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Striped structures of stable and unstable sets of expansive homeomorphisms and a theorem of K. Kuratowski on independent sets

by

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Abstract. We investigate striped structures of stable and unstable sets of expansive homeomorphisms and continuum-wise expansive homeomorphisms. The following theorem is proved: if $f : X \to X$ is an expansive homeomorphism of a compact metric space X with $\dim X > 0$, then the decompositions $\{W^{\rm s}(x) \mid x \in X\}$ and $\{W^{\rm u}(x) \mid x \in X\}$ of X into stable and unstable sets of f respectively are uncountable, and moreover there is σ (= s or u) and $\varrho > 0$ such that there is a Cantor set C in X with the property that for each $x \in C$, $W^{\sigma}(x)$ contains a nondegenerate subcontinuum A_x containing x with $\operatorname{diam} A_x \geq \varrho$, and if $x, y \in C$ and $x \neq y$, then $W^{\sigma}(x) \neq W^{\sigma}(y)$. For a continuum-wise expansive homeomorphism, a similar result is obtained. Also, we prove that if $f : G \to G$ is a map of a graph G and the shift map $\widetilde{f}$: $(G, f) \to (G, f)$ of f is expansive, then for each $\widetilde{x} \in (G, f)$, $W^{\rm u}(\widetilde{x})$ is equal to the arc component of (G, f) containing $\widetilde{x}$, and $\dim W^{\rm s}(\widetilde{x}) = 0$.

1. Introduction. All spaces under consideration are assumed to be metric. By a *compactum* we mean a compact metric space, and by a *continuum* a connected nondegenerate compactum. A homeomorphism $f : X \to X$ of a compactum X is called *expansive* [6] if there is a constant $c > 0$ (called an *expansive constant* for f) such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x, y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c\,.$$

This property has frequent applications in topological dynamics, ergodic theory and continuum theory [1, 5, 6, 25].

A homeomorphism $f : X \to X$ of a compactum X is *continuum-wise expansive* [15] if there is a constant $c > 0$ such that if A is a nondegener-

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ate subcontinuum of X, then there is an integer $n = n(A) \in \mathbb{Z}$ such that $\operatorname{diam} f^n(A) > c$. By the definitions, we can easily see that every expansive homeomorphism is continuum-wise expansive, but the converse is not true. There are many important examples of homeomorphisms which are continuum-wise expansive, but not expansive.

In [13, Theorem 3.1], we proved that if $f : X \to X$ is an expansive homeomorphism of a compactum X with $\dim X > 0$, then there is a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, the decomposition space of Z into components is upper and lower semi-continuous and all components of Z are contained in stable sets or unstable sets.

In this paper, we show more precise results. In particular, we prove that if $f : X \to X$ is an expansive homeomorphism of a compactum X with $\dim X > 0$, then the decompositions $\{W^{\mathrm{s}}(x) \mid x \in X\}$ and $\{W^{\mathrm{u}}(x) \mid x \in X\}$ of X into stable and unstable sets respectively are uncountable, and moreover there is σ ($\sigma = \mathrm{s}$ or u) and $\varrho > 0$ such that the σ-striped set $Z(\sigma, \varrho)$ of f is not empty. Hence, by using a theorem of K. Kuratowski on independent sets, it is proved that almost every Cantor set C of $Z(\sigma, \varrho)$ has the property that for each $x \in C$, $W^{\sigma}(x)$ contains a nondegenerate subcontinuum containing x and if $x, y \in C$ and $x \neq y$, then $W^{\sigma}(x) \neq W^{\sigma}(y)$.

Also, we prove that if $f : G \to G$ is a map of a graph G and the shift map $\widetilde{f} : (G, f) \to (G, f)$ of f is expansive, then for each $\widetilde{x} \in (G, f)$, $W^{\mathrm{u}}(\widetilde{x})$ is the arc component of (G, f) containing $\widetilde{x}$, and $W^{\mathrm{s}}(\widetilde{x})$ is 0-dimensional.

We refer the readers to [1], [6] and [25] for the general properties of expansive homeomorphisms.

2. Definitions and preliminaries. Let X be a metric space. Then the *hyperspaces* 2^X and $C(X)$ of X are defined as follows:

$$2^X = \{A \mid A \text{ is a nonempty compact subset of } X\},$$
$$C(X) = \{A \in 2^X \mid A \text{ is connected}\}.$$

The hyperspaces 2^X and $C(X)$ are metric spaces with the *Hausdorff metric* d_{H}, i.e., $d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid U_\varepsilon(A) \supset B \text{ and } U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A in X. Note that if X is a compactum, then the hyperspaces 2^X and $C(X)$ are also compacta (see [21]). Let A and B be subsets of X. Put $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}$.

