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Fragmentability and σ-fragmentability

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Abstract. Recent work has studied the fragmentability and σ-fragmentability properties of Banach spaces. Here examples are given that justify the definitions that have been used. The fragmentability and σ-fragmentability properties of the spaces ℓ^∞ and $\ell_c^\infty(\Gamma)$, with Γ uncountable, are determined.

1. Introduction. In a series of papers we [5–8] and others [3, 4, 9, 10, 12–15] have discussed fragmentable and σ-fragmentable spaces. In this note we discuss some examples that illuminate these concepts.

Let Z be a topological space and let ϱ be a metric on Z that is not necessarily related to the topology of Z. If $\varepsilon > 0$, the space Z is said to be *fragmented down to* ε when each non-empty subset of Z contains a non-empty relatively open subset of ϱ-diameter less than ε. The space Z is said to be *fragmented* if it is fragmented down to ε for each $\varepsilon > 0$. The space Z is said to be *σ-fragmented* if, for each $\varepsilon > 0$, Z can be written as

$$Z = \bigcup_{i=1}^{\infty} Z_i\,,$$

with each set Z_i fragmented down to ε. In the special case when Z is a norm closed bounded subset of a Banach space Y taken with its weak topology and ϱ is the restriction of the norm metric to Z, the condition that Z be fragmented by ϱ is equivalent to the point of continuity property for Z (see, for example, [9]). In many applications, and in particular when $\varrho(x,y) = \|y-x\|$ and X is a Banach space with its weak topology or a dual Banach space with its weak* topology, the metric ϱ is lower semi-continuous as a function from $X \times X$ to $\mathbb{R}$.

Our first two examples, discussed in detail in §2, relate to the definitions. It would, at first sight, seem to be more natural to define a set to be

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σ-fragmentable if it can be expressed as a countable union of fragmentable sets; and this concept might seem likely to be equivalent to the concept that we have in fact introduced. We give an example to show that these two concepts are different: our experience justifies the choice that we have made.

EXAMPLE 2.1. *Let Z be the Banach space c_0 taken with its weak topology and let ϱ be the norm metric on Z. Then Z is σ-fragmented by ϱ, but Z cannot be expressed as a countable union of sets that are fragmented by ϱ.*

In some definitions that require consideration of a space as a countable union of sets, for example, the definition of a space of the first category, one can, without loss, confine one's attention to countable unions of topologically respectable sets, indeed to countable unions of closed sets in the case of spaces of the first category. Our next example shows that this is not possible, in general, for the concept of σ-fragmentation. Recall that a set S in a topological space Z is said to be *countably determined by open sets* if it is possible to choose a sequence $G_1, G_2, \ldots$ of open sets in Z, in general depending on S, with the property that any two points of Z that lie in precisely the same sets of the sequence $G_1, G_2, \ldots$ either both lie in S or both lie in $Z \setminus S$. Note that any Borel set is countably determined by open sets. It will be convenient to say that a set D of Z is *strongly σ-fragmented by the metric ϱ* if, for each $\varepsilon > 0$, it is possible to express D as a countable union

$$D = \bigcup_{i=1}^{\infty} D_i$$

of sets that are countably determined by open sets and fragmented down to ε.

The "double arrow" space D is the set of points

$$((0,1] \times \{0\}) \cup ([0,1) \times \{1\})$$

in $\mathbb{R}^2$ endowed with the order topology from the lexicographical order: $(s,i) < (t,j)$ if either $s < t$ or $s = t$ and $i < j$. This space D is compact and Hausdorff.

EXAMPLE 2.2. *The "double arrow" space D is a compact Hausdorff space. Let ϱ be the metric that D has as a subset of $\mathbb{R}^2$. Then D is the union of two sets that are fragmented by ϱ, but D is not strongly σ-fragmented by ϱ. Further, for no metric τ on D, is D fragmented by τ.*

We remark that in the special case when X is a Banach space with a Kadec norm and ϱ is the norm metric, then X is strongly σ-fragmented (see [6, Theorem 2.3]).

Ribarska [15] has shown that if a topological space Z is σ-fragmented by a lower semi-continuous metric ϱ, then there is a second metric τ on Z with

Z fragmented by τ. It is easy to see that the metric ϱ in Example 2.2 is not lower semi-continuous for the topology of D. Example 2.2 shows that the condition that ϱ be lower semi-continuous in Ribarska's theorem cannot be omitted.

