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The Nielsen coincidence theory on topological manifolds

by

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Abstract. We generalize the coincidence semi-index introduced in [D-J] to pairs of maps between topological manifolds. This permits extending the Nielsen theory to this class of maps.

Introduction. In this paper we generalize the coincidence semi-index theory, introduced in [D-J] in the smooth case, to pairs of maps between topological manifolds. It will be based on the topological transversality lemma (1.1). To show that this new theory generalizes the previous one it will be necessary to reformulate [D-J] since the graphs of any two maps are never topologically transverse. This is done in Section 2: we give three equivalent versions of the semi-index in the smooth case and then we show that one of them coincides with the semi-index defined in Section 1. In Section 3 we prove a Wecken type theorem on realizing the above Nielsen number.

1. The coincidence semi-index on topological manifolds. Throughout this paper we consider pairs of maps $f, g : M \to N$ such that M, N are topological separable manifolds without boundary and the coincidence set $\Phi(f,g) = \{x \in M : fx = gx\}$ is compact. The construction of the semi-index we present is based on the transversality lemma (1.1) below.

Let $P \subset W$ and V be topological manifolds and ξ a normal microbundle of P in W, i.e. the total space of ξ is an open subset of W containing P. Recall that a map $h : V \to W$ is called *topologically transverse* (briefly *t-transverse*) to ξ if $h^{-1}P$ is a topological submanifold in V admitting a normal microbundle ν such that for any $x \in h^{-1}P$ a neighbourhood of x in ν_x is mapped by h homeomorphically onto a neighbourhood of hx in ξ_{hx} [K-S].

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Lemma (1.1). *Let V, W be topological manifolds, and $P \subset W$ a submanifold with a normal microbundle ξ. Let $C, D \subset V$ be closed subsets and $h : V \to W$ a continuous map transverse to ξ near C. Then there is a homotopy $\{h_t\}$ based in a given neighbourhood of $D - C$ such that $h_0 = h$ and h_1 is transverse to ξ near $C \cup D$. Moreover, if d is a metric on W, and $\varepsilon : V \to (0, \infty)$ any function then we may require $d(h_t(x), h(x)) < \varepsilon(x)$ for any $x \in V$ and $0 \leq t \leq 1$.*

Proof. Let p, v, w denote the dimensions of P, V, W respectively. For $v \neq 4 \neq w - p$ the assertion follows from Thm. (1.1) in Essay III of [K-S]. The assumption $v \neq 4$ is not necessary now since by [F] every 4-manifold is almost smoothable.

The case $w - p = 4$ follows from Thm. (5.3) of [Sc] since by [F] the Rokhlin theorem fails in the TOP category. ■

It will be convenient to start with a more general situation. For given manifolds V, W, $P \subset W$ and a map $h : V \to W$ we define a relation on $h^{-1}P$ which will also be called the *Nielsen relation*: $x \simeq y$ iff there is a path ω joining these points in V such that $h\omega$ is homotopic in W to a path in P. (Let $f, g : M \to N$ be a pair of maps: we put $h = (f, g) : V = M \to W = N \times N$, $P = \Delta N$. Then $\Phi(f, g) = h^{-1}\Delta N$ and the above relation coincides with the classical Nielsen relation [S], [D-J].) Moreover, we will always assume $h^{-1}P$ to be compact.

Now suppose that the dimensions of the manifolds $V, P \subset W$ satisfy $v + p = w$, ξ is a normal microbundle of P in W and $h : V \to W$ a map t-transverse to ξ. Since $\dim h^{-1}P = v + p - w = 0$, $h^{-1}P$ is discrete and ν splits into open sets, each projected into a point in $h^{-1}P$.

Let α, β denote two microbundles of the same dimension r and let α_x denote the fibre over a point x of the zero section of α. Then any generator $z_x \in H_r(\alpha_x, \alpha_x - x)$ will be called a (*local*) *orientation* of the microbundle α at x. Let $k : \alpha \to \beta$ be a microbundle map such that the restriction $k_{|\alpha_x}$ is a homeomorphism near $x \in \alpha_x$. Then this restriction determines an orientation $(k_|)_* z_x \in H_r(\beta_{kx}, \beta_{kx} - kx)$ which by abuse of notation will also be denoted by $k_* z_x$.

Let ω be a path establishing the Nielsen relation between $x, y \in h^{-1}P$. Let $\gamma_0 \in H_v(V, V - x)$ be an orientation of the manifold V at x. Then $h_*\gamma_0 \in H_v(\xi_{hx}, \xi_{hx} - hx)$ is an orientation of the microbundle ξ at $hx \in P$. Let γ_t denote the translation of γ_0 along $\omega(t)$.

