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Extreme points and descriptive sets

by

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Abstract. A class of closed, bounded, convex sets in the Banach space c_0 is shown to be a complete PCA set.

Introduction. Let K denote a closed, bounded, convex set in a separable B-space X, and let $\operatorname{ex} K$ denote its set of extreme points. It is possible that $\operatorname{ex} K = \emptyset$, and also that $\operatorname{ex} K$ fail to be a Borel set ([5, 6]). Hence it is natural to ask for the complexity of the set CE of sets K in X having an extreme point. This question will be answered for $X = c_0$, after a digression on the class $F(M)$ of all closed subsets of a metric space (M, d). This class can be quite mysterious ([4, 7]), but we mention only the rudiments. When $F(M)$ is provided with the Hausdorff metric—a minor adjustment is necessary when d is unbounded—certain sets $[U]$ in $F(M)$ are open. Here U is open in M and

$$A \in [U] \leftrightarrow A \cap U \neq \emptyset\,.$$

When d is totally bounded—equivalently, when $F(M)$ is separable—the sets $[U]$ generate the field of Borel sets, called the *Effros Borel structure*, and therefore the Borel structure in $F(M)$ has a definite meaning when M is separable (since then there is some totally bounded metric). Some sets are always closed, for example the subset of $M \times F(M)$ defined by $m \in A$. When X is a separable B-space, the convex sets form a G_δ. To see this, let $(U_n)_{n=1}^{\infty}$ be a basis for the open sets; then A is convex provided A meets $\frac{1}{2}(U_n + U_m)$ whenever A meets both U_n and U_m. The Hausdorff metric in $F(X)$, relative to the usual metric, will be called the *strong* metric; that relative to a totally bounded metric in X will be called a *weak* metric. (This has no relation to the weak topology.)

Let E be the subset of $F(X) \times X \times X \times X$ containing elements (A, x, y, z)

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such that

$$x \in A, \quad y \in A, \quad z \in A, \quad x \neq y, \quad x \neq z, \quad x \in \overline{yz}\,.$$

Then E is a G_δ (for any weak metric in $F(X)$) and its projection on the first and second factors is the set of pairs (A, x) such that $x \in A$ and x is not an extreme point of A. From this we conclude that CE is of class PCA (alias Σ^1_2) for a weak metric. Recent work on realization of PCA sets by means of sets in classical analysis is presented in [1, 2].

THEOREM. *Let $\mathcal{S}$ be a PCA set in a compact metric space M. Then there is a mapping $m \to K(m)$ defined on M such that*

(i) *$K(m)$ is a closed, bounded, convex subset of c_0.*
(ii) *$K(m)$ has an extreme point if and only if $m \in \mathcal{S}$.*
(iii) *The mapping is continuous from M to the strong metric in $F(c_0)$.*

Proof. This begins with some elementary topology and a summary of [6]. Being of class PCA, $\mathcal{S}$ is a continuous image $f(\mathcal{S}_1)$ of a certain CA set $\mathcal{S}_1$ in a compact metric space M_1. By a device used in [6], matters can be so arranged that f admits a continuous extension to all of M_1, mapping M_1 into M. Let $P(M_1)$ be the set of probability measures in M_1, with its w^*-topology, and T an affine homeomorphism of $P(M_1)$ onto a compact set C in c_0. Then there is a closed, bounded, convex set K_0 in $c_0 \times c_0$ such that ([6])

(i) $C \times \{0\}$ is contained in K_0.
(ii) The extreme points of K_0 are precisely the elements $(T(\delta_{m_1}), 0)$ with $m_1 \in \mathcal{S}_1$.

Let h be continuous on M_1 to $[0, 1]$ and let $K(h)$ be the convex subset of $c_0 \times c_0 \times c_0$ containing all $(T(\mu), u, v)$ such that $(T(\mu), u) \in K_0$, $\|v\| \le \int h\, d\mu$. To determine $\operatorname{ex} K(h)$, we recall that the unit ball of c_0 has no extreme points and therefore $(T(\mu), u, v)$ cannot be extreme if $\int h\, d\mu > 0$. If, then, $(T(\mu), u, v)$ is extreme, then $v = 0$, whence $(T(\mu), u)$ is extreme in K_0, and (as just observed) $\int h\, d\mu = 0$. Conversely, suppose $(T(\mu), 0)$ is extreme and $\int h\, d\mu = 0$; and suppose $(T(\mu), 0, 0) = \frac{1}{2}(T(\mu_1), u_1, v_1) + \frac{1}{2}(T(\mu_2), u_2, v_2)$. Then $\mu_1 = \mu_2 = \mu$, $u_1 = u_2 = 0$, and consequently $v_1 = v_2 = 0$. Thus, in summary

$$K(h) \text{ has an extreme point } \Leftrightarrow h \text{ has a zero in } \mathcal{S}_1\,.$$

Moreover, the Hausdorff distance between $K(h_1)$ and $K(h_2)$ is at most $\|h_1 - h_2\|$.

Let ϱ be a metric in M and suppose $\varrho \le 1$, and let $h(m, m_1) = \varrho(m, f(m_1))$. Then $h(m, \cdot)$ is continuous on M_1, and $h(m, \cdot)$ has a zero in $\mathcal{S}_1$ $\Leftrightarrow m \in f(\mathcal{S}_1) = \mathcal{S}$. Using these functions for h in $K(h)$ we obtain the theorem.

Conclusion. We mention some problems, of uncertain difficulty, related to the main result; background material is presented in [3].

(i) Find other spaces X in place of c_0. Since l^1 has RNP, the most likely candidate is L^1. Besides this, there are the separable subspaces of the non-RNP spaces of Stegall ([3], Ch. 4).

(ii) What happens when extreme points are replaced by denting points, exposed points, strongly exposed points, etc. ([3], Ch. 3)?

(iii) Classify the sets K such that $K = \overline{\text{co}}(\text{ex}\, K)$.

(iv) Fixing K, classify the set of points represented by an integral over $\text{ex}\, K$ ([3], Ch. 6).

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