In [6, Lemma 2.1] we proved that if $\varepsilon > 0$ and A and B are two sets in Z that are fragmented down to ε for some lower semi-continuous metric ϱ, then $A \cup B$ is also fragmented down to ε by ϱ. Example 2.2 shows that the condition that ϱ be lower semi-continuous cannot be omitted in this result.

In §3 we consider three examples of a rather different nature.

In [7, Theorem 4.1, (c)⇒(a)] we show that if every compact subset of a Čech-analytic space Z is fragmented by a lower semi-continuous metric ϱ, then Z is σ-fragmented by ϱ. By a well-known result of Gödel it is consistent with the usual axioms of set theory (ZFC) to suppose that the Cantor ternary set C contains an uncountable co-analytic set A, with the property that each compact subset of A is countable. The following example shows that in the result quoted from [7] the condition that Z be Čech-analytic cannot be omitted.

Example 3.1. *Let A be an uncountable co-analytic subset of the Cantor ternary set with the property that each of its compact subsets is countable. Let d be the discrete metric on A. Then d is lower semi-continuous. Each compact subset of A is fragmented by d but A is not σ-fragmented by d.*

The next example asserts that the Banach space ℓ^∞ with its weak topology is not σ-fragmented by its norm. This is proved in [6, Theorem 5.1]. However, the very complicated proof that we give in [6] was an elaboration of the much simpler proof (contained in the original version of [7]) that we include here.

Example 3.2. *Let E denote the Banach space ℓ^∞ taken with its weak topology. Then each compact subset of E is fragmented by the norm metric, but E itself is not σ-fragmented by the norm metric. However, E is fragmented by the lower semi-continuous metric τ defined by*

$$\tau(x,y) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, |x_i - y_i|\}\,.$$

Taken together with Theorem 4.1 of [7] this shows that (ℓ^∞, weak) is not Čech-analytic and so is not a Souslin ($\mathcal{F} \cup \mathcal{G}$) set within its second dual with its weak* topology. This example suggests that the condition that a Banach space with its weak topology be σ-fragmented by its norm is much stronger than the condition that it be fragmented by some lower semi-continuous metric.

Our final example is a natural space that is neither σ-fragmented by any lower semi-continuous metric nor fragmented by any metric. This example was considered by Hansell, Jayne and Talagrand [3] but their proof contains a lacuna.

EXAMPLE 3.3. *Let Γ be an uncountable set and let $E = \ell^\infty_{\mathrm{c}}(\Gamma)$ denote the Banach subspace of $\ell^\infty(\Gamma)$ consisting of all bounded real-valued functions on Γ having countable support. Then* (E, weak) *is neither σ-fragmented by any lower semi-continuous metric nor fragmented by any metric.*

We are grateful to the referee both for explaining the "folklore" result that we have included in §3 and for providing the much simpler verification for Example 3.3 that we now give in §3.

2. Examples 2.1 and 2.2

Verification of Example 2.1. Let $x_1, x_2, \ldots$ be a countable dense sequence in c_0 and let B be the unit ball. For each $\varepsilon > 0$, the sets

$$\frac{1}{3}\varepsilon B + x_i\,, \qquad i \geq 1\,,$$

form a cover of c_0 by weakly closed sets of diameter less than ε. Hence c_0 with its weak topology is σ-fragmented by the norm metric ϱ. (Of course this argument applies in each separable Banach space.)

Now suppose that we have $c_0 = \bigcup_{i=1}^\infty E_i$, with each set E_i fragmented by ϱ. Then by the Baire category theorem, at least one of the sets E_i is norm dense in some non-empty open ball. Since the property of being fragmented by ϱ is invariant under translations and dilations, one sees that the unit ball $B = \{x : \|x\| \leq 1\}$ of c_0 contains a norm dense subset D which is fragmented by ϱ. However we show that, for each $u \in D$ and for each weak open neighbourhood V of u, ϱ-diam$(V \cap D) \geq 1$. This contradiction will prove the second statement about c_0.