Definition (1.2). We will say that two points $x, y \in h^{-1}P$ are *R-related* (xRy) iff there is a path ω establishing the Nielsen relation between them such that the translation of the orientation $h_*\gamma_0$ along a path in P homotopic to $h\omega$ in W is opposite to $h_*\gamma_1$.

Let $A \subset h^{-1}P$ be a subset. It can be represented as $A = \{a_1, b_1, \ldots, a_k, b_k : c_1, \ldots, c_s\}$ where $a_i R b_i$ for any i and $c_i R c_j$ for no $i \neq j$. We will call the elements $\{c_1, \ldots, c_s\}$ *free* in this decomposition.

LEMMA (1.3). *The number of free elements does not depend on the decomposition of A.*

Proof. See the proof of (1.3) in [D-J]. ∎

We define the *semi-index* of the set $A \subset h^{-1}P$ with respect to ξ to be the number of free elements in any decomposition of A, and we denote this number by $|\mathrm{ind}|_{\mathrm{t}}(h, \xi : A)$.

LEMMA (1.4) (Homotopy invariance). *Let $H : V \times [0,1] \to W$ be a map t-transverse to a normal microbundle ξ of P in W $(v + p = w)$. Let A be a clopen subset of $H^{-1}P$ and set $A_t = \{x \in V : (x,t) \in A\}$, $H_t(x) = H(x,t)$ for $0 \leq t \leq 1$. Then $|\mathrm{ind}|_{\mathrm{t}}(H_0, \xi : A_0) = |\mathrm{ind}|_{\mathrm{t}}(H_1, \xi : A_1)$.*

Proof. By transversality, $H^{-1}P$ is a one-dimensional manifold. Consider first a connected component of $H^{-1}P$ with ends $(x,0)$, $(x',0)$. We will show that $x, x' \in H_0^{-1}P$ are R-related. Then we will show that if $(x,0), (y,1)$ are ends of a connected component and $(x',0)$, $(y',1)$ are ends of another one then $x, x' \in H_0^{-1}P$ are R-related iff $y, y' \in H_1^{-1}P$ are R-related. The above facts show that there are decompositions of A_0 and A_1 with the same number of free points, which proves our lemma.

Let ν be a normal microbundle over $H^{-1}P$ from the transversality assumption. Consider a connected component A with ends $(x,0), (y,0) \in V \times 0$. We will show that xRy in $H_0^{-1}P$.

Let $\omega(t) = (\omega_1(t), \omega_2(t)) \subset V \times I$ be a parametrization of this component. Then $(t,s) \to H(\omega_1(t), s\omega_2(t))$ is a homotopy between $H(\omega_1(t), 0) = H_0\omega_1$ and the path $H\omega \subset P$. Now ω_1 establishes the Nielsen relation between $x = \omega_1(0)$ and $y = \omega_1(1)$ in $H_0^{-1}P$.

We show that ω_1 also establishes the R-relation. Let γ_0 be a local orientation of the manifold $M \times 0$ at $(x,0)$ and let γ_t be its translation along the path $(\omega_1, 0)$. Then γ_0 is also an orientation of the microbundle ν at $(x,0)$. Let $\widehat{\gamma}_t$ denote its translation along ω. Notice that then $\widehat{\gamma}_1 = -\gamma_1$. On the other hand, $H_*\widehat{\gamma}_t$ is the translation of the orientation of ξ along the path $H\omega$ (which is contained in P and homotopic to $H_0\omega_1$ in N). For $t = 1$ we obtain $H_*\widehat{\gamma}_1 = H_{0*}\widehat{\gamma}_1 = -H_{0*}\gamma_1$, proving xRy in $H_0^{-1}P$.

Now we consider the second case: let $\omega(t) = (\omega_1(t), \omega_2(t))$, $\omega'(t) = (\omega_1'(t), \omega_2'(t))$ be parametrizations of the two components with ends $\omega(0) = (x,0)$, $\omega(1) = (y,1)$, $\omega'(0) = (x',0)$, $\omega'(1) = (y',1)$. Suppose that a path $u \subset V$ establishes the R-relation between $x, x' \in H_0^{-1}P$. We will show that $-\omega_1 + u + \omega_1'$ establishes the R-relation between $y, y' \in H_1^{-1}P$. Let $\overline{u}$ denote a path in P homotopic in W to $H_0 u$. Since

$$H_1(-\omega_1 + u + \omega_1') \simeq H(-\omega + (u,0) + \omega') \simeq H(-\omega) + \overline{u} + H\omega' \subset P\,,$$

$-\omega_1 + u + \omega_1'$ establishes the Nielsen relation between $y, y' \in H^{-1}P$.