Let $f : X \to X$ be a homeomorphism of a compactum X and let $x \in X$. Then the *stable set* $W^{\mathrm{s}}(x)$ and the *unstable set* $W^{\mathrm{u}}(x)$ are defined as follows:

$$W^{\mathrm{s}}(x) = \{y \in X \mid \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\},$$
$$W^{\mathrm{u}}(x) = \{y \in X \mid \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

Also, the *continuum-wise stable* and *unstable sets* $V^{\mathrm{s}}(x)$, $V^{\mathrm{u}}(x)$ are defined as follows:

$$V^{\mathrm{s}}(x)=\{y\in X\mid \text{there is } A\in C(X) \text{ such that } x,y\in A \text{ and } \lim_{n\to\infty}\operatorname{diam} f^n(A)=0\},$$
$$V^{\mathrm{u}}(x)=\{y\in X\mid \text{there is } A\in C(X) \text{ such that } x,y\in A \text{ and } \lim_{n\to\infty}\operatorname{diam} f^{-n}(A)=0\}.$$

Clearly, $W^\sigma(x)\supset V^\sigma(x)$, $\{W^\sigma(x)\mid x\in X\}$ and $\{V^\sigma(x)\mid x\in X\}$ are decompositions of X for both $\sigma=\mathrm{s}$ and u, i.e., $X=\bigcup\{W^\sigma(x)\mid x\in X\}$ (resp. $X=\bigcup\{V^\sigma(x)\mid x\in X\}$), and if $W^\sigma(x)\neq W^\sigma(y)$ (resp. $V^\sigma(x)\neq V^\sigma(y)$), then $W^\sigma(x)\cap W^\sigma(y)=\emptyset$ (resp. $V^\sigma(x)\cap V^\sigma(y)=\emptyset$). Also, for $0<\delta<\varepsilon$ consider the following subsets of $C(X)$:

$$\begin{aligned}
V^{\mathrm{s}}(\varepsilon)&=\{A\in C(X)\mid \operatorname{diam} f^n(A)\le\varepsilon \text{ for any } n\ge 0\},\\
V^{\mathrm{u}}(\varepsilon)&=\{A\in C(X)\mid \operatorname{diam} f^{-n}(A)\le\varepsilon \text{ for any } n\ge 0\},\\
V^{\mathrm{s}}(\delta,\varepsilon)&=\{A\in V^{\mathrm{s}}(\varepsilon)\mid \operatorname{diam} A=\delta\},\\
V^{\mathrm{u}}(\delta,\varepsilon)&=\{A\in V^{\mathrm{u}}(\varepsilon)\mid \operatorname{diam} A=\delta\},\\
V^{\mathrm{s}}&=\{A\in C(X)\mid \lim_{n\to\infty}\operatorname{diam} f^n(A)=0\},\\
V^{\mathrm{u}}&=\{A\in C(X)\mid \lim_{n\to\infty}\operatorname{diam} f^{-n}(A)=0\}.
\end{aligned}$$

We are interested in the structures of the decompositions $\{W^\sigma(x)\mid x\in X\}$ and $\{V^\sigma(x)\mid x\in X\}$ ($\sigma=\mathrm{s}$ and u) of X. Let $f:X\to X$ be an expansive homeomorphism of a compactum X with an expansive constant $c>0$ and $\dim X>0$. Let $c>\varrho>0$ be a positive number. Consider the family $\Phi(\sigma)=\{Z\mid Z$ is a closed subset of X such that (i) for each $x\in Z$ there is a subcontinuum A_x of X with $\operatorname{diam} A_x\ge\varrho$ and $x\in A_x\subset W^\sigma(x)$, and (ii) for any neighborhood U of x in X, there is $y\in Z\cap U$ such that $W^\sigma(x)\neq W^\sigma(y)\}$. By [20, p. 315], $\Phi(\sigma)$ has the maximal element $Z(\sigma,\varrho)$ $(=\mathrm{Cl}(\bigcup\{Z\mid Z\in\Phi(\sigma)\}))$. The set $Z(\sigma,\varrho)$ is said to be a *σ-striped set* of f. Note that if $0<\varrho_1<\varrho_2$, then $Z(\sigma,\varrho_1)\supset Z(\sigma,\varrho_2)$. Also, note that if $Z(\sigma,\varrho)\neq\emptyset$ for some $\varrho>0$, then X contains an uncountable collection of mutually disjoint, nondegenerate subcontinua of X each of which is contained in a different element of $\{W^\sigma(x)\mid x\in X\}$ (see (3.1)).

Let $f:X\to X$ be a map of a compactum X with metric d. Consider the following inverse limit space:

$$(X,f)=\{(x_i)_{i=0}^\infty\mid x_i\in X,\ f(x_{i+1})=x_i \text{ for each } i\ge 0\}.$$

Define a metric $\widetilde{d}$ for (G,f) by

$$\widetilde{d}(\widetilde{x},\widetilde{y})=\sum_{i=0}^\infty d(x_i,y_i)/2^i \quad \text{for } \widetilde{x}=(x_i)_{i=0}^\infty,\widetilde{y}=(y_i)_{i=0}^\infty\in(X,f).$$

The space (X,f) is called the *inverse limit of the map* f. Define a map $\widetilde{f}:(X,f)\to(X,f)$ by

$$\widetilde{f}(x_0,x_1,\dots)=(f(x_0),x_0,x_1,\dots)\quad\text{for } (x_i)_{i=0}^{\infty}\in(X,f).$$

Then $\widetilde{f}$ is a homeomorphism and it is called the *shift map* of f. Let $p_n:(X,f)\to X$ be the natural projection ($n\ge 0$), i.e., $p_n((x_i)_{i=0}^{\infty})=x_n$.

(2.1) Example. Let $f:I\to I$ be the homeomorphism as in Figure 1, where $I=[0,1]$ denotes the unit interval.