Given $u \in V \cap D$, we define a sequence $\{u_j\}$ in c_0 as follows:

$$u_j(n) = \begin{cases} u(n) & \text{if } n \neq j, \\ u(j) + \varepsilon_j & \text{if } n = j, \end{cases}$$

where $\varepsilon_j = 1$ or -1 whichever satisfies $|u(j) + \varepsilon_j| \leq 1$. The last condition is possible because $|u(j)| \leq 1$. Then $u_j \in B$ for all $j \in \mathbb{N}$, and for each $y \in \ell^1 = (c_0)^*$,

$$|\langle u_j - u, y\rangle| = |y(j)| \to 0 \qquad \text{as } j \to \infty\,.$$

Therefore, the sequence $\{u_j\}$ converges to u weakly as $j \to \infty$. Consequently, there exists a j such that $u_j \in V \cap B$, and so

$$\varrho\text{-diam}(V \cap B) \geq \|u_j - u\| = 1\,.$$

Being weak open, V is also open in the norm topology. Hence, $V \cap D$ is norm dense in $V \cap B$. It follows that

$$\varrho\text{-diam}(V \cap D) = \varrho\text{-diam}(V \cap B) \geq 1\,.$$

Verification of Example 2.2. We remark that D with its order topology is separable and hereditarily Lindelöf but is not metrizable.

First write

$$D_0 = (0,1] \times \{0\} \quad \text{and} \quad D_1 = [0,1) \times \{1\}\,.$$

Then $D = D_0 \cup D_1$. Since $\{(s,t] \times \{0\} : 0 < s < t \leq 1\}$ is a base for the induced topology on D_0, D_0 is fragmented by ϱ. Similarly, D_1 is fragmented by ϱ. Consequently, D is σ-fragmented by ϱ.

Recall that order intervals of the form

$$[(0,1),(t,j))\,, \quad ((s,i),(t,j)) \quad \text{or} \quad ((s,i),(1,0)]$$

constitute a base for the order topology of D. Call such an interval a *basic interval*. Then each basic interval I has the property that the projections of $I \cap D_0$ and $I \cap D_1$ on $[0,1]$ differ by at most a finite number of points. Since D is hereditarily Lindelöf, each open subset G of D is the union of a countable family of basic intervals, and therefore the projections of $G \cap D_0$ and $G \cap D_1$ on $[0,1]$ differ by at most a countable number of points.

Now consider any set C in D that is countably determined by open sets. Let $G_1, G_2, \ldots$ be a sequence of open sets in D with the property that two points that lie in just the same sets of the sequence $G_1, G_2, \ldots$ either both lie in C or both lie in $D \setminus C$. Let S be the union for $i \geq 1$, of the countable sets in $[0,1]$ where the projections of $G_i \cap D_0$ and $G_i \cap D_1$ on $[0,1]$ differ. Then S is countable and the projections of the sets

$$(C \setminus (S \times \{0,1\})) \cap D_0\,, \qquad (C \setminus (S \times \{0,1\})) \cap D_1\,,$$

on $[0,1]$ coincide.

Suppose that we can write

$$D = \bigcup_{i=1}^{\infty} E_i\,,$$

with each set E_i, $i \geq 1$, countably determined by open sets and with the property that each non-empty subset has a relatively open subset that is non-empty and of ϱ-diameter less than 1. We seek a contradiction. Choose $i \geq 1$ so that E_i is uncountable. Since E_i is countably determined by open sets, we can choose an uncountable set H in $[0,1]$ with $H \times \{0,1\} \subset E_i$. We use the real topology on H. By removing from H at most a countable set, we obtain an uncountable subset K of H with the property that each point of K is both a limit point of an increasing sequence in K and also a limit point of a decreasing sequence in K (see, e.g. [11, p. 59]). Consider the subset

$$F = K \times \{0, 1\}$$

of E_i. Each non-empty relatively open subset of F contains a pair of points $(k, 0)$ and $(k, 1)$ with $k \in K$. Hence this relatively open subset has ϱ-diameter greater than or equal to 1. This contradiction shows that D is not strongly σ-fragmented by ϱ.

Finally, suppose that τ is any metric on D and that D is fragmented by τ. We again seek a contradiction. We construct, inductively, sequences $I_1, I_2, \ldots$ and $J_1, J_2, \ldots$ of subsets of D of the forms

$$I_i = (a_i, b_i) \times \{0, 1\}, \quad J_i = [c_i, d_i] \times \{0, 1\}, \quad i \geq 1,$$

with $I_1 \supset J_1 \supset I_2 \supset J_2 \supset \ldots$, and

$$\tau\text{-diam}\, I_i < 2^{-i}, \quad i \geq 1.$$

We start by taking $J_0 = D$. When J_i has been defined for $i \geq 0$, since D is fragmented by τ, we can choose a non-empty relatively open subset K_i of $[c_i, d_i] \times \{0, 1\}$ of τ-diameter less than 2^{-i-1}. We can choose $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$ with $a_{i+1} < c_{i+1} < d_{i+1} < b_{i+1}$ and $(a_{i+1}, b_{i+1}) \times \{0, 1\} \subset K_i \subset J_i$.

