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Movability and limits of polyhedra

by

V. F. Laguna (Madrid), **M. A. Moron** (Madrid),
Nguyen To Nhu (Hanoi) and **J. M. R. Sanjurjo** (Madrid)

Abstract. We define a metric d_{S}, called the *shape metric*, on the hyperspace 2^X of all non-empty compact subsets of a metric space X. Using it we prove that a compactum X in the Hilbert cube is movable if and only if X is the limit of a sequence of polyhedra in the shape metric. This fact is applied to show that the hyperspace $(2^{\mathbb{R}^2}, d_{\mathrm{S}})$ is separable. On the other hand, we give an example showing that $2^{\mathbb{R}^2}$ is not separable in the fundamental metric introduced by Borsuk.

Introduction. For a metric space X let 2^X denote the collection of all non-empty compact subsets of X. There have been several methods developed for imposing a metric topology on 2^X. The best known makes use of the Hausdorff metric d_{H} (see [16] for a wide information). The Hausdorff metric plays an important role in topology, though the topological properties of compacta have little influence on such a distance.

There have been several attempts (see [2], [4], [5], [7]) to introduce on 2^X other metrics with the property that if compacta $A_1, A_2, \ldots$ converge to a compactum A_0 then some topological or shape properties of A_n pass onto the limit A_0. In other words, to find properties which define closed subsets in the corresponding topology in 2^X.

K. Borsuk introduced in [4] the fundamental metric d_{F}, which is a rather natural modification of the metric of continuity. In its definition Borsuk used, instead of maps, fundamental sequences, which are a basic concept for the theory of shape. The fundamental metric has many good properties of shape-theoretical nature. For instance [6], if $\{A_n\}$ is a sequence of compacta such that $d_{\mathrm{F}}(A_n, A) \to 0$ and $A \in \mathrm{FANR}$ then $\mathrm{sh}(A) \leq \mathrm{sh}(A_n)$ for almost all n. If A is a more general compactum then some good properties are still formulable in terms of quasi-domination [4].

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The authors feel that the property of being the limit of a sequence of polyhedra or ANR's in a metric designed for the study of the shape properties of compacta should be a shape invariant property. In other words, if A is a limit of polyhedra and $\mathrm{Sh}(A) = \mathrm{Sh}(A')$ then A' should also be a limit of polyhedra. However, this is not the case with the fundamental metric, where the property of being a limit of polyhedra characterizes the class of Clapp's AANR's (see [10]) and this property is not a shape invariant.

On the other hand, it is commonly known that the Warsaw circle is not a limit of topological circles in the fundamental metric and this certainly goes against the usual intuition in the context of shape theory.

In this paper we propose a new definition of a metric suitable for the study of the shape properties of compacta. We modify Borsuk's original definition making it more flexible. Roughly speaking, we change the rigid notion of ε-pushes in the definition of the fundamental metric by the more flexible one of homotopies in small neighbourhoods of compacta. In this way we obtain the shape metric d_{S} which conserves all the good properties of the fundamental metric established by Borsuk, Boxer, Čerin, Sher, Šostak and others. Moreover, we prove that a compactum in the Hilbert cube is movable if and only if it is the limit of a sequence of polyhedra. As a consequence, the property of being a limit of polyhedra in d_{S} is a shape invariant property. This result also gives a new characterization of movability. As a corollary of this result we deduce that $2^{\mathbb{R}^2}$ is separable if we use the metric d_{S}, in contrast to the fact, also proved in this paper, that $2^{\mathbb{R}^2}$ is not separable in the fundamental metric. We also prove, among other results, that a compactum X is of trivial shape if and only if it is the limit, in the shape metric, of a sequence of arcs. This result is also false for the fundamental metric. Our metric also makes possible the representation of the Warsaw circle as a limit of topological circles.

Finally, we formulate (see Proposition 3.9) in terms of convergence in 2^Q_{S} one of Borsuk's oldest unsolved problems in shape theory.

Basic references for shape theory are [3], [11], [15].

In this paper "map" is a continuous function, "compactum" is a compact metric space. We denote by $i_{A,B} : A \to B$ the inclusion map and by $S(f)$ the shape morphism induced by f.