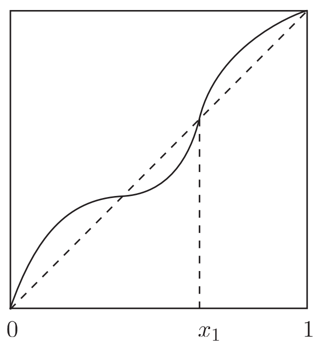

Fig. 1

Then $\{W^{\mathrm{s}}(x)\mid x\in I\}=\{\{0\},(0,x_1),\{x_1\},(x_1,1]\}$ is finite, because $W^{\mathrm{s}}(0)=\{0\}$, $W^{\mathrm{s}}(y)=(0,x_1)$ for $y\in(0,x_1)$, $W^{\mathrm{s}}(x_1)=\{x_1\}$ and $W^{\mathrm{s}}(x)=(x_1,1]$ for $x\in(x_1,1]$. Similarly, $\{W^{\mathrm{u}}(x)\mid x\in I\}$ is finite. Hence $Z(\sigma,\varrho)=\emptyset$ for each $\varrho>0$ ($\sigma=\mathrm{s}$ and u).

(2.2) Example. Let S^1 be the unit circle and let $f:S^1\to S^1$ be the natural covering map with degree 2. Consider the inverse limit (S^1,f) of f and the shift map $\widetilde{f}:(S^1,f)\to(S^1,f)$. The continuum (S^1,f) is well-known as the 2-adic solenoid and $\widetilde{f}$ is an expansive homeomorphism (see [26]). In this case, for each $\widetilde{x}\in(S^1,f)$, $W^{\mathrm{u}}(\widetilde{x})=V^{\mathrm{u}}(\widetilde{x})$ is the arc component of (S^1,f) containing $\widetilde{x}$. Also, $V^{\mathrm{s}}(\widetilde{x})=\{\widetilde{x}\}\subsetneq W^{\mathrm{s}}(\widetilde{x})$ for each $\widetilde{x}\in(S^1,f)$. Then the decomposition $\{W^{\sigma}(\widetilde{x})\mid\widetilde{x}\in(S^1,f)\}$ ($\sigma=\mathrm{s}$ and u) is uncountable. Note that $\dim W^{\mathrm{s}}(\widetilde{x})=0$, because $W^{\mathrm{s}}(\widetilde{x})$ is an F_σ-set and does not contain a nondegenerate subcontinuum (see (3.10) below). Note that the continuum (S^1,f) itself is a u-striped set $Z(\mathrm{u},\varrho)$ of $\widetilde{f}$ for some $\varrho>0$, but $Z(\mathrm{s},\varrho)=\emptyset$ for each $\varrho>0$.

(2.3) Example. (a) *There is an expansive homeomorphism* $f:X\to X$ *such that* $\mathrm{Int}_X\,W^{\sigma}(x)\ne\emptyset$ *for some* $x\in X$. Let G be the one-point union of the unit interval I and a circle S^1, i.e., $(G,*)=(I,1)\vee(S^1,*)$. Define a map $g:G\to G$ such that $g|S^1:S^1\to S^1$ is the natural covering map with degree 2 and $g(0)=0$, $g(1)=*$ and $g(I)=G$. We can choose $g:G\to G$

so that $\widetilde{g} : X = (G, g) \to X = (G, g)$ is expansive (see [10, Theorem 4.3]). Then $W^{\mathrm{u}}(\widetilde{0})$ is a dense open set of X, where $\widetilde{0} = (0, 0, \ldots)$. Hence X itself is not a u-striped set of $\widetilde{g}$.

(b) *There is an expansive homeomorphism h of a continuum Y such that there is a point $x_0 \in Y$ such that if A is any nondegenerate subcontinuum of Y containing x_0, then $A \notin V^{\mathrm{s}} \cup V^{\mathrm{u}}$.* Let G and $g : G \to G$ be the same as in (a), and let X_i $(i = 1, 2)$ be the copies of the space X as in (a). Let $(Y, \widetilde{0}) = (X_1, \widetilde{0}) \vee (X_2, \widetilde{0})$ be the one-point union of X_1 and X_2 (see Figure 2).

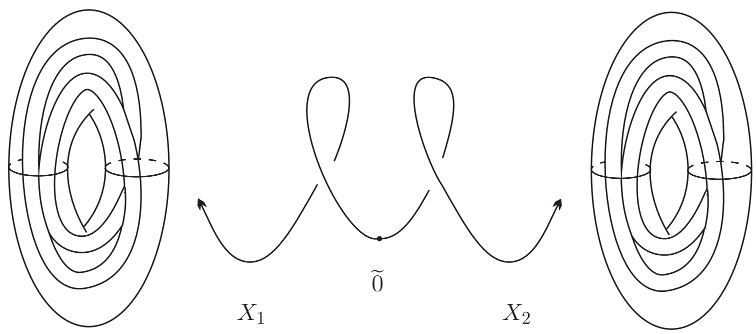

Fig. 2

Take a natural injection $i : \mathbb{R} \to Y$ such that $i(0) = \widetilde{0}$, where $\mathbb{R}$ is the set of real numbers. Let $k : \mathbb{R} \to \mathbb{R}$ be defined by $k(x) = 2x + 1$ $(x \geq 0)$, $k(x) = \frac{1}{2}x + 1$ $(x \leq 0)$. Define a homeomorphism $h : Y \to Y$ by $h(x) = \widetilde{g}(x)$ if $x \in (S^1, g|S^1) \subset X_1$, $h(x) = \widetilde{g}^{-1}(x)$ if $x \in (S^1, g|S^1) \subset X_2$ and $h(x) = i \circ k \circ i^{-1}(x)$ if $x \in i(\mathbb{R})$. Then h is an expansive homeomorphism. Note that if $x_0 \in i(\mathbb{R})$ and A is any nondegenerate subcontinuum containing x_0, then $A \notin V^{\mathrm{s}} \cup V^{\mathrm{u}}$.