Let γ_1 be an orientation of $M \times 1$ at $(y,1)$. It is also an orientation of the microbundle ν at $(y,1) = \omega(1)$; let γ_t be its translation along $\omega(t)$. Then γ_0 is an orientation of $M \times 0$ at $(x,0)$; let γ_t' be its translation along $(u,0)$ to $(x',0)$. Then γ_1' is an orientation of ν at $(x',0) = \omega'(0)$; let γ_t'' be its translation along ω'. Notice that γ_1'' is the translation of the orientation γ_1 of $M \times 1$ along $-\omega_1 + u + \omega_1'$.

It remains to show that $H_{1*}\gamma_1''$ is opposite to the translation of the orientation $H_{1*}\gamma_1$ of ξ along a path lying in P and homotopic in W to $H_1(-\omega_1 + u + \omega_1')$. Notice that $H(-\omega) + \overline{u} + H\omega'$ is such a path. Now

- the translation of $H_*\gamma_1$ along $-H\omega$ gives $H_*\gamma_0 = H_{0*}\gamma_0$,
- the translation of $H_{0*}\gamma_0$ along $\overline{u}$ gives $-H_{0*}\gamma_0' = -H_*\gamma_0''$ since xRx',
- the translation of $-H_*\gamma''$ along $H\omega'$ gives $-H_*\gamma_1'' = -H_{1*}\gamma_1''$

hence we obtain the orientation opposite to $H_{1*}\gamma_1''$, which proves yRy' in $H_1^{-1}P$. ∎

Let A_0 from the above lemma be a Nielsen class of H_0. Then A_1 is contained in a Nielsen class A_1' of H_1 and $A_1' - A_1$ is the union of pairs of points where each pair is the boundary of a component of $H^{-1}P$. Thus $A_1' - A_1$ splits into pairs of R-related points, which implies that the semi-indices of A_0 and A_1 are the same. Since by (1.1) any homotopy between transverse maps may be deformed to a transverse homotopy, we obtain:

LEMMA (1.5). *Let $H : V \times I \to W$ be a Φ-compact homotopy between the maps $H_0, H_1 : V \to W$ transverse to a normal bundle ξ of P in W. Let $A_i \subset H_i^{-1}P$ be Nielsen classes corresponding under this homotopy* $(i = 0,1)$. *Then*

$$|\mathrm{ind}|_{\mathrm{t}}(H_0, \xi : A_0) = |\mathrm{ind}|_{\mathrm{t}}(H_1, \xi : A_1)\,. \ \blacksquare$$

A Nielsen class A of a t-transverse map h will be called *essential* if $|\mathrm{ind}|_{\mathrm{t}}(h, \xi : A) \neq 0$. Define the *Nielsen number* $N(h)$ of h to be the number of essential classes; in fact, $N(h)$ also depends on P and ξ. By Lemma (1.5), $N(h)$ is a correctly defined Φ-homotopy invariant. This definition implies $\# h^{-1}P \geq N(h)$.

Let $h : V \to W$ be a Φ-compact map (not necessarily transverse). We define $N(h) = N(h')$ where h' is any transverse map Φ-compactly homotopic to h. By (1.5) this is a correctly defined Φ-compact homotopy invariant.

THEOREM (1.6). *Let $h : V \to W$ be Φ-compact. Then $h^{-1}P$ contains at least $N(h)$ points.*

P r o o f. This is evident when h is transverse to P. In the general case suppose that $\# h^{-1}P = k < N(h)$. For any $x \in h^{-1}P$ take a contractible neighbourhood W_{hx} of hx in W such that $W_{hx} \cap P$ is a deformation retract of W_{hx}. Then take a neighbourhood V_x of x in V such that $h(\operatorname{cl} V_x) \subset W_{hx}$. Let $\{h_t\}$ be a compact homotopy, supported in $\bigcup\{V_x : x \in h^{-1}P\}$, from h to a map t-transverse to ξ. Since this homotopy may be arbitrarily small, we may assume $h_t(V_x) \subset W_{hx}$ hence $h_1^{-1}P \cap V_x$ belongs to a Nielsen class. Now $h_1^{-1}P$ contains at most k nonempty Nielsen classes, contradicting $N(h_1) = N(h) > k$. ■

Now we may apply the above results to coincidences: let $f, g : M \to N$ be a Φ-compact map of n-manifolds. We put $V = M$, $W = N \times N$, $h(x) = (fx, gx)$, $P = \Delta N$ and ξ the microbundle $N \times N \ni (x, y) \to (x, x) \in \Delta N$. We define the *coincidence Nielsen number* of the pair f, g by $N(f, g) = N(h)$. Since $\Phi(f, g) = h^{-1}(\Delta N)$ and both Nielsen relations coincide, Thm. (1.6) implies

COROLLARY (1.7). *$N(f, g)$ is a Φ-compact homotopy invariant and $\Phi(f, g)$ contains at least $N(f, g)$ points.* ■

Next we show that in the oriented case our semi-index equals the absolute value of the ordinary coincidence index [S], [V], [D-K].