The corresponding sets I_{i+1} and J_{i+1} then satisfy our requirements, and the construction follows by induction. Now we can choose a real number ℓ with

$$\{\ell\} \times \{0, 1\} \subset J_i \subset I_i \quad \text{for } i \geq 1.$$

Hence

$$0 < \tau((\ell, 0), (\ell, 1)) \leq \tau\text{-diam}\, I_i < 2^{-i}$$

for all $i \geq 1$. This contradiction shows that D is not fragmented by τ.

3. Examples 3.1, 3.2 and 3.3

Verification of Example 3.1. As we have already remarked, by a well-known result of Gödel, it is consistent with the usual axioms of set theory to suppose that the Cantor ternary set C contains an uncountable co-analytic set A, with the property that each compact subset of A is countable. We suppose that A is such a set in C.

Let K be a non-empty compact subset of A and let L be a non-empty subset of K. If L has no isolated point, then $\operatorname{cl} L$, and so also K, must be uncountable, contrary to the choice of A. Hence L must have an isolated point, and so K is fragmented by any metric on C.

Let d be the discrete metric on C with $d(x, y) = 1$ whenever $x \neq y$. This metric d is lower semi-continuous on C. Consider any representation of A as a countable union $A = \bigcup_{i=1}^{\infty} A_i$. Since A is uncountable we can choose $i \geq 1$ so that A_i is uncountable. Then A_i has an uncountable subset, B_i say, that is dense-in-itself. Hence B_i has no non-empty relatively open subset with d-diameter less than 1. This shows that A is not σ-fragmented by d.

Before we give the detailed verification of Example 3.2 we describe some of the properties of the Banach space ℓ^∞ consisting of all bounded sequences $x = x_1, x_2, \ldots$ of real numbers, with the supremum norm. It is convenient, when considering the dual space $(\ell^\infty)^*$ of ℓ^∞, to identify ℓ^∞ with the space $C(\beta\mathbb{N})$ of continuous functions on the Stone–Čech compactification of $\mathbb{N}$. Then $(\ell^\infty)^*$ becomes the space $\mathcal{M}$ of signed Radon measures on $\beta\mathbb{N}$.

If $x \in \ell^\infty$ and $\mu \in \mathcal{M}$, then

$$\mu(x) = \int_{\beta\mathbb{N}} \widehat{x}\, d\mu\,,$$

where $\widehat{x}$ is the continuous extension of x on $\beta\mathbb{N}$. It follows that the sets of the form

$$\{x \in \ell^\infty : |\lambda_i(x)| < 1\,,\ i = 1, \ldots, n\}\,,$$

with $\lambda_1, \ldots, \lambda_n$ finite *positive* Radon measures on $\beta\mathbb{N}$, form a base for weak neighbourhoods of 0 in ℓ^∞.

Our first approach to the verification of Example 3.2 was based on an attempt to prove that the Baire Category Theorem holds for the unit ball B of ℓ^∞ with its weak topology. This we could not do and we had to formulate and establish a weak and peculiar form of the Baire Category Theorem for B.

We are grateful to the referee for providing us with the statement and proof of the following result that he attributes to "folklore".

If K is any infinite compact Hausdorff space, then the unit ball B of $C(K)$ with its weak topology does not satisfy the Baire Category Theorem.

Proof. First consider the case when K is not a scattered space. Then there is a continuous map p of K onto the unit interval $I = [0,1]$ (see [16, §8.5.4, (i)⇔(ii)]). By Zorn's lemma we can choose a minimal compact set P that is mapped by p onto I. Let $V_1, V_2, \ldots$ be an open basis for I, and write

$$G_i = \{f \in B : \text{for some } x \in p^{-1}(V_i)\,,\ f(x) > 3/4\}\,,$$
$$H_i = \{f \in B : \text{for some } x \in p^{-1}(V_i)\,,\ f(x) < 1/4\}\,,$$

for $i = 1, 2, \ldots$ Then G_i and H_i are weakly open in B. If $\mu_1, \ldots, \mu_j$ is any finite set of Radon measures on K, since V_i is uncountable, there is always a point t in V_i with

$$\mu_k(p^{-1}(t)) = 0\,, \qquad 1 \le k \le j\,.$$

This enables us to show that both G_i and H_i are weakly dense in B. If the Baire Category Theorem were to hold for B, then the intersection of the sets G_i, H_i, $i \ge 1$, would be dense in B. Consider any function f in

