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Almost split sequences for non-regular modules

by

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Abstract. Let A be an Artin algebra and let $0 \to X \to \bigoplus_{i=1}^{r} Y_i \to Z \to 0$ be an almost split sequence of A-modules with the Y_i indecomposable. Suppose that X has a projective predecessor and Z has an injective successor in the Auslander–Reiten quiver Γ_A of A. Then $r \le 4$, and $r = 4$ implies that one of the Y_i is projective-injective. Moreover, if $X \to \bigoplus_{j=1}^{t} Y_j$ is a source map with the Y_j indecomposable and X on an oriented cycle in Γ_A, then $t \le 4$ and at most three of the Y_j are not projective. The dual statement for a sink map holds. Finally, if an arrow $X \to Y$ in Γ_A with valuation (d, d') is on an oriented cycle, then $dd' \le 3$.

Let A be a fixed Artin algebra, $\operatorname{mod} A$ the category of finitely generated left A-modules and $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$. Denote by Γ_A the Auslander–Reiten quiver of A. The shape of a connected component of Γ_A without projectives or without injectives is fairly well understood [5, 8, 12]. The results of this paper will give some information on connected components of Γ_A which contain both a projective module and an injective module.

The notion of an almost split sequence, which was introduced by Auslander and Reiten in [1], plays a fundamental role in the representation theory of algebras (see, for example, [10]). Let $0 \to X \to \bigoplus_{i=1}^{r} Y_i \to Z \to 0$ be an almost split sequence in $\operatorname{mod} A$ with the Y_i indecomposable. Then the number r measures the complication of the maps in $\operatorname{mod} A$ starting with X and those ending with Z. Therefore it is interesting to find the number of the indecomposable summands of the middle term of an almost split sequence. The well-known Bautista–Brenner theorem [3] states that if A is of finite representation type, then the middle term of an almost split sequence in $\operatorname{mod} A$ has at most four indecomposable summands, and the number four occurs only in the case where one indecomposable summand is projective-injective. Our main result clearly generalizes this theorem. Moreover, we will also discuss almost split sequences for modules on oriented cycles in Γ_A.

We begin with the following easy observation.

LEMMA 1. *Let $g : Y \to Z$ be an irreducible epimorphism with Z indecomposable, and let*

$$Z_n \to Z_{n-1} \to \ldots \to Z_1 \to Z_0 = Z$$

be a sectional path in Γ_A with $n \geq 1$. If there is an irreducible map from $Y \oplus Z_1$ to Z, then Z_i is not projective for $0 \leq i \leq n$ and there is an irreducible epimorphism $g_i : D\,\mathrm{Tr}\,Z_i \to Z_{i+1}$ for $0 \leq i < n$.

Proof. The lemma follows from the easy facts that if

$$0 \to X \xrightarrow{(f,f')} Y \oplus Y' \xrightarrow{\binom{g}{g'}} Z \to 0$$

is an exact sequence, then g is epic if and only if f' is epic, and that if $p : M \to N$ is an epimorphism, then so is the co-restriction of p to a summand of N.

We have the following immediate consequence.

COROLLARY 2. *Let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence with the Y_i indecomposable. If the co-restriction of f to Y_i is epic for $1 \leq i \leq r$, then any sectional path in Γ_A ending with Z contains no projective module.

Proof. Assume that the co-restriction of f to Y_i is epic for all $1 \leq i \leq r$. Let

$$Z_n \to Z_{n-1} \to \ldots \to Z_1 \to Z_0 = Z$$

be a sectional path in Γ_A with $n > 0$. Then $Z_1 \cong Y_{i_0}$ for some $1 \leq i_0 \leq r$ and $Z_2 \ncong X$ if $n \geq 2$. Now there is an irreducible epimorphism $h : X \to Z_1$ by assumption. Hence Z_1 is not projective, and if $n > 1$, then Z_j with $2 \leq j \leq n$ is not projective by Lemma 1.

We quote the following lemma from [9].