LEMMA (1.8). *Let $f, g : M \to N$ be a Φ-compact pair, M, N oriented manifolds, and $A \subset \Phi(f, g)$ a Nielsen class. Then*

$$|\operatorname{ind}(f, g : A)| = |\operatorname{ind}|_{\mathrm{t}}(f, g : A) .$$

P r o o f. Since both ind and $|\operatorname{ind}|_{\mathrm{t}}$ are Φ-homotopy invariant, we may assume that the pair f, g is transverse. Consider $x_0, x_1 \in \Phi(f, g)$. We will show that $x_0 R x_1$ iff $\operatorname{ind}(f, g : x_0) = -\operatorname{ind}(f, g : x_1)$. Let u be a path establishing the Nielsen relation between x_0 and x_1. We notice that in the oriented case the translation of a local orientation $\alpha_0 \in H_n(M, M - x_0)$ along u gives $\alpha_1 \in H_n(M, M - x_1)$ such that both α_0, α_1 determine the same global orientation $\alpha \in H_n M$. Similarly, the translation of a generator $\beta_0 \in H_n(y_0 \times (N, N - y_0))$ (along a path in ΔN from (y_0, y_0) to (y_1, y_1)) gives $\beta_1 \in H_n(y_1 \times (N, N - y_0))$ such that β_0 and β_1 correspond to the same generator $\beta \in H_n(N \times N, N \times N - \Delta N)$. Thus the points x_0, x_1 are R-related iff the fixed global orientations are preserved by the pair f, g at one of these points and reversed at the other. But the last means exactly $\operatorname{ind}(f, g : x_0) = -\operatorname{ind}(f, g : x_1)$ as claimed. ■

2. The semi-index on smooth manifolds. In [D-J] we considered pairs of maps $f, g : M \to N$ between two smooth closed n-manifolds. For each Nielsen class we defined a semi-index which we will denote here by $|\operatorname{ind}|_{\mathrm{d}}$. To do this we replaced the given pair by a transverse one (in the smooth category), here called a *d-transverse pair*, and we defined on $\Phi(f,g)$ a reducibility relation (here termed the R_{d}-*relation*). Then any Nielsen class $A \subset \Phi(f,g)$ was represented as $A = \{a_1, b_1, \ldots, a_k, b_k; c_1, \ldots, c_s\}$ where $a_i R_{\mathrm{d}} b_i$ for any $i = 1, \ldots, k$, but $c_i R_{\mathrm{d}} c_j$ for no $i, j = 1, \ldots, s$, $i \neq j$, and we defined $|\mathrm{ind}|_{\mathrm{d}}(f, g : A) = s$. However, the approach from [D-J] is not convenient for comparison with the method from Section 1 of the present paper: in [D-J] we considered transversality in $M \times N$, while here in $N \times N$. To overcome this difference we will give three equivalent versions (A), (B), (C), of d-transversality and of the R_{d}-relation. Method (A) is the easiest to formulate: it does not involve any product spaces. Method (B) is the one given in [D-J]. We will show that in the smooth case, $|\mathrm{ind}|_{\mathrm{t}}$ coincides with $|\mathrm{ind}|_{\mathrm{d}}$ obtained by using method (C). Then the equivalence of (B) and (C) implies the desired equality $|\mathrm{ind}|_{\mathrm{d}} = |\mathrm{ind}|_{\mathrm{t}}$.

Consider a Φ-compact, smooth pair of maps $f, g : M \to N$ between smooth n-manifolds. Then for any $x \in \Phi(f,g)$ the following three conditions are equivalent:

(A) The difference of the tangent homomorphisms $f_* - g_* : T_x M \to T_{fx} N$ is an isomorphism.

(B) Let $\Gamma_f = \{(x,y) \in M \times N : y = fx\}$ denote the graph of f. Then the tangent spaces $T_{(x,fx)}\Gamma_f$, $T_{(x,gx)}\Gamma_g$ span the whole $T_{(x,fx)} M \times N$.

(C) The intersection of the subspaces $\mathrm{im}(f,g)_*$ and $T_{(fx,gx)}\Delta N$ in $T_{(fx,gx)} N \times N$ is zero.

Notice that (B) means that the pair f, g is transverse in the sense of [D-J]; here we will call it *d-transverse*. Assume that $f, g : M \to N$ is d-transverse and ω is a path establishing the Nielsen relation between the points $x, y \in \Phi(f,g)$. Fix an ordered basis $a_1^0, \ldots, a_n^0$ of $T_x M$ and let $a_1^t, \ldots, a_n^t \in T_{r(t)} M$ be its translation along ω. Then

(A′) $(f_* - g_*)a_1^0, \ldots, (f_* - g_*)a_n^0$ form a basis of $T_{fx} N$,

$(f_* - g_*)a_1^1, \ldots, (f_* - g_*)a_n^1$ form a basis of $T_{gx} N$.