Remark. Instead of the solenoid (S^1, g), one can construct the examples above with the help of an Anosov diffeomorphism, say $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the 2-dimensional torus T^2, and a curve outside T^2 "unwinding" from an unstable manifold in T^2.

A subset E of a space X is an *F_σ-set* in X if E is the union of a countable collection of closed subsets of X. A subset E of X is an *$F_{\sigma\delta}$-set* in X if it is the intersection of a countable collection of F_σ-sets.

In this paper, we use a theorem of K. Kuratowski on independent sets [19]. A subset F of X is said to be *independent in $R \subset X^n$* if for every system $x_1, \ldots, x_n$ of different points of F the point $(x_1, \ldots, x_n) \in F^n$ never belongs to R. In [19], K. Kuratowski proved the following theorem.

(2.4) Theorem ([19, Main theorem and Corollary 3]). *If X is a complete space and $R \subset X^n$ is an F_σ-set of the first category, then the set $J(R)$ of* *all compact subsets F of X independent in R is a dense G_δ-set in the space 2^X of all compact subsets of X. Moreover, if X has no isolated points, then almost every Cantor set of X is independent in R.*

(2.5) Proposition. *Let $f : X \to X$ be a homeomorphism of a compactum X. Then $W^\sigma(x)$ is an $F_{\sigma\delta}$-set in X ($\sigma = \mathrm{s}, \mathrm{u}$).*

Proof. We prove the case $\sigma = \mathrm{u}$. Let $x \in X$. For any natural numbers $m, n \geq 1$, consider the set

$$W_{m,n}(x) = \{y \in X \mid d(f^{-i}(x), f^{-i}(y)) \leq 1/n \text{ for } i \geq m\}.$$

Then $W_{m,n}(x)$ is closed and $W_n(x) = \bigcup_{m=1}^\infty W_{m,n}(x)$ is an F_σ-set. Hence $W^\mathrm{u}(x) = \bigcap_{n=1}^\infty W_n(x)$ is an $F_{\sigma\delta}$-set in X.

(2.6) Proposition. *Let $f : X \to X$ be an expansive homeomorphism of a compactum X. Then $W^\sigma(x)$ is an F_σ-set in X ($\sigma = \mathrm{s}, \mathrm{u}$).*

Proof. We prove the case $\sigma = \mathrm{u}$. Let $c > 0$ be an expansive constant for f and let $0 < \varepsilon < c/2$. Note that if $y, y' \in X$ and $d(f^{-i}(y), f^{-i}(y')) \leq \varepsilon$ for each $i \geq m$ (m is some natural number), then $\lim_{i\to\infty} d(f^{-i}(y), f^{-i}(y')) = 0$ (see [20, p. 315]). For any $m = 1, 2, \ldots$, put

$$W_{m,\varepsilon}(x) = \{y \in X \mid d(f^{-i}(x), f^{-i}(y)) \leq \varepsilon \text{ for } i \geq m\}.$$

Since $W^\mathrm{u}(x) = \bigcup_{m=1}^\infty W_{m,\varepsilon}(x)$ and $W_{m,\varepsilon}(x)$ is closed, $W^\mathrm{u}(x)$ is an F_σ-set in X.

(2.7) Proposition. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X. Then $V^\sigma(x)$ is an F_σ-set in X ($\sigma = \mathrm{s}, \mathrm{u}$).*

Proof. We prove the case $\sigma = \mathrm{u}$. Let $c > 0$ be a continuum-wise expansive constant for f and let $0 < \varepsilon < c/2$. Note that if $A \in C(X)$ and $\operatorname{diam} f^{-i}(A) \leq \varepsilon$ for any $i \geq m$, then $\lim_{i\to\infty} \operatorname{diam} f^{-i}(A) = 0$ (see [15, (2.1)]). For each $m = 1, 2, \ldots$, put

$$V_{m,\varepsilon}(x) = \bigcup\{A \in C(X) \mid x \in A \text{ and } \operatorname{diam} f^{-i}(A) \leq \varepsilon \text{ for } i \geq m\}.$$

Then $V^\mathrm{u}(x) = \bigcup_{m=1}^\infty V_{m,\varepsilon}(x)$ is an F_σ-set in X.

3. Striped structures of stable and unstable sets. In this section, we study striped structures of stable and unstable sets of expansive homeomorphisms and continuum-wise expansive homeomorphisms. The main result of this section is the following theorem.

(3.1) Theorem. *Let $f : X \to X$ be an expansive homeomorphism of a compactum X with $\dim X > 0$. Then the decomposition $\{W^\sigma(x) \mid x \in X\}$ ($\sigma = \mathrm{s}$ and u) of X is uncountable. Moreover, there exists σ ($\sigma = \mathrm{s}$ or u) and $\varrho > 0$ such that the σ-striped set $Z(\sigma, \varrho)$ is not empty. In particular, almost* *every Cantor set C of $Z(\sigma, \varrho)$ has the property that for any $x \in C$, there exists a nondegenerate subcontinuum A_x of X such that $x \in A_x \subset W^\sigma(x)$, and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$.*

To prove (3.1), we need the following facts. The next lemma is obvious.