LEMMA 3. *Let $p : M \to Y$ be a non-zero map with Y indecomposable, and let $f : Y \to Z_1 \oplus Z_2$ be an irreducible map with Z_1, Z_2 indecomposable. If $pf = 0$, then Y, Z_1, Z_2 are not projective, moreover, there is a map $q : M \to D\,\mathrm{Tr}\,Y$ in $\mathrm{mod}\,A$, a map $v : D\,\mathrm{Tr}\,Y \to Y$ in $\mathrm{rad}(\mathrm{mod}\,A)$ and a source map*

$$(h_1, h_2, h) : D\,\mathrm{Tr}\,Y \to D\,\mathrm{Tr}\,Z_1 \oplus D\,\mathrm{Tr}\,Z_2 \oplus U$$

such that $p = qv$ and $qh = 0$.

In the case where there is an irreducible epimorphism $f : P \to Z$ with P indecomposable projective, Auslander and Reiten described in [1] the almost split sequence ending with Z. Thus the following fact is of interest.

COROLLARY 4. *If $f : P \to Z$ is an irreducible epimorphism with P indecomposable projective, then Z is indecomposable. Dually, if $g : X \to I$ is an irreducible monomorphism with I indecomposable injective, then X is indecomposable.*

Proof. Assume that $f : P \to Z$ is an irreducible epimorphism. Let $k : K \to P$ be the kernel of f; then clearly $kf = 0$. Thus Z is indecomposable by Lemma 3.

An indecomposable module X in $\operatorname{mod} A$ is said to be *left stable* if $D\operatorname{Tr}^n X \neq 0$ for all $n \geq 0$, and *right stable* if $\operatorname{Tr} D^n X \neq 0$ for all $n \geq 0$. Let ${}_l\Gamma_A$ be the full subquiver of Γ_A generated by the left stable modules, and ${}_r\Gamma_A$ the full subquiver generated by the right stable modules. We call the connected components of ${}_l\Gamma_A$ *left stable components* of Γ_A, and those of ${}_r\Gamma_A$ *right stable components* of Γ_A [8].

For a module M in mod A, we denote by $\ell(M)$ its composition length.

LEMMA 5. *Let $f : X \to \bigoplus_{i=1}^4 Y_i$ be an irreducible map with X indecomposable and the Y_i indecomposable non-projective. If f is epic or $\ell(X) \geq \ell(\operatorname{Tr} DX)$, then*

(1) *X has no projective predecessor in Γ_A*;
(2) *$\ell(D\operatorname{Tr}^n X)$ monotone grows to infinity*;
(3) *X is not on any oriented cycle in Γ_A.*

Proof. Assume that f is epic or $\ell(X) \geq \ell(\operatorname{Tr} DX)$. We claim that $2\ell(X) \geq \sum_{i=1}^4 \ell(Y_i)$.

Indeed, this is clear if f is epic. Otherwise $\operatorname{Tr} DX \neq 0$ and $\ell(X) \geq \ell(\operatorname{Tr} DX)$. Hence $2\ell(X) \geq \ell(X) + \ell(\operatorname{Tr} DX) \geq \sum_{i=1}^4 \ell(Y_i)$.

Let $h : D\operatorname{Tr} X \to W$ be an irreducible map with W indecomposable. If $W \not\cong D\operatorname{Tr} Y_i$ for all $1 \leq i \leq 4$, then

$$\begin{aligned}\ell(D\operatorname{Tr} X) &\geq \ell(W) + \sum_{i=1}^4 \ell(D\operatorname{Tr} Y_i) - \ell(X)\\ &\geq \ell(W) + \sum_{i=1}^4 (\ell(X) - \ell(Y_i)) - \ell(X) > \ell(W)\,.\end{aligned}$$

If $W \cong D\operatorname{Tr} Y_i$ for some i, say $W \cong D\operatorname{Tr} Y_1$, then

$$\begin{aligned}\ell(D\operatorname{Tr} X) &\geq \sum_{i=1}^4 \ell(D\operatorname{Tr} Y_i) - \ell(X)\\ &\geq \ell(W) + \sum_{i=2}^4 (\ell(X) - \ell(Y_i)) - \ell(X) \geq \ell(W)\,.\end{aligned}$$