$\bigcap_{i=1}^{\infty} G_i \cap H_i$, and any point q in P. Since the map p is irreducible, for any open neighbourhood N of q, the open set $I \setminus p(P \setminus N)$ is non-empty, and so contains the set V_i for some i. Thus $p^{-1}(V_i) \subset N$, and we can choose points x, y in N so that the chosen f satisfies $f(x) > 3/4$, $f(y) < 1/4$. Hence the oscillation of f at q is at least $1/2$ contrary to the continuity of f at q. This shows that the Baire Category Theorem does not hold.

The case of an infinite compact scattered space K is similar but simpler. Every non-empty closed set C in such a space has a singleton that is a clopen set in C. A simple argument (see for example [12, Lemma 5.3]) shows that each infinite sequence in K has a convergent subsequence. Thus we can choose a sequence $k_1, k_2, \ldots$ of isolated points of K converging to a point q of K. For each $i \geq 1$ we introduce the sets

$$G_i = \{f \in B : \text{for some } j \geq i,\ f(k_j) > 3/4\},$$
$$H_i = \{f \in B : \text{for some } j \geq i,\ f(k_j) < 1/4\}.$$

It is easy to check that each set G_i, $i \geq 1$, H_i, $i \geq 1$, is weakly open and weakly dense in B. If the Baire Category Theorem were to hold for B, then there would be a function f in $\bigcap\{G_i \cap H_i : i \geq 1\}$, which would have to have oscillation at least $1/2$ at the point q. Thus the Baire Category Theorem does not hold.

We return from this digression to the verification of Example 3.2. It will be convenient to say that a set S in ℓ^∞ is a *set with infinitely many free coordinates the others being fixed* or just a *coordinate set* if for some ξ in ℓ^∞ and some subset M of $\mathbb{N}$ with $\mathbb{N} \setminus M$ infinite, we have

$$S = \{x : x_m = \xi_m \text{ for } m \in M\}.$$

We prove a lemma about the intersections of such sets with the unit ball B of ℓ^∞.

LEMMA 3.1. *Let U be a weakly open set in ℓ^∞, and let S be a coordinate set in ℓ^∞ with $B \cap S \cap U \neq \emptyset$. Then there is a coordinate set T with $T \subset S$ and*

$$\emptyset \neq B \cap T \subset B \cap S \cap U.$$

Proof. We take S to be the coordinate set

$$S = \{x : x_m = \xi_m \text{ for } m \in M\},$$

with ξ in ℓ^∞ and M a subset of $\mathbb{N}$ with $\mathbb{N} \setminus M$ infinite. Take η to be any point in the non-empty set $B \cap S \cap U$. Then $\|\eta\| \leq 1$ and

$$\eta_m = \xi_m \quad \text{for } m \in M.$$

Let V be a weakly open neighbourhood of η contained in U and of the form

$$V = \{x : |\lambda_i(x) - \lambda_i(\eta)| < 1 \text{ for } i = 1, \ldots, n\}$$

where $\lambda_1, \ldots, \lambda_n$ are positive Radon measures on $\beta\mathbb{N}$.

We claim that there is an infinite subset P of $\mathbb{N}\setminus M$ such that $\lambda_i(\overline{P}) < 1/2$ for $i = 1, \ldots, n$ where $\overline{P}$ is the closure of P in $\beta\mathbb{N}$. First partition $\mathbb{N} \setminus M$ as

$$\mathbb{N} \setminus M = \bigcup_{k=1}^{\infty} P_k\,,$$

where each P_n is infinite and $P_k \cap P_l = \emptyset$ if $k \neq l$. Then $\{\overline{P}_k : k \in \mathbb{N}\}$ is a mutually disjoint family of subsets of $\beta\mathbb{N}$. Since $\lambda = \sum_{i=1}^{n} \lambda_i$ is a finite positive measure, $\lambda(\overline{P}_k) < 1/2$ for some k. Let $P = P_k$. Then for each i, $\lambda_i(\overline{P}) \le \lambda(\overline{P}) < 1/2$ as desired.

