I^{(2)} \cap I^{(3)}$.

If $s = s' \in I^{(3)}$ then $s_1, t \in I^{(3)}$ and we prove the statement as in (1).

Suppose that $s = s' \in I^{(2)}$. Then $s_1, t \in I^{(2)}$. Let

$$
\beta_{s_2s_3}, \ldots, \beta_{sm, r_p} \quad \text{and} \quad \beta_{t_2t_3}, \ldots, \beta_{t_l, r_q}
$$

be sequences of short pairs in $I^{(2)}$ ending at $r_p, r_q \in \mathbb{R}_2$ whose $\varrho$-cosets are the sources of the arrows $b_p, b_q$ respectively. The sinks of these arrows are $\mathfrak{t}_p, \mathfrak{t}_q$. Let $u_p$ and $u_q$ be paths composed of the $\varrho$-cosets of short pairs in
\[
I^{(3)} \text{ connecting } \mathbf{s}_p \text{ and } \mathbf{s}_q \text{ with } \mathbf{t} \text{ respectively. Then}
\]
\[
(b_t t_1, \ldots, b_{s_1}, b_{s_2}, \ldots, b_{s_n} b_{u_p})
\]
is an \( \Omega \)-contour and since \( \bar{b}_{s, s_1} \in T, \bar{b}_p \equiv \bar{b}_q \pmod{N_{\Omega}} \) and by the induction hypothesis we get \( \bar{b} \in N_{\Omega} \).

Suppose now that \( s \neq s' \). Then \( r_q(s) > 1 \), hence \( s \in \text{rsk}(I_q) \). If \( s \in \mathbb{R}_2 \) then, since \( b \in Q' \), \( t \in I^{(2)} \). We have a path \( u_p \) composed of the \( \mathbb{Q} \)-cosets of short pairs from \( I^{(2)} \) connecting \( \mathbf{t} \) with the source \( \mathbf{t}_p \) of a skew arrow \( b_p \) such that \( r_p \in \mathbb{R}_2 \). From the rib convexity of \( \mathbb{R}_2 \) we get the existence of a rib path \( v \) from \( \mathbf{t} \) to \( \mathbf{t}_p \). Then
\[
(v b_p, b u_p b_p)
\]
is an \( \Omega \)-contour and since \( \mathbf{v} \in N_{\Omega} \) and by the induction hypothesis \( \mathbf{u}_p \in N_{\Omega} \), we get \( \mathbf{b} \in N_{\Omega} \).

If \( s \in \mathbb{R}_3 \) then \( s, t \in I^{(3)} \) and we prove the statement as in (1).

(4) We show that \( \bar{b}_p \equiv \bar{b}_q \pmod{N_{\Omega}} \) for any \( p, q \). Note that it is enough to show that \( \bar{b}_p \equiv \bar{b}_q \pmod{N_{\Omega}} \) if \( r_p \not\in \mathbb{R}_2, r_q \in \mathbb{R}_2 \) and \( r_p < r_q \). Let \( v \) be a path composed of the \( \mathbb{Q} \)-cosets of short pairs in \( I^{(2)} \) from \( \mathbf{t}_q \) to \( \mathbf{t}_q \), and \( u_p, u_q \) be the paths in \( Q'' \) composed of the \( \mathbb{Q} \)-cosets of short pairs connecting \( \mathbf{t}_p \) and \( \mathbf{t}_q \) with \( \mathbf{t} \) respectively. Then \( (v b_p, v b_q u_q) \) is an \( \Omega \)-contour and by (1) and (3) we get \( \bar{b}_p \bar{b}_q^{-1} \in N_{\Omega} \).