(3.2) Lemma. *Let $f : X \to X$ be a map of a compactum X and let $N \geq 1$ be a natural number. Suppose that there is $\gamma > 0$ such that $d(f^{iN}(x), f^{iN}(y)) \geq \gamma$ for each $i = 0, 1, 2, \ldots$ Then there is $\eta > 0$ such that $d(f^i(x), f^i(y)) \geq \eta$ for each $i = 0, 1, 2, \ldots$*

(3.3) Lemma ([15, (2.3)]). *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$ and let $0 < \varepsilon < c/2$. Then there is $\delta > 0$ such that if A is any nondegenerate subcontinuum of X such that* $\operatorname{diam} A \leq \delta$ *and* $\operatorname{diam} f^m(A) \geq \varepsilon$ *for some integer $m \in \mathbb{Z}$, then one of the following conditions holds*:

(a) *If $m \geq 0$, then* $\operatorname{diam} f^n(A) \geq \delta$ *for each $n \geq m$. More precisely, there is a subcontinuum B of A such that* $\operatorname{diam} f^j(B) \leq \varepsilon$ *for $0 \leq j \leq n$ and* $\operatorname{diam} f^n(B) = \delta$.

(b) *If $m < 0$, then* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each $n \geq -m$. More precisely, there is a subcontinuum B of A such that* $\operatorname{diam} f^{-j}(B) \leq \varepsilon$ *for $0 \leq j \leq n$ and* $\operatorname{diam} f^{-n}(B) = \delta$.

(3.4) Lemma ([15, (2.4)]). *Let f, c, ε, δ be as in* (3.3). *Then for any $\gamma > 0$, there is $N > 0$ such that if $A \in C(X)$ and* $\operatorname{diam} A \geq \gamma$, *then either* $\operatorname{diam} f^n(A) \geq \delta$ *for each $n \geq N$, or* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each $n \geq N$.*

Proof of (3.1). Let c, ε, δ be positive numbers as in (3.3). Suppose that there exists no nondegenerate subcontinuum A of X such that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$, i.e., $V^{\mathrm{u}} = \{\{x\} \mid x \in X\}$. Let C be a nondegenerate component of X. Then for any $x \in C$, $W^{\mathrm{u}}(x) = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed (see (2.6)). Note that $\operatorname{Int}_C(F_i \cap C) = \emptyset$ for all i. By the Baire category theorem, $\{W^{\mathrm{u}}(x) \mid x \in C\}$ is uncountable, and hence $\{W^{\mathrm{u}}(x) \mid x \in X\}$ is uncountable. The case $\sigma = \mathrm{s}$ is similar.

Next, we shall show the existence of a nonempty σ-striped set $Z(\sigma, \varrho)$. By [20, Lemma 3], there is a nondegenerate subcontinuum $A \in V^{\mathrm{s}} \cup V^{\mathrm{u}}$. From now on, we assume that there is a nondegenerate subcontinuum A such that $\lim_{i\to\infty} \operatorname{diam} f^{-i}(A) = 0$. In this case, $V^{\mathrm{u}}(\delta, \varepsilon) \neq \emptyset$ (see (3.3)). Note that if $A \in V^{\mathrm{u}}$, then $W^{\mathrm{u}}(x) = W^{\mathrm{u}}(y)$ for all $x, y \in A$.

For any closed subset M of $V^{\mathrm{u}}(\varepsilon)$, we define

$$M^f = \{A \in C(X) \mid \text{for any neighborhood } \mathbf{U} \text{ of } A \text{ in } C(X) \text{ there is } A' \in M \text{ such that } A' \in \mathbf{U} \text{ and } W^{\mathrm{u}}(a) \cap W^{\mathrm{u}}(a') = \emptyset \text{ for all } a \in A \text{ and } a' \in A'\}.$$

We can easily see that $M^f \subset M$ is a closed subset of $C(X)$ and $(M^f)^f \subset M^f$. For any ordinal numbers, define $M_0 = M$, $M_1 = M^f$, $M_{\alpha+1} = (M_\alpha)^f$, and $M_\lambda = \bigcap_{\alpha<\lambda} M_\alpha$, where λ is a limit ordinal. From now on, we assume that

$$M = M(\delta) = V^{\mathrm{u}}(\delta, \varepsilon).$$

Now, we need the following claim which is directly proved by transfinite induction.

Claim (λ). *Let λ be a countable ordinal. If $A \in M_\lambda$, then there are two subcontinua A_1 and A_2 of A such that $d(A_1, A_2) \geq \delta/3$ and $A_1, A_2 \in (M')_\lambda$, where $M' = M(\gamma)$ $(= V^{\mathrm{u}}(\gamma, \varepsilon))$ and $0 < \gamma < \delta/3$.*

By transfinite induction, we shall prove $M_\lambda \neq \emptyset$ for any ordinal λ. Choose $\gamma > 0$ such that if A and B are any subsets of X with $\operatorname{diam} A \geq \delta$ and $\operatorname{diam} B \geq \delta$, then there are $a \in A$ and $b \in B$ such that $d(a, b) \geq 3\gamma$. Let N be a natural number such that if $A \in C(X)$ and $\operatorname{diam} A \geq \gamma$, then either $\operatorname{diam} f^n(A) \geq \delta$ $(n \geq N)$ or $\operatorname{diam} f^{-n}(A) \geq \delta$ $(n \geq N)$ (see (3.4)). We may assume $\gamma \leq \delta/3$.