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1. The shape metric. Let H denote the Hilbert space of all square summable sequences of real numbers. If X is a subset of H we denote by 2^X the family of all non-empty compact subsets of X. If $Z \subset X$ we denote by $\mathbf{B}(Z,\varepsilon)$ $(\varepsilon > 0)$ the set

$$\mathbf{B}(Z,\varepsilon) = \{x \in H : d(x,Z) < \varepsilon\}\,.$$

We define what we call the *shape metric* on 2^X as follows:

1.1. Definition. Let $A, B \in 2^X$. We consider the non-empty set of positive numbers

$$\begin{aligned} S(A,B) = \{\varepsilon > 0 : A \subset \mathbf{B}(B,\varepsilon),\ B \subset \mathbf{B}(A,\varepsilon) &\text{ and there exist shape} \\ &\text{morphisms } \underline{f} : A \to B \text{ and } \underline{g} : B \to A \text{ with} \\ &S(i_{B,\mathbf{B}(A,\varepsilon)})\underline{f} = S(i_{A,\mathbf{B}(A,\varepsilon)}) \text{ and} \\ &S(i_{A,\mathbf{B}(B,\varepsilon)})\underline{g} = S(i_{B,\mathbf{B}(B,\varepsilon)})\}\,. \end{aligned}$$

We define $d_{\mathrm{S}}(A,B) = \inf S(A,B)$.

By using standard arguments it is easy to prove

1.2. Proposition. d_{S} *is a metric on* 2^X.

To compare d_{S} with other known metrics on 2^X we have

1.3. Proposition. *Let X be a subset of H. Then*

$$d_{\mathrm{H}} \le d_{\mathrm{S}} \le d_{\mathrm{F}} \le d_{\mathrm{C}} \quad \textit{in } 2^X\,,$$

where d_{C} is the metric of continuity [2].

Proof. We only have to prove that

$$d_{\mathrm{S}}(A,B) \le d_{\mathrm{F}}(A,B) \quad \text{for every } A, B \in 2^X\,.$$

Assume that $d_{\mathrm{F}}(A,B) < \varepsilon$. Then there are two fundamental sequences $\underline{\alpha} = \{\alpha_k, A, B\}$ and $\underline{\beta} = \{\beta_k, B, A\}$ and a neighbourhood (U,V) of (A,B) such that for almost all k, $d(x,\alpha_k(x)) < \varepsilon$ for every $x \in U$ and $d(y,\beta_k(y)) < \varepsilon$ for every $y \in V$. Then $A \subset \mathbf{B}(B,\varepsilon)$ and $B \subset \mathbf{B}(A,\varepsilon)$. Since H is convex the segment joining x and $\alpha_k(x)$ is contained in $\mathbf{B}(A,\varepsilon)$ for every $x \in A$. It follows that

$$S(i_{B,\mathbf{B}(A,\varepsilon)})\underline{f} = S(i_{A,\mathbf{B}(A,\varepsilon)})\,,$$

where $\underline{f}$ is the shape morphism generated by $\underline{\alpha}$. Analogously we have

$$S(i_{A,\mathbf{B}(B,\varepsilon)})\underline{g} = S(i_{B,\mathbf{B}(B,\varepsilon)})\,,$$

where $\underline{g}$ is the shape morphism generated by $\underline{\beta}$. Consequently, $d_{\mathrm{S}}(A,B) \le \varepsilon$ and the proposition is proved.

1.4. Examples. (a) It is well known that in the Hausdorff metric every compactum is the limit of a sequence of finite sets. On the other hand, it is easy to see that no circle is the limit of a sequence of finite sets in the shape metric. Therefore the function assigning to every compactum $A \in 2^X_{\mathrm{H}}$ the same compactum $A \in 2^X_{\mathrm{S}}$ is not always continuous.

(b) As we know (see [8] for example), the limit of a sequence of ANR-spaces in the fundamental metric is an $\mathrm{AANR_C}$. On the other hand, it is easy to see that the Warsaw circle, which does not belong to the class $\mathrm{AANR_C}$, is the limit of a sequence of circles in the shape metric. Therefore the metric d_{S} is essentially stronger than d_{F} and the two metrics are not topologically equivalent.