Let T be the coordinate set

$$T = \{x : x_m = \eta_m \text{ for } m \in \mathbb{N} \setminus P\}\,.$$

Then, since $M \subset \mathbb{N} \setminus P$ and $\eta_m = \xi_m$ for $m \in M$, $T \subset S$. Hence $B \cap T \subset B \cap S$. Furthermore, if $x \in B \cap T$, then $\widehat{x} - \widehat{\eta} = (x - \eta)^\wedge \equiv 0$ on $(\mathbb{N} \setminus P)^-$. Hence for each $i \in \{1, \ldots, n\}$,

$$|\lambda_i(x) - \lambda_i(\eta)| = \Big|\int (\widehat{x} - \widehat{\eta})\, d\lambda_i\Big| = \Big|\int_{\overline{P}} (\widehat{x} - \widehat{\eta})\, d\lambda_i\Big|$$

$$\le \int_{\overline{P}} |\widehat{x} - \widehat{\eta}|\, d\lambda_i \le 2\lambda_i(\overline{P}) < 1\,.$$

It follows that $\eta \in B \cap T \subset V \subset U$ and therefore

$$\emptyset \neq B \cap T \subset B \cap S \cap U\,.$$

Our next lemma provides the form of the Baire Category Theorem that we have promised. We say that a subset E of B is *nowhere dense on coordinate sets* if for every coordinate set S with $B \cap S \neq \emptyset$, there is a weakly open set U of ℓ^∞ with

$$B \cap S \cap U \neq \emptyset \quad \text{but} \quad E \cap S \cap U = \emptyset\,,$$

that is, $E \cap S$ is not weakly dense in $B \cap S$.

LEMMA 3.2. *The unit ball of ℓ^∞ cannot be contained in a countable union of sets that are nowhere dense on coordinate sets.*

Proof. Let S_0 be an arbitrary coordinate set and suppose that

$$\emptyset \neq B \cap S_0 \subset \bigcup_{i=1}^{\infty} E_i\,,$$

where, for each i, $E_i \subset B$ and E_i is nowhere dense on coordinate sets. We seek a contradiction. Since E_1 is nowhere dense on coordinate sets, there is a weakly open subset U_1 of ℓ^∞ with

$$B \cap S_0 \cap U_1 \neq \emptyset \quad \text{and} \quad E_1 \cap S_0 \cap U_1 = \emptyset .$$

By Lemma 3.1, there is a coordinate set S_1 with

$$S_1 \subset S_0 \quad \text{and} \quad \emptyset \neq B \cap S_1 \subset B \cap S_0 \cap U_1 .$$

Next, since E_2 is nowhere dense on coordinate sets there is a weakly open subset U_2 of ℓ^∞ with

$$B \cap S_1 \cap U_2 \neq \emptyset \quad \text{and} \quad E_2 \cap S_1 \cap U_2 = \emptyset .$$

Then by Lemma 3.1, there is a coordinate set S_2 with

$$S_2 \subset S_1 \quad \text{and} \quad \emptyset \neq B \cap S_2 \subset B \cap S_1 \cup U_2 .$$

Proceeding in this way we can find a decreasing sequence $\{S_i : i = 0, 1, 2, \ldots\}$ of coordinate sets and a sequence $\{U_i : i \in \mathbb{N}\}$ of weakly open sets in ℓ^∞ such that for each $i \in \mathbb{N}$,

$$E_i \cap S_{i-1} \cap U_i = \emptyset \quad \text{and} \quad \emptyset \neq B \cap S_i \subset B \cap S_{i-1} \cap U_i .$$

Therefore $E_i \cap S_i = \emptyset$ for each i, which implies that

$$B \cap \bigcap_{i=0}^{\infty} S_i \subset \Big(\bigcup_{i=1}^{\infty} E_i \Big) \cap \bigcap_{i=1}^{\infty} S_i = \emptyset .$$

But we show that $B \cap \bigcap_{i=0}^\infty S_i \neq \emptyset$, which establishes the lemma. For each i, since $B \cap S_i \neq \emptyset$, there are $\xi^{(i)} \in B \cap S_i$ and $M_i \subseteq \mathbb{N}$ with $\mathbb{N} \setminus M_i$ infinite such that

$$S_i = \{x : x_m = \xi^{(i)}_m \text{ for } m \in M_i\} .$$

We may assume that $\xi^{(i)}_m = 0$ for $m \notin M_i$. Since $S_{i+1} \subset S_i$, we must have

$$M_i \subset M_{i+1} \quad \text{and} \quad \xi^{(i+1)}_m = \xi^{(i)}_m \quad \text{for } m \in M_i .$$

Define $\xi \in \ell^\infty$ by $\xi_m = \lim_{i\to\infty} \xi^{(i)}_m$ for each $m \in \mathbb{N}$. Then ξ is well-defined and clearly $\xi \in B \cap \bigcap_{i=0}^\infty S_i$.