Let α_0 and α_1 denote the orientations determined by these bases.

(B′) $(a_1^t, f_* a_1^t), \ldots, (a_n^t, f_* a_n^t)$ form a basis of $T_{(\omega(t), f\omega(t))}\Gamma_f$,

$(a_1^t, g_* a_1^t), \ldots, (a_n^t, g_* a_n^t)$ form a basis of $T_{(\omega(t), g\omega(t))}\Gamma_g$.

Let $\beta_t(f)$ and $\beta_t(g)$ denote the orientations determined by these bases. Let $\beta_0 = \beta_0(f) \wedge \beta_0(g)$ and $\beta_1 = \beta_1(f) \wedge \beta_1(g)$.

(C′) $(f_* a_1^0, g_* a_1^0), \ldots, (f_* a_n^0, g_* a_n^0)$ form a basis of the fibre of the normal bundle $\nu = T(N \times N)/T(\Delta N)$ of $\Delta N \subset N \times N$ at the point (fx, gx), and $(f_* a_1^1, g_* a_1^1), \ldots, (f_* a_n^1, g_* a_n^1)$ form a basis of the fibre of ν at (fy, gy).

Let γ_0, γ_1 denote the orientations determined by these bases.

THEOREM (2.1). *Under the above notations the following three conditions are equivalent*:

(A$''$) α_1 *is opposite to the translation of* α_0 *along* $f\omega \subset N$.
(B$''$) β_1 *is opposite to the translation of* β_0 *along* $(\omega, f\omega) \subset M \times N$.
(C$''$) γ_1 *is opposite to the translation of* γ_0 *along* $(f\omega, f\omega) \subset N \times N$.

Proof. The proof will be based on the following

LEMMA (2.2). *Let* V, W *be* n*-dimensional real linear spaces*, $a_1, \ldots, a_n$, *a basis of* V *and* $b_1, \ldots, b_n$, $c_1, \ldots, c_n \in W$ *such that* $b_1 - c_1, \ldots, b_n - c_n$ *is a basis of* W. *Then the orientations of* $V \times W$ *given by the ordered bases* $(a_1, b_1), \ldots, (a_n, b_n)$, $(a_1, c_1), \ldots, (a_n, c_n)$ *and* $(0, b_1 - c_1), \ldots, (0, b_n - c_n)$, $(a_1, 0), \ldots, (a_n, 0)$ *coincide.*

Proof. Consider the homotopies

$$t \to (a_1, b_1 - tc_1), \ldots, (a_n, b_n - tc_n), (a_1, (1-t)c_1), \ldots, (a_n, (1-t)c_n)$$

and

$$t \to (ta_1, b_1 - c_1), \ldots, (ta_n, b_n - c_n), (a_1, 0), \ldots, (a_n, 0)$$

in the Stiefel space of ordered bases.

Proof of Theorem (2.1). (A$''$)$\Leftrightarrow$(B$''$). We compare β_1 and the translation of β_0. The orientation β_1 is given by the basis

$$(a_1^1, f_\# a_1^1), \ldots, (a_n^1, f_\# a_n^1),\ (a_1^1, g_\# a_1^1), \ldots, (a_n^1, g_\# a_n^1)\,,$$

which by (2.2) is equivalent to

$$(0, (f_\# - g_\#)a_1^1), \ldots, (0, (f_\# - g_\#)a_n^1),\ (a_1^1, 0), \ldots, (a_n^1, 0)\,.$$

For the same reasons β_0 is equivalent to

$$(0, (f_\# - g_\#)a_1^0), \ldots, (0, (f_\# - g_\#)a_n^0),\ (a_1^0, 0), \ldots, (a_n^0, 0)\,.$$

Now let $(0, b_1^t), \ldots, (0, b_n^t)$, $(a_1^t, 0), \ldots, (a_n^t, 0)$ denote the translation of the last basis along $(\omega, f\omega)$ in $M \times N$. For $t = 1$ we obtain $(0, b_1^1), \ldots, (0, b_n^1)$, $(a_1^1, 0), \ldots, (a_n^1, 0)$, which agrees with β_1 iff the orientations of $T_{fy}N$ given by the bases $(f_\# - g_\#)a_1^1, \ldots, (f_\# - g_\#)a_n^1$ and $b_1^1, \ldots, b_n^1$ are equal, i.e. α_1 equals the translation of α_0.