First, choose $A \in V^{\mathrm{u}}(\delta, \varepsilon)$. Since $\operatorname{diam} A = \delta$, we can choose two subcontinua A_1, B_1 of A such that $A_1, B_1 \in M' = V^{\mathrm{u}}(\gamma, \varepsilon)$ and $d(A_1, B_1) \geq \delta/3 \geq \gamma$ (see Claim (0)). Since $\operatorname{diam} f^N(A_1) \geq \delta$ and $\operatorname{diam} f^N(B_1) \geq \delta$, we choose a subcontinuum A_2 of $f^N(A_1)$ and a subcontinuum B_2 of $f^N(B_1)$ such that $A_2, B_2 \in V^{\mathrm{u}}(\gamma, \varepsilon)$ and $d(A_2, B_2) \geq \gamma$. By induction, we can choose two sequences $\{A_n\}$ and $\{B_n\}$ of $C(X)$ such that $A_n \subset f^N(A_{n-1})$, $B_n \subset f^N(B_{n-1})$, $d(A_n, B_n) \geq \gamma$ and $A_n, B_n \in V^{\mathrm{u}}(\gamma, \varepsilon)$. Also, choose a subsequence $n_1 < n_2 < \ldots$ of natural numbers such that $\lim_{i\to\infty} A_{n_i} = A'$ and $\lim_{i\to\infty} B_{n_i} = B'$. Then $d(f^{-Ni}(A'), f^{-Ni}(B')) \geq \gamma$ for each $i \geq 0$. By (3.2), $d(f^{-i}(a), f^{-i}(b)) \geq \eta$ for all $a \in A'$, $b \in B'$ and $i \geq 0$, and hence $W^{\mathrm{u}}(a) \neq W^{\mathrm{u}}(b)$. Note that for each $a_{n_i} \in A_{n_i}$ and $b_{n_i} \in B_{n_i}$, $W^{\mathrm{u}}(a_{n_i}) = W^{\mathrm{u}}(b_{n_i})$. Hence either $A' \in (M')^f = (M')_1$ or $B' \in (M')^f = (M')_1$. We assume that $A' \in (M')_1$. By (3.3), $f^N(A')$ contains a subcontinuum A_1 such that $A_1 \in M_1$, which implies that $M_1 \neq \emptyset$.

For a countable ordinal λ, we may assume that for any $\alpha < \lambda$, M_α is not empty. We must consider the following two cases.

(I) $\lambda = \alpha + 1$. Claim (α) and an argument similar to the above show that M_λ is not empty.

(II) λ is a limit ordinal. In this case, take an increasing sequence $\alpha_1 < \alpha_2 < \ldots$ of countable ordinals such that $\lim_{i\to\infty} \alpha_i = \lambda$. Also, choose $A_i \in M_{\alpha_i}$ for each i. We may assume that $\lim_{i\to\infty} A_i = A_\infty$. Then $A_\infty \in \bigcap_{\alpha<\lambda} M_\alpha = M_\lambda$.

Thus we proved that $M_\lambda \neq \emptyset$ for any countable ordinal λ. Hence there is a countable ordinal α such that $M_\alpha = M_{\alpha+1}$ $(\neq \emptyset)$. Put $Z = \bigcup\{A \mid A \in M_\alpha\}$.

Since M_α is closed in $C(X)$, Z is also closed in X. We can easily see that $Z = Z(\mathrm{u}, \delta)$ is a u-striped set of f. Put

$$A(n, \varepsilon) = \{(x, y) \in Z \times Z \mid d(f^{-i}(x), f^{-i}(y)) \le \varepsilon \text{ for each } i \ge n\}.$$

Then $A(n, \varepsilon)$ is a closed subset of $Z \times Z$; put $R = \bigcup_{n=1}^{\infty} A(n, \varepsilon)$. Note that $\mathrm{Int}_Z\, A(n, \varepsilon) = \emptyset$. Hence R is an F_σ-set of the first category in $Z \times Z$. By the theorem of K. Kuratowski on independent sets (see (2.4)), $\mathbf{S} = \{S \in 2^Z \mid S$ is independent in $R\} = \{S \in 2^Z \mid$ for any $x, y \in S$ with $x \ne y, W^{\mathrm{u}}(x) \ne W^{\mathrm{u}}(y)\}$ is a dense G_δ-set in 2^Z. By (2.4), almost every Cantor set of Z is contained in $\mathbf{S}$. This completes the proof.

(3.5) Corollary. *Under the assumption of* (3.1), *if moreover* V^{s} *and* V^{u} *contain nondegenerate subcontinua*, *then for both* $\sigma = \mathrm{s}$ *and* $\sigma = \mathrm{u}$, *the* σ*-striped set* $Z(\sigma, \varrho)$ *of* f *is not empty for some* $\varrho > 0$.

By (2.7) and an argument similar to the above, we can prove the following theorem on continuum-wise expansive homeomorphisms.