Verification of Example 3.2. We know that each weakly compact subset of E is fragmented by the norm metric. Since $E = \bigcup_{n=1}^\infty nB$, it suffices to prove that B is not σ-fragmented by the norm metric.

We suppose that $B = \bigcup_{i=1}^\infty B_i$, where, for each $i \geq 1$, each non-empty subset of B_i has a non-empty relatively weakly open subset of norm diameter less than 1. By Lemma 3.2 we can choose $i \geq 1$ so that $B_i \neq \emptyset$ and B_i fails to be nowhere dense on coordinate sets. This ensures the existence of a coordinate set S with $B \cap S \neq \emptyset$ such that $B_i \cap S$ is weakly dense in $B \cap S$. By the choice of the decomposition of B, we can choose a non-empty relatively weakly open subset of $B_i \cap S$ of the form $B_i \cap S \cap U$, with U weakly open in ℓ^∞, and of diameter less than 1. Since $B_i \cap S \cap U$ is weakly dense in $B \cap S \cap U$ and since the norm is weakly lower semi-continuous,

$$\operatorname{diam}(B \cap S \cap U) = \operatorname{diam}(B_i \cap S \cap U) < 1\,.$$

By Lemma 3.1, we can choose a coordinate set T with $T \subset S$ and

$$\emptyset \neq B \cap T \subset B \cap S \cap U\,.$$

Then $\operatorname{diam}(B \cap T) < 1$.

Now we can choose points ξ^+, ξ^- in $B \cap T$ of the form

$$\begin{aligned} \xi_n^\pm &= \xi_n \quad \text{for } n \neq m\,,\\ \xi_m^\pm &= \pm 1\,. \end{aligned}$$

Hence $\operatorname{diam}(B \cap T) \geq \|\xi_m^+ - \xi_m^-\| = 2$. This contradiction establishes the first statement of the example.

It is easy to see that each point of ℓ^∞ has weak neighbourhoods of arbitrarily small τ-diameter. Hence the second statement.

Before we give the detailed verification of Example 3.3 we describe some of the properties of the Banach space $E = \ell_{\mathrm{c}}^\infty(\Gamma)$. We take Γ to be an uncountable discrete set. As usual, $\ell^\infty(\Gamma)$ denotes the Banach space of all bounded real-valued maps $x : \Gamma \to \mathbb{R}$ with the supremum norm

$$\|x\| = \sup\{|x_\gamma| : \gamma \in \Gamma\}\,.$$

The support of an element x of $\ell^\infty(\Gamma)$ is defined to be the set

$$\operatorname{supp} x = \{\gamma : \gamma \in \Gamma \text{ and } x_\gamma \neq 0\}\,.$$

We take $\ell_{\mathrm{c}}^\infty(\Gamma)$ to be the set of all elements x of $\ell^\infty(\Gamma)$ with countable support. It is easy to verify that $\ell_{\mathrm{c}}^\infty(\Gamma)$ is a closed linear subspace of $\ell^\infty(\Gamma)$ and so is a Banach space in its own right under the supremum norm. This Banach space $E = \ell_{\mathrm{c}}^\infty(\Gamma)$ is studied in §5 of the paper [3] by Hansell, Jayne and Talagrand. In particular, they show that, for each linear functional μ in the dual E^* of E, there is a countable subset $\Theta = \Theta(\mu)$ of Γ such that

$$\mu(x) = 0 \quad \text{for all } x \text{ in } E \text{ with } (\operatorname{supp} x) \cap \Theta = \emptyset\,.$$

It is convenient to introduce another topology τ on $E = \ell_{\mathrm{c}}^\infty(\Gamma)$ whose base consists of all the sets of the form

$$U(x, \Theta) = \{y \in E : y(\gamma) = x(\gamma) \text{ for all } \gamma \in \Theta\}\,,$$

where $x \in E$ and Θ is a countable subset of Γ. Suppose that $\mu \in E^*$ and $\Theta(\mu)$ is the corresponding countable set provided by the result quoted at the end of the last paragraph. Then, for $y \in E$, the function μ is constant on the τ-open neighbourhood $U(y, \Theta(\mu))$ of y. Thus μ is τ-continuous. Hence the τ-topology is at least as strong as the weak topology on E. From the definition of τ it is clear that a τ-G_δ-set is τ-open. We need the following two lemmas, both of which are known.