Thus h is epic. By Corollary 2, any sectional path in Γ_A ending with X contains no projective module. Moreover, we have

$$\begin{aligned}\ell(D\operatorname{Tr}X) &\geq \sum_{i=1}^{4}\ell(D\operatorname{Tr}Y_i)-\ell(X)\\ &\geq \sum_{i=1}^{4}(\ell(X)-\ell(Y_i))-\ell(X)\geq \ell(X)\,.\end{aligned}$$

By induction we have $\ell(D\operatorname{Tr}^{n+1}X)\geq \ell(D\operatorname{Tr}^{n}X)>0$ for all $n\geq 0$, and any sectional path in Γ_A ending with $D\operatorname{Tr}^n X$ contains no projective module. Thus X has no projective predecessor in Γ_A.

Since $2\ell(X)\geq \sum_{i=1}^{4}\ell(Y_i)$, either $\ell(X)\geq \ell(Y_1)+\ell(Y_2)$ or $\ell(X)\geq \ell(Y_3)+\ell(Y_4)$. Thus we may assume that the co-restriction $g: X\to Y_1\oplus Y_2$ of f is epic. Let $k: K\to X$ be the kernel of g. By Lemma 3, there is a map $k_1: K\to D\operatorname{Tr}X$ in $\operatorname{mod}A$, a map $v_1: D\operatorname{Tr}X\to X$ in $\operatorname{rad}(\operatorname{mod}A)$ and an irreducible epimorphism $g_1: D\operatorname{Tr}X\to D\operatorname{Tr}Y_3\oplus D\operatorname{Tr}Y_4$ such that $k=k_1v_1$ and $k_1g_1=0$. By induction, for all $n>0$, there is a map $k_n: K\to D\operatorname{Tr}^n X$ and a map $v_n: D\operatorname{Tr}^n X\to D\operatorname{Tr}^{n-1}X$ in $\operatorname{rad}(\operatorname{mod}A)$ such that $k=k_nv_n\ldots v_1$. Hence $\ell(D\operatorname{Tr}^n X)$ tends to infinity by the Harada–Sai lemma [6]. In particular, X is not $D\operatorname{Tr}$-periodic.

Let Γ be the left stable component of Γ_A containing X. Then Γ contains no $D\operatorname{Tr}$-periodic module since X is not. Note that all predecessors of X in Γ_A are left stable, hence in Γ. In particular, the $D\operatorname{Tr}Y_i$ are in Γ. So Γ contains no oriented cycle [8, (2.3)]. Thus X is not on any oriented cycle in Γ_A. The proof is complete.

We also need the following lemma.

LEMMA 6. *Let X be an indecomposable module in* $\operatorname{mod}A$ *such that there is a sectional path from X to an injective module in Γ_A. Assume that $f: X\to\bigoplus_{i=1}^{r}Y_i$ is a source map with the Y_i indecomposable. If $r>4$ or $r=4$ with all Y_i non-projective, then X has no projective predecessor and is not on any oriented cycle in Γ_A.*

P r o o f. Let $r\geq 4$, and let

$$(*)\qquad X=X_0\to X_1\to\ldots\to X_{t-1}\to X_t$$

be a shortest sectional path in Γ_A with X_t injective. If $t=0$, then X is injective. Therefore f is epic. Thus the lemma holds by Lemma 5.

Suppose now that $t>0$ and $X_1\cong Y_1$. Then X_j is not injective for $0\leq j<t$, and there is an irreducible epimorphism $f_t: X_t\to\operatorname{Tr}DX_{t-1}$. By Lemma 1, there is an irreducible epimorphism $f_1: Y_1\to\operatorname{Tr}DX$. It follows then that the co-restriction of f to $\bigoplus_{i=2}^{r}Y_i$ is epic. If $r>4$, then the lemma follows from Lemma 5. Assume that $r=4$ with all Y_i non-projective. Note that X is not projective by Corollary 4. By the dual of Lemma 5, we have $\ell(D\,\mathrm{Tr}\,X) \geq \ell(X)$ since X has an injective successor.