The next proposition is very useful in the case when we may use ambient spaces, such as Euclidean space $\mathbb{R}^n$ or the Hilbert cube Q instead of H.

1.5. PROPOSITION. *Let C be a closed convex set of H and $X \subset C$. Then the metric defined on 2^X exactly in the same way as in Definition* 1.1, *but using C instead of H as ambient space*, *agrees with the shape metric.*

We leave to the reader the proof of this proposition which depends on the fact that there exists a retraction $r : H \to C$ such that $d(x, r(x)) = d(x, C)$ for every $x \in H$. When $x \in C$ this retraction maps the ball $B(x, \varepsilon)$ taken in H onto the ball with the same centre and radius taken in C.

The following observation shows that 2^X_{S} is a metric invariant of X.

1.6. PROPOSITION. *Let X and X' be subsets of H. If there is an isometry between X and X' then 2^X_{S} and $2^{X'}_{\mathrm{S}}$ are isometric. If X and X' are homeomorphic then so are 2^X_{S} and $2^{X'}_{\mathrm{S}}$.*

Proof. The first assertion is a consequence of Proposition 1.5 and the fact, proved in [19], p. 50, that every isometry between compact subsets of H can be extended to an isometry between their closed linear spans. The second assertion is a consequence of the fact that every homeomorphism between compact subsets of H can be extended to a homeomorphism of H (see [14]).

2. Some properties of the fundamental metric retained by the shape metric. In this brief section we examine some properties of the fundamental metric (see [4], [9] for example) that remain true for the shape metric. Proofs similar to those in [4] and [9] are skipped.

2.1. PROPOSITION. *Let $\{A_n\}$ be a sequence of compacta in H such that $A_n \to A$ in the shape metric. If B is a compactum such that $B \overset{\mathrm{q}}{\geq} A_n$ for every $n \in \mathbb{N}$ then $B \overset{\mathrm{q}}{\geq} A$, where $\overset{\mathrm{q}}{\geq}$ is the quasi-domination relation.*

2.2. PROPOSITION. *Let X be a subset of H. Let $\{A_n\}$ be a sequence of compacta lying in X such that $d_{\mathrm{S}}(A_n, A) \to 0$. If $A \in \mathrm{FANR}$ then $\mathrm{sh}(A) \leq \mathrm{sh}(A_n)$ for almost all n.*

2.3. PROPOSITION. *Let X be a metric space. Then the family $T(X)$ of all compacta of trivial shape in X is closed in 2^X_{S}.*

P r o o f. Assume that $A \in \operatorname{cl} T(X)$. Then there is a sequence $\{A_n\}$ of compacta of trivial shape in H such that $A_n \to A$ in the shape metric. It follows from Proposition 2.1 that A is quasi-dominated by a point and, hence, A is of trivial shape.

As we have seen before, $d_{\mathrm{H}}(A,B) \le d_{\mathrm{S}}(A,B)$ for every $A, B \in 2^X$ and the two metrics are not topologically equivalent. However, for compacta with trivial shape we have the following:

2.4. PROPOSITION. *For all compacta A and B of trivial shape we have $d_{\mathrm{H}}(A,B) = d_{\mathrm{S}}(A,B)$.*

P r o o f. Assume that $d_{\mathrm{H}}(A,B) < \varepsilon$. Then $A \subset \mathbf{B}(B,\varepsilon)$ and $B \subset \mathbf{B}(A,\varepsilon)$. Since A and B are of trivial shape, if $\underline{f} : A \to B$ and $\underline{g} : B \to A$ are shape morphisms induced by constant maps we have $S(i_{B,\mathbf{B}(A,\varepsilon)})\underline{f} = S(i_{A,\mathbf{B}(A,\varepsilon)})$ and $S(i_{A,\mathbf{B}(B,\varepsilon)})\underline{g} = S(i_{B,\mathbf{B}(B,\varepsilon)})$. Hence $d_{\mathrm{S}}(A,B) \le \varepsilon$ and the proof is finished.

2.5. R e m a r k. In connection with Proposition 2.4 one may ask whether $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ implies $d_{\mathrm{H}}(A,B) = d_{\mathrm{S}}(A,B)$. As the reader can easily check, this is not the case even if A and B are homeomorphic. In the case of Figure 1, A and B are homeomorphic, but $d_{\mathrm{H}}(A,B) < d_{\mathrm{S}}(A,B)$.