(5) We show as in (2) that \( \hat{d}_p = \hat{d}_q \pmod{N_{\Omega}} \) for all 1-3-skew arrows \( d_p, d_q \) whose sources are the \( \mathbb{Q} \)-cosets of elements of \( \mathbb{R}_1 \); \( \hat{d}_p = \hat{a}_r, \bar{b}_q \pmod{N_{\Omega}} \) for any 1-3-skew arrow \( d_p, 1 \)-2-skew arrow \( a_r \) and 2-3-skew arrow \( b_q \) such that the sources of \( d_p \) and \( a_r \) are the \( \mathbb{Q} \)-cosets of elements of \( \mathbb{R}_1 \), and \( \hat{a}_p = \hat{a}_r \pmod{N_{\Omega}} \) for any 1-2-skew paths \( a_p, \hat{a}_r \) whose sources are the \( \mathbb{Q} \)-cosets of elements of \( \mathbb{R}_1 \).

(6) We show as in (3) that if \( b = \bar{b}_{st} \) is an arrow in \( Q^- \) then \( \bar{b} \in N_{\Omega} \).

(7) We show as in (4) that
\[
\hat{d}_p \equiv \hat{a}_r, \quad \hat{a}_q \equiv \hat{a}_r, \quad \hat{d}_p \equiv \hat{a}_r \hat{b}_q \pmod{N_{\Omega}}
\]
for arbitrary \( p, q, r \). Note that there exists at least one 2-3-skew arrow and at least one 1-3-skew arrow or 1-2-skew arrow.

(8) We show that \( \hat{\beta}_{ij}^t \in N_{\Omega} \) for any \( i, j \) such that the arrow \( \hat{\beta}_{ij}^t \) exists. There is a rib path \( u \) from \( \bar{t} \) to \( \bar{t} \) and a nonzero path \( v \) from \( \bar{t} \) to \( \bar{t} \) composed of the \( \mathbb{Q} \)-cosets of short pairs in \( I^* \). Then \( (u v, \hat{\beta}_{ij}^t v) \) is an \( \Omega \)-contour and since \( \hat{u}, \hat{v} \in N_{\Omega} \) we get \( \hat{\beta}_{ij}^t \in N_{\Omega} \) as well.
We have shown that \( \hat{a}_1 N_\Omega, \hat{b}_1 N_\Omega \) generate the group \( \Pi_1(Q, \Omega) \) if there exists a 1-2-skew arrow \( a_1 \), and \( \hat{d}_1 N_\Omega, \hat{b}_1 N_\Omega \) generate \( \Pi_1(Q, \Omega) \) if there exists a 1-3-skew arrow \( d_1 \).

(9) Now we prove that \( \{\hat{a}_1 N_\Omega, \hat{b}_1 N_\Omega\} \) (or \( \{\hat{d}_1 N_\Omega, \hat{b}_1 N_\Omega\} \)) is a set of free generators of \( \Pi_1(Q, \Omega) \).

Suppose that \( a_1 \) exists. We have to show that no word of the form
\[
\kappa = \hat{a}_1^s \hat{b}_1^t \ldots \hat{a}_1^s \hat{b}_1^t
\]
such that \( s_{i+1} \neq 0 \neq t_i \) for \( i = 1, \ldots, l-1 \) or \( s_1 \neq 0 \) or \( t_l \neq 0 \) belongs to \( N_\Omega \).

Suppose that \( \omega = \lambda_1 \omega_1 + \ldots + \lambda_m \omega_m \) is a minimal relation in \( \Omega \) such that \( m \geq 2 \). Then all the \( \omega_i \) have a common sink and a common source. Moreover, \( \omega \) is a sum of elements of the form \( a_1 b a_2 \) where \( a_1, a_2 \in K Q \) and \( b \) is a relation of type (a), (b), (c), (d), (e), (f) or (g) (see (3.3)). Since \( \omega \) is minimal and \( m \geq 2 \) we have \( \omega = a_1 b a_2 \) where \( a_1, a_2 \in K Q \) are paths in \( Q \) and \( b \) is a relation of one of the above types. Thus the following types of \( \Omega \)-contours are possible:

- \((\gamma_1, \gamma_2), (\gamma_1 b_i \gamma_2, \gamma_3 b_j \gamma_4), (\gamma_1 a_i \gamma_2, \gamma_3 a_j \gamma_4)\),
- \((\gamma_1 d_i \gamma_2, \gamma_3 d_j \gamma_4), (\gamma_1 d_i \gamma_2 a_j a_j \gamma_4 b_k \gamma_5)\)

(induced by relations of type (b)), and

- \((\gamma_i, \gamma_j)\),

(induced by relations of types (c) to (g)), where the \( \gamma_s \) denote paths in \( Q \) which do not contain skew arrows.