(A$''$)$\Leftrightarrow$(C$''$). We compare γ_1 and the translation of γ_0 along $(f\omega, f\omega)$. The homotopy $t \to (\ldots, (f_\# a_i^0 - t g_\# a_i^0, (1-t) g_\# a_i^0), \ldots)$ shows that γ_0 may also be represented by $(b_1^0, 0), \ldots, (b_n^0, 0)$ where $b_1^0, \ldots, b_n^0$ represents α_0. Let $b_1^t, \ldots, b_n^t$ denote again the translation of the last basis along $f\omega$. Then $(b_1^t, 0), \ldots, (b_n^t, 0)$ represents the translation of γ_0 along $(f\omega, f\omega)$. On the other hand, γ_1 is represented by $(\ldots, ((f_\# - g_\#)a_i^1, 0), \ldots)$, which agrees with the above translation iff α_1 agrees with the translation of α_0. ∎

We will say that two points $x, y \in \Phi(f,g)$ are R_{d}-*related* if there is a path establishing the Nielsen relation between them and for which one (hence each) of the conditions (A″), (B″), (C″) from (2.1) holds. We notice that (C″) coincides with the reducibility relation from [D-J]. Now we are in a position to prove equality of $|\mathrm{ind}|_{\mathrm{d}}$ and $|\mathrm{ind}|_{\mathrm{t}}$. This will be done in the following theorem where we approximate the given pair by a pair simultaneously d- and t-transverse and use the version (C) to prove the equality of the semi-indices in this case.

THEOREM (2.3). *Let A be a Nielsen class of a Φ-compact pair $f, g : M \to N$. Then $|\mathrm{ind}|_{\mathrm{t}}(f, g : A) = |\mathrm{ind}|_{\mathrm{d}}(f, g : A)$.*

Proof. Since both semi-indices are homotopy invariants we may assume that f, g is a smooth d-transverse pair. Then $\Phi(f,g) = \{x_1, \ldots, x_k\}$ is finite. We will show that f, g may be replaced by a pair which is simultaneously d- and t-transverse. Fix euclidean neighbourhoods U_i such that $fx_i = gx_i = 0 \in U_i \subset N$ ($i = 1, \ldots, k$). Since d-transversality of f, g means that the difference of the tangent maps $f_* - g_*$ is an isomorphism, by the Implicit Function Theorem there exist disjoint balls U_i' in M such that $x_i \in U_i' = B(x_i, 2\varepsilon_i) \subset M$, $g - f$ is a diffeomorphism on U_i' and $fU_i' \cup gU_i' \subset U_i$. Let $\eta_i : \mathbb{R} \to \mathbb{R}$ be a non-increasing smooth function satisfying $\eta_i(-\infty, \varepsilon_i] = 1$, $\eta_i[2\varepsilon_i, \infty) = 0$.

Define

$$(f', g') = \begin{cases} ((1 - \eta_i(|x - x_i|))f(x), g(x) - \eta_i(|x - x_i|)f(x)) & \\ & \text{for } x \in U_i',\ i = 1, \ldots, k\,, \\ (fx, gx) & \text{otherwise.} \end{cases}$$

The pair f', g' is homotopic to f, g, $\Phi(f', g') = \Phi(f, g)$ and moreover f', g' is simultaneously t-transverse and d-transverse. It remains to show that for f', g' the relations R_{t} and R_{d} coincide. Until the end of this proof we will write f, g instead of f', g'. Let $x_0, x_1 \in \Phi(f,g)$ and let a path ω establish the Nielsen relation between these points. Set $y_i = fx_i = gx_i$ ($i = 0, 1$) and fix an orientation of M at x_0: $\gamma_0 \in H_n(M, M - x_0) \simeq H_n(T_{x_0}M, T_{x_0}M - x_0)$. Then

$$((f,g)_*\gamma_0)^{\mathrm{t}} \in H_n(y_0 \times N, y_0 \times (N - y_0))$$

and

$$\begin{aligned} &((f,g)_*\gamma_0)^{\mathrm{d}} \\ &\quad \in H_n((T(N \times N)/T(\Delta N))_{(y_0,y_0)}, (T(N \times N) - T(\Delta N)/T(\Delta N))_{(y_0,y_0)}) \\ &\quad = H_n(T(N \times N)_{(y_0,y_0)}, (T(N \times N) - T(\Delta N))_{(y_0,y_0)}) \\ &\quad = H_n(U_0 \times U_0, U_0 \times U_0 - \Delta U_0) \end{aligned}$$