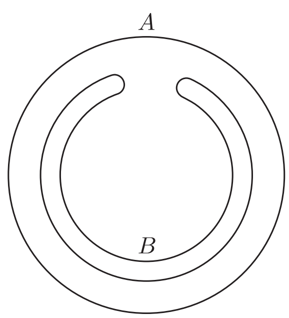

Fig. 1. $A =$ circle, $B =$ circle

2.6. PROPOSITION. *A compactum X lying in the Hilbert cube Q is of trivial shape if and only if X is the limit of a sequence of arcs in 2^Q_{S}.*

P r o o f. The "if" part is a consequence of Proposition 2.3.

Assume that X is of trivial shape. Let L_n be an arc in $\mathbf{B}(X,1/n)$ which is $1/n$-dense in $\mathbf{B}(X,1/n)$. By 2.4 we have $d_{\mathrm{S}}(L_n,X) = d_{\mathrm{H}}(L_n,X) \le 1/n$, and hence $L_n \to X$.

2.7. R e m a r k. Proposition 2.6 does not hold for the fundamental metric. If we consider the plane continuum $X = Y \cup Z$, where Y is a segment, Z is the topologist sine curve and $Y \cap Z = \{p\}$ with $p \in$ limit segment of Z

then X is of trivial shape but not an $\mathrm{AANR_C}$ and hence it is not a limit of arcs in the fundamental metric.

2.8. PROPOSITION. *Let $A_0, A_1, A_2, \ldots$ be a sequence of compacta such that $\lim d_{\mathrm{S}}(A_k, A_0) = 0$. If $p_n(A_k) \le m$ for every $k = 1, 2, \ldots$, where p_n denotes the n-th Betti number in the Vietoris homology theory, then $p_n(A_0) \le m$.*

3. On limits of polyhedra in the shape metric. In this section we give the main results discussed in the introduction.

3.1. THEOREM. *Let X be a compact subset of the Hilbert cube $Q = \prod_{i=1}^{\infty} I_i$. Then the following conditions are equivalent.*

(a) *X is movable.*

(b) *For every $\varepsilon > 0$ there exists a closed neighbourhood U of X in Q such that $d_{\mathrm{S}}(X, U) < \varepsilon$.*

(c) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $Y \subset \mathbf{B}(X, \delta)$ is a compactum containing X then $d_{\mathrm{S}}(X, Y) < \varepsilon$.*

(d) *There exists a sequence (P_n) of polyhedra in Q such that $\lim_{n\to\infty} d_{\mathrm{S}}(X, P_n) = 0$.*

Proof. We prove that (a)⇒(c)⇒(d)⇒(a). The implication (c)⇒(b) is obvious, and the proof of (b)⇒(a) is given below in the last sentence of the proof of (d)⇒(a).

(a)⇒(c). Assume that X is movable and let $\varepsilon > 0$ be given. By [18], X is uniformly movable and, hence, there exists a neighbourhood U of X in $\mathbf{B} = \mathbf{B}(X, \varepsilon)$ and a shape morphism $\underline{g} : U \to X$ such that $S(i_{X,\mathbf{B}})\underline{g} = S(i_{U,\mathbf{B}})$. If $Y \subset U$ is a compactum containing X then $\underline{g}S(i_{Y,U})$ and $S(i_{X,Y})$ demonstrate $d_{\mathrm{S}}(X, Y) < \varepsilon$.

(c)⇒(d). Select polyhedra $P_n \subset \prod_{i=1}^{n} I_i$, $n = 1, 2, \ldots$, so that $U_n = \{(x_i) \in Q : (x_1, \ldots, x_n, 0, 0, \ldots) \in P_n\}$ is a descending sequence of neighbourhoods of X (see [3]). We clearly have $d_{\mathrm{S}}(P_n, U_n) \to 0$, and therefore $d_{\mathrm{S}}(P_n, X) \to 0$.