Hence we get the following types of generators of \( N_\Omega \):

\[
\hat{\gamma}_i, \hat{\gamma}_1 \hat{b}_i^{-1} \gamma_2 b_j \gamma_3, \hat{\gamma}_1 \hat{a}_i^{-1} \gamma_2 a_j \gamma_3, \hat{\gamma}_1 \hat{d}_i^{-1} \gamma_2 d_j \gamma_3, \hat{\gamma}_1 \hat{d}_i \gamma_2 \hat{b}_j^{-1} \gamma_3 \hat{a}_k^{-1} \gamma_4,
\]

where the \( \hat{\gamma}_i \) are elements of the free group \( \Pi_1(Q) \) which are words without the letters \( \hat{a}_i, \hat{b}_i, \hat{d}_i \).

Consider the group homomorphism

\[
h : \Pi_1(Q) \rightarrow \mathbb{Z} a * \mathbb{Z} b
\]

defined by \( h(\hat{\gamma}_i) = 1, h(\hat{a}_i) = a, h(\hat{b}_i) = b, h(\hat{d}_i) = ab \). Note that all the generators of \( N_\Omega \) listed above are contained in \( \text{Ker}(h) \). Hence \( N_\Omega \subset \text{Ker}(h) \).

If \( \kappa \) is as above then \( h(\kappa) \neq 1 \), so \( \kappa \not\in N_\Omega \).

If there is no 1-2-skew arrow \( a_i \) in \( Q \) then we prove in a similar way that \( \{\hat{d}_1 N_\Omega, \hat{b}_1 N_\Omega\} \) freely generates \( \Pi_1(Q, \Omega) \). This finishes the proof of (a).

The statement (b) follows from the above considerations and from the construction of the universal cover described in (2.6). Since \( \Pi_1(Q, \Omega) = \mathbb{Z} a * \mathbb{Z} \beta \) it is easy to see that the construction in our case coincides with the construction (4.1) applied to \( G = \Pi_1(Q, \Omega) \).
5. Three-partite posets and the associated three-peak bound quivers. In this section we discuss some special case of three-separate posets, namely the three-partite posets in the sense of Definition 5.1 below.

If $I_1, I_2 \subseteq I$ are subposets then we write $I_1 < I_2$ if for all $i_1 \in I_1$ and $i_2 \in I_2$ we have $i_1 < i_2$. We say that $I_1$ is connected if it is connected with respect to the equivalence relation generated by the following relation:

\[ i \sim j \iff \text{either } i < j \text{ or } j < i \text{ is a minimal relation in } I. \]

**Definition 5.1** (compare with [S4, Def. 4.1]). A three-separate poset $I^*_e$ with a three-separation $I^{(1)} + I^{(2)} + I^{(3)}$ and a unique maximal element * is called **three-partite** if

(a) $I^{(k)}$ is the disjoint union of subposets $C^{(k)}$ and $J^{(k)}$ such that $C^{(k)}$ is either empty or it is a chain

\[ C^{(k)} : c_1^{(k)} \rightarrow c_2^{(k)} \rightarrow \ldots \rightarrow c_m^{(k)} \]

for $k = 2, 3$, $I^{(1)} < J^{(2)} < J^{(3)}$ and $C^{(2)} < C^{(3)}$.

(b) The stratified poset $I_e$ is rib-convex.