(where U_0 is a euclidean neighbourhood of x_0) are orientations of the microbundle $N \times N \to \Delta N$ and of the normal bundle $T(N \times N)/T(\Delta N)$ at (y_0, y_0). Then the inclusion $y_0 \times U_0 \subset U_0 \times U_0$ induces a homomorphism

$$H_n(y_0 \times N, y_0 \times (N - y_0)) = H_n(y_0 \times U_0, y_0 \times (U_0 - y_0)) \\ \to H_n(U_0 \times U_0, U_0 \times U_0 - \Delta U_0)$$

sending $((f,g)_*\gamma_0)^{\mathrm{t}}$ to $((f,g)_*\gamma_0)^{\mathrm{d}}$. Let (y_s, y_s) be a path in ΔN homotopic in $N \times N$ to $(f\omega, g\omega)$ and U_s be the translation of U_0 along (y_s, y_s). Then the translations of the above orientations $((f,g)_*\gamma_0)^{\mathrm{t}}_s$, $((f,g)_*\gamma_0)^{\mathrm{d}}_s$ along (y_s, y_s) are compatible under the maps induced by the inclusions $y_s \times U_s \subset U_s \times U_s$. On the other hand, let γ_1 be the translation of γ_0 along $\omega \subset M$. Then the orientations $((f,g)_*\gamma_1)^{\mathrm{t}}$, $((f,g)_*\gamma_1)^{\mathrm{d}}$ (defined as for γ_0) correspond under the inclusion $y_1 \times U_1 \subset U_1 \times U_1$, hence $((f_*, g_*)\gamma_0)^{\mathrm{t}}_1 = ((f_*, g_*)\gamma_1)^{\mathrm{t}}$ iff $((f_*, g_*)\gamma_0)^{\mathrm{d}}_1 = ((f_*, g_*)\gamma_1)^{\mathrm{d}}$. This means that the relations R_{t} and R_{d} coincide. ∎

In particular, the Nielsen number introduced in Section 1 coincides, in the smooth case, with the number from [D-J].

3. A Wecken type theorem. We will show that the Nielsen number from Section 1 is the best lower bound on the number of coincidence points in dimensions ≥ 3 (compare [Ji1], [D-K], [D-J]). We will follow the scheme from [Ji1].

Consider the following setting:

(∗) $P \subset W$ and V are topological manifolds whose dimensions satisfy $p + v = w$, $p \geq 2$, $v \geq 3$, and ξ is a normal microbundle of P in W.

The crucial step is the folowing

LEMMA (3.1) (A Whitney type lemma). *Let D denote a v-dimensional ball, $h : D \to W$ a map t-transverse to ξ, $h^{-1}P = \{x_0, x_1\}$ and suppose the points x_0, x_1 are R-related. Then h is homotopic rel* $\mathrm{bd}\, D$ *to a map into $W - P$.*

Proof. We may assume that $hx_0 \neq hx_1$ since otherwise we may compose h near x_1 with a local isotopy of W near hx_1. Let $\omega : [0,1] \to D$ be the straight line segment from x_0 to x_1. Then $h\omega$ is homotopic (rel ends) to a map ω' into P. Since $\dim P \geq 2$ and $hx_0 \neq hx_1$, we may assume that ω' is a locally flat (hence flat) regular arc in P. Let U denote a euclidean neighbourhood of ω' in W.

We will show that h is homotopic rel $\mathrm{bd}\, D$ to a map h' satisfying $h'^{-1}P = h^{-1}P = \{x_0, x_1\}$ and $h'(\omega[0,1]) \subset U$. Then we take a euclidean neighbourhood U_0 of $\omega' \subset P$ and we put $U = U_0 \times \mathbb{R}^d$, a neighbourhood of the zero section of ξ restricted to $U_0 \subset P$. Since $h'(\omega[0,1]) \subset U$, a euclidean neighbourhood D_0 of $\omega[0,1] \subset D$ is also sent by h' into U. We consider the composition $\widehat{h} : D_0 \xrightarrow{h'} U = U_0 \times \mathbb{R}^d \to \mathbb{R}^d$. The zeros of $\widehat{h}$ correspond to the elements of $h'^{-1}P$, hence $\widehat{h}^{-1}(0) = \{x_0, x_1\}$ and $\widehat{h}$ is t-transverse to $0 \subset \mathbb{R}^d$. The assumption $x_0 R x_1$ implies that the degrees of $\widehat{h}$ at x_0 and x_1 are opposite, hence the induced homomorphism $\widehat{h}_* : H_d(D_0, \operatorname{bd} D_0) \to H_d(\mathbb{R}^d, \mathbb{R}^d - 0)$ is zero and by the Hopf lemma $\widehat{h}$ is homotopic rel $\operatorname{bd} D_0$ to a map into $\mathbb{R}^d - 0$. The last gives a homotopy rel $\operatorname{bd} D_0$ from h' to a map into $W - P$.

It remains to construct h'. Fix ball neighbourhoods K_i of $x_i \in \operatorname{int} D$ so small that $h(K_i) \subset U$ and let $x_i' = \omega(t_i)$ be the unique common point of $\omega[0,1]$ and $\operatorname{bd} K_i$ $(i = 0, 1)$.