(d)⇒(a). Assume that $P_n \to X$ in 2^Q_{S}, where the P_n are polyhedra. Let U be a neighbourhood of X in Q. Consider $\varepsilon > 0$ such that $\mathbf{B}(X, \varepsilon) \subset U$ and for sufficiently large n we have $\mathbf{B}(P_n, \varepsilon) \subset U$ and such that there are shape morphisms $\underline{f}' : P_n \to X$ and $\underline{g}' : X \to P_n$ with

$$S(i_{X,\mathbf{B}(P_n,\varepsilon)})\underline{f}' = S(i_{P_n,\mathbf{B}(P_n,\varepsilon)}) \quad \text{and} \quad S(i_{P_n,\mathbf{B}(X,\varepsilon)})\underline{g}' = S(i_{X,\mathbf{B}(X,\varepsilon)})\,.$$

Since P_n is a polyhedron, there exists a neighbourhood U_0 of X in $\mathbf{B}(X, \varepsilon)$ and an extension $\underline{g} : U_0 \to P_n$ of $\underline{g}'$ such that

$$S(i_{P_n,\mathbf{B}(X,\varepsilon)})\underline{g} = S(i_{U_0,\mathbf{B}(X,\varepsilon)})\,.$$

Then we define $\underline{f} : U_0 \to X$ by $\underline{f} = \underline{f}'\underline{g}$. Clearly, we have

$$S(i_{X,U})\underline{f} = S(i_{X,U})\underline{f}'\underline{g} = S(i_{P_n,U})\underline{g} = S(i_{U_0,U})\,.$$

This shows that X is uniformly movable and hence movable. The proof of Theorem 3.1 is finished. ∎

3.2. Remark. It can be proved in an analogous way that a compactum $X \subset \mathbb{R}^n$ is movable if and only if X is the limit of a sequence of polyhedra in $\mathbb{R}^n$.

As consequences of Theorem 3.1 we obtain the following

3.3. Corollary. *The family of all movable compacta in Q is closed in 2^Q_{S}.*

Since the family of all compact polyhedra in Q is separable with the metric of continuity and hence separable with the shape metric, from Theorem 3.1 we get

3.4. Corollary. *The family of all movable compacta in Q is separable.*

Since all plane compacta are movable (see Borsuk [3]), from Theorem 3.1 we get

3.5. Corollary. $2^{\mathbb{R}^2}_{\mathrm{S}}$ *is separable.*

As we shall see in Section 4, Corollary 3.5 does not hold true for the fundamental metric.

Since zero-dimensional compacta are movable we have

3.6. Corollary. *The family of all zero-dimensional compacta in Q is separable in the shape metric.*

One may ask if Corollary 3.5 holds true for higher dimensions. The following theorem shows that the answer is no.

3.7. Theorem. $2^{\mathbb{R}^3}_{\mathrm{S}}$ *is not separable.*

Proof. Consider an uncountable family of sequences of prime numbers such that for any sequences $\underline{k} = \{k_n\}$ and $\underline{k}' = \{k'_n\}$ with $\underline{k} \neq \underline{k}'$ we have $\underline{k} \setminus \underline{k}' \neq \emptyset$ and $\underline{k}' \setminus \underline{k} \neq \emptyset$. For every $\underline{k}$ we construct the solenoid $S(k_1, k_2, \ldots)$ as an intersection of a sequence of solid tori $T_1, T_2 \ldots$ (see Godlewski [12]). We can assure that all of them lie in a solid torus T (the same for all the solenoids) and the winding number of T_i in T is $k_1 k_2 \ldots k_i$ and that the generalized ball $\mathbf{B}(T_i, 1)$ is contained in T for every $i = 1, 2, \ldots$

We shall prove that if $\underline{k} \neq \underline{k}'$ then $d_{\mathrm{S}}(S(\underline{k}), S(\underline{k}')) \geq 1$. Otherwise there is a fundamental sequence $\underline{\alpha} = \{\alpha_n, S(\underline{k}), S(\underline{k}')\}$ such that $\underline{\alpha}$ is homotopic in T to the inclusion $i : S(\underline{k}) \to T$. Then if we take a $k'_i \in \underline{k}' \setminus \underline{k}$, there exists a solid torus T_j which is a neighbourhood of $S(\underline{k})$ in T and an index n such that α_n maps T_j into a solid torus T'_i which is a neighbourhood of $S(\underline{k}')$ in T

and whose winding number is $k'_1 k'_2 \ldots k'_i$ and such that $\alpha_n|T_j$ is homotopic in T to the inclusion $i : T_j \to T$. But this is impossible since the winding number of T_j in T is $k_1 k_2 \ldots k_j$ and $k'_i \in \underline{k}' \backslash \underline{k}$. Therefore $d_{\mathrm{S}}(S(\underline{k}), S(\underline{k}')) \geq 1$ and as a consequence $2^{\mathbb{R}^3}_{\mathrm{S}}$ is not separable. This completes the proof.