(c) There exist connected subposets $I_0^{(1)} \subseteq I^{(1)}, I_0^{(2)} \subseteq J^{(2)}, I_0^{(3)} \subseteq J^{(3)} \subseteq J^{(3)}$ and poset isomorphisms $\sigma_1 : I_0^{(1)} \rightarrow I_0^{(2)}, \sigma_2 : I_0^{(2)} \rightarrow I_0^{(3)}$ and $\sigma_3 : J_0^{(2)} \rightarrow J_0^{(3)}$ satisfying the following conditions:

(i) $\sigma_2$ is the restriction of $\sigma_3$ to $I_0^{(2)}$,

(ii) $r_e(i) = 3$ if and only if $i$ belongs to $I_0^{(k)}$ for some $k = 1, 2, 3$, and $r_e(i) = 2$ if and only if $i$ belongs to $J_0^{(k)} \setminus I_0^{(k)}$ for some $k = 2, 3$,

(iii) $(i,j) \in (\sigma_1(i), \sigma_1(j)) \in (\sigma_2 \sigma_1(i), \sigma_2 \sigma_1(j))$ provided $i \sim j$, $i, j \in I_0^{(1)}$, $[i,j] = \{i,j\}$, and $(i,j) \in (\sigma_3(i), \sigma_3(j))$ provided $i \sim j$, $i, j \in J_0^{(2)}$, $[i,j] = \{i,j\}$.

We visualize this notion in Fig. 5.2.

Following an idea in [S4] we associate with any three-partite stratified poset $I^*_e$ a three-peak bound quiver

\[ I^*_{e} = (Q^+, \Omega^+) \]

defined as follows:

For the quiver $Q^+$ we take the disjoint union of $Q^-$ (see (4.1)) and two chains:

\[ C^+ : c_1^{(3)+} \rightarrow c_2^{(3)+} \rightarrow \ldots \rightarrow c_m^{(3)+} \rightarrow +, \]

\[ C^\times : c_1^{(2)\times} \rightarrow \ldots \rightarrow c_m^{(2)\times} \rightarrow c_1^{(3)\times} \rightarrow \ldots \rightarrow c_m^{(3)\times} \rightarrow \times, \]

connected by the following arrows (we use the notation from Section 4):

(i) $\beta_{r_i^+} : \tau_i \rightarrow +$ if $r_i$ and $c_m^{(3)}$ are unrelated,
Fig. 5.2
(ii) $\beta_{p_i, \times}: \mathcal{T}_i \to \times$ if $p_i$ and $c_{m_3}^{(3)}$ are unrelated and $\beta_{t_i, \times}: \mathcal{T}_i \to \times$ if $t_i$ and $c_{m_3}^{(3)}$ are unrelated.

(iii) $a_i^\times: \mathcal{T}_i \to \mathcal{T}_i^\times$ if $q_i \in C^{(2)}$.

(iv) $b_i^+: \mathcal{T}_i \to \mathcal{T}_i^{(+)}$ if $s_i \in C^{(3)}$.

(v) $d_i^+: \mathcal{T}_i \to \mathcal{T}_i^{(+)}$ if $u_i \in C^{(3)}$.

(see Fig. 5.2).

For $\Omega^{+\times}$ we take the set $\Omega^-$ with the following additional relations:

(1) $wu_\beta$ and $wb_j^+$ if $w$ is neither a 3-rib path nor a 2-rib path nor an $I^{(2)}$-path,

(2) $w\beta_{t_i, \times}$, $w\beta_{b_i, \times}$, $wa_i^\times$ and $wd_i^\times$ if $w$ is neither a 3-rib path nor an $I^{(1)}$-path,

(3) $wu - vu'$, where $w: \mathcal{P} \to \mathcal{Q}_i$ is a 3-rib path, a 2-rib path or an $I^{(2)}$-path, and where $v: \mathcal{P} \to \mathcal{Q}_j$ and $u$ and $u'$ have a common sink in $C^+$.  

(4) $wu - vu'$, where $w: \mathcal{P} \to \mathcal{Q}_i$ is a 3-rib path or an $I^{(1)}$-path, and where $v: \mathcal{P} \to \mathcal{Q}_j$, $u$ and $u'$ have a common sink in $C^\times$.