The assumption that $h\omega$ is homotopic to $\omega' \subset P$ implies a homotopy from $h\omega_{|[t_0,t_1]}$ rel $\{t_0, t_1\}$ to a map into U (if only K_0, K_1 are small enough). Let $H : [t_0, t_1] \times [0,1] \to W$ denote this homotopy. We may assume that $H((t_0,t_1)\times(0,1)) \cap P = \emptyset$: we apply Lemma (1.1) to $V = D = (t_0, t_1) \times (0,1)$, $C = \emptyset$ and

$$\varepsilon(x,t) = \operatorname{dist}((x,t), \operatorname{bd}([t_0,t_1] \times [0,1]))\,.$$

We may, moreover, assume that $H(t,1) \notin P$ for any $t \in [t_0, t_1]$ (consider H restricted to $[t_0, t_1] \times [0, s_0]$ for some s_0 close enough to 1). Then we define a homotopy

$$H' : D \times 0 \cup (K_0 \cup \omega[t_0,t_1] \cup K_1) \times I \to W$$

by

$$H'(x,s) = \begin{cases} h(x) & \text{for } x \in K_0 \cup K_1 \text{ or } s = 0, \\ H(t,s) & \text{for } x = \omega(t),\ t \in [t_0,t_1]. \end{cases}$$

Let $\varrho : D \times I \to D \times 0 \cup (K_0 \cup \omega[t_0,t_1] \cup K_1) \times I$ be a retraction satisfying $\varrho^{-1}(x,t) = (x,t)$ for $x \in \operatorname{int}(K_0 \cup K_1)$. Then the composition $H'' = H'\varrho : D \times I \to W$ satisfies $H''(x,0) = h(x)$, $H''^{-1}P = \{x_0, x_1\} \times [0,1]$, $H''(\omega[0,1],1) \subset U$. If $\delta : D \to [0,1]$ is an Urysohn function satisfying $\delta(\omega[t_0,t_1]) = 1$, $\delta(\operatorname{bd} D) = 0$ then $\overline{H}(x,t) = H''(x, \delta(x)t)$ is moreover a homotopy rel $\operatorname{bd} D$. We put $h'(x) = \overline{H}(x,1)$. ∎

Now following the Creating and Cancelling Procedures from [Ji1] we obtain

THEOREM (3.2) (A Wecken type theorem). *Under the assumptions* $(*)$ *any* Φ*-compact map* $h : V \to W$ *is* Φ*-compactly homotopic to a map* h' *so that* $h'^{-1}P$ *contains exactly* $N(h)$ *coincidence points.* ∎

We may assume that the homotopy from (3.2) has compact carrier, i.e. is constant outside a compact set. To see this let h_t be a homotopy from (3.2) and let $\lambda : V \to [0,1]$ be a function equal to 1 on the compact set $\{x \in V : h_t x \in P \text{ for some } t \in [0,1]\}$ and to zero outside a compact set. We put $h_t' = h_{\lambda(x)t}(x)$.

COROLLARY (3.3). *Let $f, g : M \to N$ be a Φ-compact pair of maps between n-manifolds, $n \geq 3$. Then there is a homotopy $\{f_t\}$ starting from $f_0 = f$, constant outside a compact set and such that $\Phi(f_1, g)$ contains exactly $N(f, g)$ elements.*

Proof. We put in (3.2): $V = M$, $W = N \times N$, $h = (f, g)$, $P = \Delta N$, $\xi = (N \times N \ni (x, y) \to (x, x) \in \Delta N)$. We get a homotopy (f_t, g_t) with compact carrier from f, g to a pair with exactly $N(f, g)$ coincidence points. Now Theorem 1 of [B] yields a homotopy f'_t such that $f'_1 = f_1$ and $\#\Phi(f'_0, g) = N(f, g)$. It remains to show that the homotopy f'_t has a compact carrier. But we notice that if the carrier of the homotopy g_t from the Lemma in Section 2 of [B] is compact then so is the carrier of the homotopy h_t constructed there. ∎

Finally, let us notice that all the results of [Je1] and [Je2] remain valid after replacing smooth manifolds by topological manifolds. In particular, we obtain a theorem expressing the Nielsen number of a fibre map by the Nielsen numbers of pairs between base spaces and fibres ([Je1], Thm. (4.3)) as well as a theorem expressing $N(f, g)$ by $N(\widetilde{f}, \widetilde{g})$ where $\widetilde{f}, \widetilde{g}$ are lifts to finite coverings ([Je2], Thm. (2.5)). These generalizations are evident, except possibly for Section 3 of [Je1]; but in fact the main result of that section (Thm. (3.13)) can be proved in a much simpler way in the TOP category.

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