One of Borsuk's oldest unsolved problems in shape theory is the following (see [3]):

3.8. Problem. Let $\{X_n\}$ be a decreasing sequence of compact ANR-spaces such that there exists a retraction r_n from X_n onto X_{n+1} for every $n \in \mathbb{N}$. Is it true that the intersection $\bigcap_{n=1}^{\infty} X_n$ is an FANR-space?

Some partial answers to this problem have been given (see [3], [13]). Our shape metric gives a reformulation of Problem 3.8.

3.9. Proposition. *Let $\{X_n\}$ be as in* 3.8. *Then the intersection $\bigcap_{n=1}^{\infty} X_n$ is an FANR-space if and only if it is the limit of $\{X_n\}$ in the shape metric.*

Proof. Let $X = \bigcap_{n=1}^{\infty} X_n$ and suppose that $X_n \to X$ (in the shape metric). Then by Corollary 3.3, X is movable. On the other hand, X is a WANR-space (see [1]). It follows that X is an FANR-space.

Conversely, if $X \in$ FANR then X is movable and hence $X = \lim_{n\to\infty} X_n$ by Theorem 3.1.

3.10. Remark. Let us note that, concerning Proposition 3.9, the situation is far from being the same in the fundamental metric: We are able to prove that if $\{X_n\}$ is a sequence of ANR-spaces in Q satisfying all hypotheses of Problem 3.8 then the following conditions are equivalent (recall that $X = \bigcap_{n=1}^{\infty} X_n$):

(a) X_n converges to X in the metric of continuity;
(b) X_n converges to X in the fundamental metric;
(c) X is an $\mathrm{AANR}_{\mathrm{C}}$;
(d) X is an $\mathrm{AANR}_{\mathrm{N}}$ (AANR in the sense of Noguchi [17]);

Now we are going to show that, in some cases, proximity in d_{S} implies shape equivalence.

3.11. Proposition. *Let K be an FANR-space in the Hilbert cube Q and let $\mathcal{K}$ denote the family of all fundamental retracts of K. Then there exists an $\varepsilon > 0$ such that if $K', K'' \in \mathcal{K}$ and $d_{\mathrm{S}}(K', K'') < \varepsilon$ then* $\mathrm{sh}(K') = \mathrm{sh}(K'')$.

The $\varepsilon > 0$ should be chosen such that K is a shape retract of $\mathbf{B}(K, \varepsilon)$ in Q. We leave the proof of Proposition 3.11 to the reader.

As a consequence of Proposition 3.11 we have

3.12. Corollary. *Let K be an FANR-space in the Hilbert cube Q. Then there exists an $\varepsilon > 0$ such that if K' is a retract of K and* $\mathrm{diam}(K') < \varepsilon$ *then K' has trivial shape.*

4. The plane hyperspace with the fundamental metric. In contrast to Corollary 3.5, in this section we present an example showing that the hyperspace of compacta in the plane equipped with the fundamental metric is not separable. Namely, we prove the following theorem:

4.1. THEOREM. *There exists an uncountable family $\{S_p\}$ of compacta in the plane such that $d_{\mathrm{F}}(S_p, S_{p'}) \geq 1/8$ for every $p \neq p'$.*

We use the following notation: For every $n \in \mathbb{N}$ put

(1) $C_n = \{x \in \mathbb{R}^2 : \|x\| = 5 + n^{-1}\}$ and $C_\infty = \{x \in \mathbb{R}^2 : \|x\| = 5\}$;

(2) $a_n = (0; 5 + n^{-1}) \in C_n$, $\quad b_n = (n^{-1}; [n^{-1}(25n + 10)]^{1/2}) \in C_n$.