Let $h : D\,\mathrm{Tr}\,X \to \bigoplus_{j=1}^{n} W_j$ be a source map with the W_j indecomposable, and $W_j = D\,\mathrm{Tr}\,Y_j$ for $1 \leq j \leq 4$. Since the co-restriction of f to $Y_3 \oplus Y_4$ is epic, by Lemma 3, the co-restriction of h to W_j with $j \neq 3, 4$ is epic. Similarly considering separately the co-restrictions of f to $Y_2 \oplus Y_4$ and $Y_2 \oplus Y_3$ which are epic, we deduce that the co-restrictions of h to W_3, W_4 are epic. Therefore any sectional path in Γ_A ending with X contains no projective module by Corollary 2. In particular, $D\,\mathrm{Tr}\,Y_i$ is not projective for $1 \leq i \leq r$. Hence $D\,\mathrm{Tr}\,X$ has no projective predecessor and is not on any oriented cycle in Γ_A by Lemma 5. Therefore X admits no projective predecessor in Γ_A. Moreover, X is not on any oriented cycle in Γ_A since $D\,\mathrm{Tr}\,X$ is not.

We are ready to get our main result.

THEOREM 7. *Let A be an Artin algebra, and let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in $\mathrm{mod}\,A$ *with the Y_i indecomposable. Assume that X has a projective predecessor and Z has an injective successor in Γ_A. Then $r \leq 4$, and $r = 4$ implies that one of the Y_i is both projective and injective, whereas the others are neither.*

Proof. Let $r \geq 4$. We consider the first case where $\ell(Z) \geq \ell(X)$. Then by the dual of Lemma 5, one of the Y_i is injective. By Lemma 6, we infer that $r = 4$ and one of the Y_i is projective. It is now easy to see that one of the Y_i is both projective and injective, and the others are neither. A dual argument will show that the theorem holds in the case where $\ell(X) \geq \ell(Z)$.

Remark. It is well-known that if A is of finite representation type, then any indecomposable module has a projective predecessor and an injective successor in Γ_A. Hence the above result generalizes the Bautista–Brenner theorem [3].

PROPOSITION 8. *Let A be an Artin algebra, and let X be an indecomposable module in* $\mathrm{mod}\,A$ *which is on an oriented cycle in Γ_A. If $f : X \to \bigoplus_{i=1}^{r} Y_i$ is a source map with the Y_i indecomposable then $r \leq 4$, and $r = 4$ implies that one of the Y_i is projective. Dually, if $g : \bigoplus_{j=1}^{t} Z_j \to X$ is a sink map with the Z_j indecomposable then $t \leq 4$, and $t = 4$ implies that one of the Z_j is injective.*

Proof. Assume that $f : X \to \bigoplus_{i=1}^{r} Y_i$ is a source map with the Y_i indecomposable and $r \geq 4$. Let

$$X = X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n = X$$

be an oriented cycle in Γ_A with $n \geq 2$. If there is a sectional path from X to an injective module in Γ_A, then we are done by Lemma 6.

Assume now that there is no sectional path from X to an injective module in Γ_A. By a result of Bautista and Smalø [4], there is a minimal $m \leq t$ such that $X_m = \operatorname{Tr} DX_{m-2}$. Then X_j is not injective for all $0 \leq j < m$. Thus $\operatorname{Tr} DX$ is also on an oriented cycle in Γ_A. If $\ell(\operatorname{Tr} DX) > \ell(X)$ then, by the dual of Lemma 5, we infer that one of the Y_i is injective, which is a contradiction. Hence $\ell(X) \geq \ell(\operatorname{Tr} DX)$. By Lemma 5, one of the Y_i is projective. Using now the dual of Lemma 6, we deduce that $r = 4$. The proof is complete.