LEMMA 3.3. *The space (E,τ) is Baire.*

Proof. In fact, (E,τ) is α-favourable in the sense of Choquet [1]. A winning strategy is given by associating to each non-empty set V a basic open set contained in V. That this function is winning is based on the fact that if $\{U(x_n,\Theta_n) : n \in \mathbb{N}\}$ is a decreasing sequence of basic open sets, then its intersection is non-empty. Indeed, in this case

$$\Theta_n \subset \Theta_{n+1} \quad \text{and} \quad x_{n+1}|\Theta_n = x_n|\Theta_n\,,$$

for each n. Hence if x is defined to be the common extension of $\{x_n|\Theta_n : n \in \mathbb{N}\}$ on $\Theta = \bigcup_{n=1}^{\infty}\Theta_n$ and to be null on $\Gamma\setminus\Theta$, then $x \in U(x_n,\Theta_n)$ for each n. Using this property of basic open sets and mimicking the usual proof of the Baire Category Theorem, one can also prove directly that (E,τ) is Baire.

The next lemma is an immediate consequence of (a)$\Rightarrow$(b) in Theorem 3.1 of [6]. However, we give a proof for the convenience of the reader.

LEMMA 3.4. *Let ϱ be a lower semi-continuous metric on a Hausdorff Baire space Z. If Z is σ-fragmented by ϱ, then there is a dense G_δ-subset D of Z such that the identity map $Z \to (Z,\varrho)$ is continuous at each point of D.*

Proof. For $\varepsilon > 0$, let

$$O_\varepsilon = \bigcup\{U : U \text{ open in } Z \text{ and } \varrho\text{-diam}\, U < \varepsilon\}\,.$$

Since Z is a Baire space, it suffices to prove that O_ε is dense in Z. Since Z is σ-fragmented by ϱ, $Z = \bigcup_{n=1}^{\infty} Z_n$ where each Z_n is fragmented by ϱ down to ε. Let U be a non-empty open set in Z. Then U is a Baire space in the induced topology, and hence, for some n, the closure of Z_n contains a non-empty open subset V of U, i.e. $Z_n \cap V$ is dense in V. Then there is an open subset W of Z such that

$$Z_n \cap V \cap W \neq \emptyset \quad \text{and} \quad \varrho\text{-diam}(Z_n \cap V \cap W) < \varepsilon\,.$$

Thus $V \cap W \neq \emptyset$, and since $Z_n \cap V \cap W$ is dense in $V \cap W$ and ϱ is lower semi-continuous,

$$\varrho\text{-diam}(V \cap W) = \varrho\text{-diam}(Z \cap V \cap W) < \varepsilon\,.$$

It follows that $V \cap W \subset O_\varepsilon$ and hence $O_\varepsilon \cap U \neq \emptyset$.

Verification of Example 3.3. Assume that $(E$, weak$)$ is σ-fragmented by a lower semi-continuous metric ϱ. Then since τ is stronger than the weak topology, ϱ is τ lower semi-continuous and (E,τ) is σ-fragmented by ϱ. By Lemma 3.3, (E,τ) is a Baire space, and therefore by Lemma 3.4 the identity map $(E,\tau) \to (E,\varrho)$ is continuous at some point, say x, in E. Then the singleton $\{x\}$ is a τ-G_δ-set and hence τ-open. But this is impossible, since Γ is uncountable. This proves that (E, weak) is not σ-fragmented by a lower semi-continuous metric.

Similarly suppose now that (E, weak) is fragmented by a metric ϱ, not necessarily lower semi-continuous. Then (E,τ) is fragmented by ϱ. By induction, we can choose a decreasing sequence $\{U(x_n, \Theta_n) : n \in \mathbb{N}\}$ of basic τ-open sets with $\varrho\text{-diam}\, U(x_n, \Theta_n) \to 0$. As remarked in the proof of Lemma 3.3, the intersection of this sequence is open, and not empty; it is also of ϱ-diameter 0. Thus we are again led to a contradiction.

As we have already remarked, Hansell, Jayne and Talagrand [3] discuss this example and give a proof, that is not quite complete, of the second statement. The lacuna is on page 218 where they claim that $W \subset V_n$ apparently because they believe that they have ensured that z_0 belongs to V_n, a belief that seems hard to justify.

An alternative verification for Example 3.3 can be obtained by proving just the second assertion in the example by the method used above and by then appealing to the general result of Ribarska [15] mentioned in the introduction.

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