(3.6) Theorem. *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X > 0$. *Then the decompositions* $\{V^\sigma(x) \mid x \in X\}$ ($\sigma = \mathrm{s}$ *and* u) *are uncountable. Moreover*, *there is* σ ($\sigma =$ s *or* u) *and a positive number* $\varrho > 0$ *such that there is a nonempty closed set* Z' *of* X *such that* (i) *for each* $x \in Z'$ *there is a subcontinuum* A_x *of* X *with* $\operatorname{diam} A_x \ge \varrho$ *and* $x \in A_x \subset V^\sigma(x)$, (ii) *for any neighborhood* U *of* x *in* X, *there is* $y \in Z' \cap U$ *such that* $V^\sigma(x) \ne V^\sigma(y)$. *In particular*, *almost every Cantor set* C *of* $Z(\sigma)$ *has the property that for any* $x \in C$, *there is a nondegenerate subcontinuum* A_x *of* X *with* $x \in A_x \subset V^\sigma(x)$, *and if* $x, y \in C$ *and* $x \ne y$, *then* $V^\sigma(x) \ne V^\sigma(y)$.

(3.7) Theorem. *Let* X *be a locally connected continuum* (= *Peano continuum*). *If* $f : X \to X$ *is an expansive homeomorphism* (*resp. a continuum-wise expansive homeomorphism*) *of* X, *then there is an uncountable subset* Z *of* X *such that* $\mathrm{Cl}(Z) = X$, *and*

(1) *for each* $x \in Z$ *and* $\sigma = \mathrm{s}$ *and* u, *there is a nondegenerate subcontinuum* $A_x \in V^\sigma$ *with* $x \in A_x$ *and* $\operatorname{diam} A_x \ge \delta$ *for some* $\delta > 0$,

(2) *if* $x, y \in Z$ *and* $x \ne y$, *then* $W^\sigma(x) \ne W^\sigma(y)$ (*resp.* $V^\sigma(x) \ne V^\sigma(y)$) *for both* $\sigma = \mathrm{s}$ *and* u.

To prove (3.7), we need the following.

(3.8) Lemma ([16, (1.6)]). *Let* $f : X \to X$ *be a continuum-wise expansive homeomorphism of a Peano continuum* X. *Then there is* $\delta > 0$ *such that for each* $x \in X$, *there are two subcontinua* A_x *and* B_x *such that* $x \in A_x \cap B_x$, $A_x \in V^{\mathrm{s}}$, $B_x \in V^{\mathrm{u}}$, $\operatorname{diam} A_x = \delta$ *and* $\operatorname{diam} B_x = \delta$. *In particular*, $\mathrm{Int}_X(W^\sigma(x)) = \emptyset$ *for each* $x \in X$ *and* $\sigma = \mathrm{s}$, u.

Proof of (3.7). Suppose that f is an expansive homeomorphism. The case of continuum-wise expansive homeomorphism is similarly proved. Let $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ be a base of X. We use the Baire category theorem. By induction, we obtain a countable subset Z_ω of X such that $U_i \cap Z_\omega \neq \emptyset$ and if $x, y \in Z_\omega$, then $W^\sigma(x) \neq W^\sigma(y)$ for $\sigma = \mathrm{s}$ and u, because $\mathrm{Int}_X(W^\sigma(x)) = \emptyset$ and $W^\sigma(x)$ is an F_σ-set (see (2.6)). By transfinite induction, for any countable ordinal λ we have a countable set Z_λ such that (1) if $x, y \in Z_\lambda$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$ for both $\sigma = \mathrm{s}$ and u, and (2) if $\alpha < \beta$, then $Z_\alpha \subsetneq Z_\beta$. Put $Z = \bigcup_{\lambda<\omega_1} Z_\lambda$. Clearly, Z is the desired set.

Let $f : X \to X$ be a homeomorphism of a compactum X. For $\sigma = \mathrm{s}$ and u, let $M_\sigma(f)$ be the maximal element of the family $\{Z \mid Z$ is a closed subset of X such that for any $x \in Z$ and any neighborhood U of x in X there is $y \in Z \cap U$ such that $W^\sigma(x) \neq W^\sigma(y)\}$. Clearly, $M_\sigma(f)$ is f-invariant. It is called the *σ-mixed set* of f.

By (3.1) and (3.7), we have the following corollary.

(3.9) COROLLARY. *If $f : X \to X$ is an expansive homeomorphism of a compactum X with $\dim X > 0$, then the σ-mixed set $M_\sigma(f)$ is not empty and hence it is a perfect set. Moreover, if X is a Peano continuum, then $M_\sigma(f) = X$.*

Proof. By (3.1), there is an uncountable subset H_σ of X such that if $x, y \in H_\sigma$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$. Then $\mathrm{Cl}(H_\sigma)$ is a closed and uncountable set, hence it contains a Cantor set C. Then $C \subset M_\sigma(f)$.

For the case of inverse limits of graphs, we have the following theorem.