Recall that if $X \to Y$ is an arrow in Γ_A, then its valuation (d, d') is defined so that d' is the multiplicity of X in the domain of the sink map for Y and d is the multiplicity of Y in the codomain of the source map for X.

A path $X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n$ in Γ_A is said to be *pre-sectional* if $D \operatorname{Tr} X_{i+1} = X_{i-1}$ for some $0 < i < n$ implies that the multiplicity of X_{i-1} in the domain of the sink map for X_i is greater than one [7].

LEMMA 9. *Let $X \to Y$ be an arrow in Γ_A with valuation (d, d'). Assume that both d and d' are greater than one. Then neither X nor Y is on an oriented cycle. Moreover, either Y has no projective predecessor or X has no injective successor in Γ_A.*

Proof. Let $f : X \to Y$ be an irreducible map. First assume that f is an epimorphism. Then Y is not projective. Let $g : D \operatorname{Tr} Y \to X \oplus X_1$ be a source map. Then the co-restriction of g to X_1 is an epimorphism. Note that X is a summand of X_1 since $d' > 1$. The co-restriction h of g to X is an epimorphism. By Corollary 2, any sectional path in Γ_A ending with Y contains no projective module. Since $d > 1$ and there is an irreducible epimorphism $h : D \operatorname{Tr} Y \to X$, we similarly conclude that X is not projective and there is an irreducible epimorphism $f_1 : D \operatorname{Tr} X \to D \operatorname{Tr} Y$. Note that the valuation of the arrow $D \operatorname{Tr} X \to D \operatorname{Tr} Y$ is also (d, d').

By induction we have $D \operatorname{Tr}^n Y \neq 0$ for all $n \geq 0$, and any sectional path in Γ_A ending with $D \operatorname{Tr}^n Y$ contains no projective module. Therefore Y has no projective predecessor in Γ_A. Now the arrow $X \to Y$ is contained in a left stable component of Γ_A, say Γ. For all $n > 0$, there is a pre-sectional path

$$D \operatorname{Tr}^n X \to D \operatorname{Tr}^n Y \to D \operatorname{Tr}^{n-1} X \to \ldots \to D \operatorname{Tr} Y \to X \to Y$$

in Γ_A. Thus Y is not $D \operatorname{Tr}$-periodic [7, (1.16)]. Therefore Γ contains no oriented cycle since $X \to Y$ has non-trivial valuation (d, d') [8, (2.3)]. Thus Y is not on any oriented cycle in Γ_A, and hence X is not either. Dually, if f is a monomorphism, then X has no injective successor and neither X nor Y is on an oriented cycle in Γ_A.

Finally, we have the following.

PROPOSITION 10. *Let A be an Artin algebra, and let $X \to Y$ be an arrow in Γ_A with valuation (d, d'). If the arrow $X \to Y$ is on an oriented cycle in Γ_A, then $dd' \le 3$.*

Proof. Suppose that the arrow $X \to Y$ is on an oriented cycle in Γ_A. Assume that $d \ge 4$. By Proposition 8, we infer that $d = 4$ and there is a source map $f : X \to \bigoplus_1^4 Y$ with Y projective. Hence we have an almost split sequence

$$0 \to X \xrightarrow{f} \bigoplus_1^4 Y \xrightarrow{g} \operatorname{Tr} DX \to 0\,.$$

Since the co-restriction of f to $\bigoplus_1^3 Y$ is a monomorphism, so is the restriction of g to Y. Hence by the dual of Corollary 2, we infer that any sectional path in Γ_A starting with X contains no injective module. Since X is on an oriented cycle, using the Bautista–Smalø theorem, we deduce that $\operatorname{Tr} DX$ is also on an oriented cycle. Hence Y is injective by Proposition 8, which is a contradiction. Thus $d \le 3$. Dually $d' \le 3$. Moreover, by Lemma 9, either $d = 1$ or $d' = 1$. Therefore $dd' \le 3$.

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