Analogously to the bipartite case [S4] there is an algebra isomorphism

$$KI_g^{+\times} \simeq \xi R \xi,$$

where $\tilde{R} = K(\tilde{Q}, \tilde{\Omega})$ and 

$$\xi = \sum_{t \in Q_0^{(e)}} e_t + \sum_{t \in C^{(2)}} e_{(t, \alpha)} + \sum_{t \in C^{(3)}} (e_{(t, \beta)} + e_{(t, \beta \alpha)}) + e_{(\alpha, \beta)} + e_{(\beta, \alpha)}$$

and we consider the diagram

$$\begin{align*}
\text{mod}_{sp}(\xi \tilde{R} \xi) & \xrightarrow{L_\xi} \text{mod}_{sp}(\nu \tilde{R} \nu) & \xrightarrow{T_\nu} \text{mod}_{sp}(\tilde{R}) \\
\text{mod}_{sp}(KI_g^{+\times}) & \xrightarrow{f_{+\times}} \text{mod}_{sp}(R)
\end{align*}$$

where $\iota$ is the natural equivalence, $f_{+\times}$ is the covering functor (see [Ga], [S4, 4.20]), $\nu = \sum_t e_t$ where $t$ runs over the set of vertices of the union of all quivers $Q_i^{(\omega)}$ for $\omega$ of the form $\omega = \alpha^{i_1} \beta^{i_1} \ldots \alpha^{i_m} \beta^{i_m}$, where $s_i, t_i \geq 0$ for $i = 1, \ldots, m$, $L_\xi$ and $T_\nu$ are the lower and upper induction functors respectively (see [S4, S5]), $f_{+\times}$ is the composed functor $f_{sp} \circ T_\nu \circ L_\xi \circ \iota$.

By the Splitting Theorem of [S3], Proposition 4.3 above, Theorem 4.19 and Remark 4.21 of [S4] we get the following.

**Theorem 5.5.** If $I_g^+$ is a three-partite stratified poset then:

(a) The functor $f_{+\times}: \text{mod}_{sp}(KI_g^{+\times}) \to \text{mod}_{sp}(R)$ is exact, faithful, dense and preserves indecomposability.
(b) The category $\text{mod}_{sp}(KI_e^{++})$ is of finite representation type if and only if so is the category $\text{mod}_{sp}(R)$.

(c) If $K$ is an algebraically closed field then $\text{mod}_{sp}(KI_e^{++})$ is of tame (resp. wild) representation type if and only if $\text{mod}_{sp}(R)$ is of tame (resp. wild) representation type.

Applying arguments similar to those used in [S4, Proposition 4.9] and Proposition 4.3 above one can prove the following.

**Theorem 5.6.** Let $I_e^*$ be a three-partite poset and let $I_e^{++}$ be the associated three-peak bound quiver (5.3).

(a) The fundamental group $\Pi_1(I_e^{++})$ is trivial. If in addition every vertex of $I_e^{++}$ is separating then the Auslander–Reiten quiver $\Gamma_{sp}(KI_e^{++})$ of $\text{mod}_{sp}(KI_e^{++})$ has a preprojective component.

(b) If the Auslander–Reiten quiver $\Gamma_{sp}(KI_e^{++})$ of $\text{mod}_{sp}(KI_e^{++})$ has a preprojective component then $\text{mod}_{sp}(KI_e^*)$ is of finite representation type if and only if $I_e^{++}$ contains no Weichert’s critical forms (see [W]).

Let us finish with a simple corollary from the above considerations.

**Corollary 5.7.** If $I_e^*$ is a three-partite stratified poset and $\text{mod}_{sp}(KI_e^*)$ is of finite representation type then $I_e$ does not contain any rib.

**Proof.** It is easy to check that if $I_e$ contains a rib then $I_e^{++}$ contains a subquiver of type $\tilde{D}_4$ which is of infinite representation type. Thus the statement follows from Theorem 5.6 above.

**References**


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