Let (a_n, b_n) denote the small arc from a_n to b_n in C_n. Put

$$B_n = \overline{C_n \setminus (a_n, b_n)} \subset \mathbb{R}^2 . \tag{3}$$

For every $a, b \in \mathbb{R}^2$, let $[a, b] = \{ta + (1 - t)b : 0 \leq t \leq 1\} \subset \mathbb{R}^2$. For every $n \geq m$ we set

$$A^k_{m,n} = \begin{cases} \displaystyle\bigcup_{i=m}^{n} B_i \cup \bigcup_{i=m}^{n-1} [b_i, a_{i+1}] \cup [b_n, b_{n+1}] & \text{if } k \text{ is odd,} \\ \displaystyle\bigcup_{i=m}^{n} B_i \cup \bigcup_{i=m}^{n-1} [a_i, b_{i+1}] \cup [a_n, a_{n+1}] & \text{if } k \text{ is even.} \end{cases} \tag{4}$$

Let $p = \{n_k\}$ be an increasing sequence of natural numbers. We define

$$S_p = C_\infty \cup \bigcup_{k=1}^{\infty} A^k_{m_{k-1}+1, m_k} \subset \mathbb{R}^2 \tag{5}$$

where $m_0 = 0$ and $m_k = n_1 + \ldots + n_k$.

Observe that for an increasing sequence $p = \{n_k\}$ of natural numbers the compactum S_p is a spiral constructed in the following way: First we go n_1 rounds from a_1 to b_{n_1} (through the segments $[b_i, a_{i+1}]$, $i = 1, \ldots, n_1 - 1$).

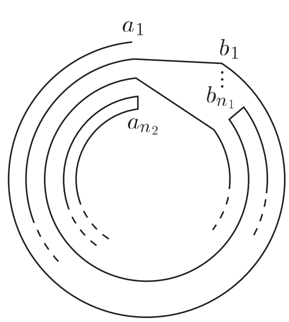

Fig. 2.

Then at b_{n_1} we change the direction and go in the opposite direction n_2 rounds from b_{n_1} to a_{n_2} (through the segments $[a_i, b_{i+1}]$, $i = n_1+1, \ldots, n_2-1$) and so on, see Figure 2.

Let P denote the family of all increasing sequences of natural numbers. Observe that P is uncountable. We now show that the family $\{S_p\}_{p\in P}$ satisfies the condition of Theorem 4.1.

Suppose $d_{\mathrm{F}}(S_p, S_{p'}) < 1/8$. Fix $n \in \mathbb{N}$. There exist neighbourhoods U of S_p and V of $S_{p'}$ in $\mathbb{R}^2$ and retractions $r : U \to \mathbf{B}(C_\infty, 1/n) \cup S_p$ and $r : V \to \mathbf{B}(C_\infty, 1/n) \cup S_{p'}$ which are also 1/8-pushes. By definition of d_{F} (see [4]), there exist 1/8-pushes f_k of S_p into V and g_ℓ of $S_{p'}$ into U. Consequently, there exist 1/4-pushes f of S_p into $\mathbf{B}(C_\infty, 1/n) \cup S_{p'}$ and g of $S_{p'}$ into $\mathbf{B}(C_\infty, 1/n) \cup S_p$. It is easy to see that $f(a_1) \in C_1$ and $g(a_1) \in C_1$ and then, that $S_p \setminus \mathbf{B}(C_\infty, 2/n) = S_{p'} \setminus \mathbf{B}(C_\infty, 2/n)$. Hence $S_p = S_{p'}$.

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V. F. Laguna
DEPARTAMENTO DE
MATEMATICA FUNDAMENTAL
FACULTAD DE CIENCIAS, U.N.E.D.
28040 MADRID, SPAIN

M. A. Moron
DEPARTAMENTO DE MATEMATICAS
E.T.S.I. DE MONTES
UNIVERSIDAD POLITECNICA
28040 MADRID, SPAIN

Nguyen To Nhu
INSTITUTE OF MATHEMATICS
P.O. BOX 631
BO HO
HANOI, VIETNAM

J. M. R. Sanjurjo
DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA
FACULTAD DE MATEMATICAS
UNIVERSIDAD COMPLUTENSE
28040 MADRID, SPAIN

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