(3.10) THEOREM. *Let $f : G \to G$ be a map of a graph G (= finite connected 1-dimensional polyhedron). Suppose that the shift map $\widetilde{f} : (G, f) \to (G, f)$ is expansive. Then for each $\widetilde{x} \in (G, f)$,* (a) *$W^{\mathrm{u}}(\widetilde{x})$ is equal to the arc component $A(\widetilde{x})$ of (G, f) containing $\widetilde{x}$, and* (b) *$W^{\mathrm{s}}(\widetilde{x})$ is 0-dimensional.*

To prove (3.10), we need the following notations. Let A be a closed subset of a compactum X. A map $f : X \to X$ is called *positively expansive on A* if there is $c > 0$ such that if $x, y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > c$. If a map $f : X \to X$ is positively expansive on the whole space X, we say f is *positively expansive*. Let $\mathcal{A}$ be a finite closed covering of X. A map $f : X \to X$ is *positively pseudo-expansive with respect to $\mathcal{A}$* if the following conditions hold:

(P_1) f is positively expansive on A for each $A \in \mathcal{A}$.

(P_2) For all $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$, either f is positively expansive on $A \cup B$, or there is a natural number $k \geq 1$ such that for any $A', A'' \in \mathcal{A}$ with $A' \cap A'' \neq \emptyset$, either

$$f^k(A' \cup A'') \cap (A - B) = \emptyset \quad \text{or} \quad f^k(A' \cup A'') \cap (B - A) = \emptyset\,.$$

(3.11) THEOREM ([13, (2.5)]). *Let G be a graph and let $f : G \to G$ be an onto map. Then the shift map $\widetilde{f} : (G, f) \to (G, f)$ is expansive if and only if f is positively pseudo-expansive with respect to $\mathcal{A}$, where $\mathcal{A} = \{e \mid e$ is an edge of some simplicial complex K with $|K| = G\}$.*

(3.12) PROPOSITION ([13, (2.9)]). *Let $f : G \to G$ be an onto map of a graph G. If the shift map $\widetilde{f} : (G, f) \to (G, f)$ is expansive, then there is $\alpha > 0$ such that if A is a subcontinuum of (G, f) with* $\operatorname{diam} A \le \alpha$, *then $A \in V^{\mathrm{u}}$, i.e.,* $\lim_{n\to\infty} \operatorname{diam} \widetilde{f}^{-n}(A) = 0$.

Proof of (3.10). We may assume that $f : G \to G$ is an onto map, since so is $f|G' : G' \to G'$, where $G' = p_n((G, f))$ and $p_n : (G, f) \to G$ is the natural projection.

(a) Let $\widetilde{y}$ be any point of the arc component $A(\widetilde{x})$ of (G, f) containing $\widetilde{x}$. Choose an arc A from $\widetilde{x}$ to $\widetilde{y}$ in $A(\widetilde{x})$. Choose points $\widetilde{x} = \widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_m = \widetilde{y}$ of A such that $\operatorname{diam}[\widetilde{x}_i, \widetilde{x}_{i+1}] \le \alpha$ for each $i = 0, 1, \ldots, m-1$, where $\alpha > 0$ is as in (3.12) and $[\widetilde{x}_i, \widetilde{x}_{i+1}]$ denotes the arc from $\widetilde{x}_i$ to $\widetilde{x}_{i+1}$ in A. By (3.12), $[\widetilde{x}_i, \widetilde{x}_{i+1}] \in V^{\mathrm{u}}$ for each i, hence $\widetilde{y} \in W^{\mathrm{u}}(\widetilde{x})$. This implies that $A(\widetilde{x}) \subset W^{\mathrm{u}}(\widetilde{x})$.

We show the converse inclusion. Let $\widetilde{y} \in W^{\mathrm{u}}(\widetilde{x})$. Suppose that $\widetilde{x} = (x_i)_{i=0}^{\infty}$ and $\widetilde{y} = (y_i)_{i=0}^{\infty}$. Since $\lim_{i\to\infty} d(x_i, y_i) = 0$, there is $m \ge 0$ such that if $n \ge m$, then $x_n \in e_n$ and $y_n \in e'_n$, where K is a simplicial complex as in (3.11) and e_n, e'_n are edges of K such that $e_n \cap e'_n \ne \emptyset$. Also, we assume that $d(x_n, y_n) < \min\{d(e, e') \mid e$ and e' are edges of K with $e \cap e' = \emptyset\}$ for each $n \ge m$.

Since $f : G \to G$ is positively pseudo-expansive with respect to $\mathcal{A} = \{e \mid e$ is an edge of $K\}$, it is positively expansive on $e_n \cup e'_n$ for each $n \ge m$. We may assume that $f([x_n, y_n])$ does not contain a simple closed curve, where $[x_n, y_n]$ is the arc in $e_n \cup e'_n$ from x_n to y_n. It follows that $f([x_{n+1}, y_{n+1}]) = [x_n, y_n]$ and $f|[x_{n+1}, y_{n+1}] : [x_{n+1}, y_{n+1}] \to [x_n, y_n]$ is a homeomorphism for each $n \ge m$, because $f|[x_n, y_n]$ is locally injective and $f([x_n, y_n])$ does not contain a simple closed curve. Consider the subset $A = \{(z_i)_{i=0}^{\infty} \mid z_i \in G,\ z_n \in [x_n, y_n]$ for each $n \ge m$ and $f(z_{i+1}) = z_i$ for each $i \ge 0\}$ in (G, f). Clearly, A is an arc from $\widetilde{x}$ to $\widetilde{y}$ in (G, f). Hence $\widetilde{y} \in A(\widetilde{x})$. Note that $V^{\mathrm{u}}(\widetilde{x}) = W^{\mathrm{u}}(\widetilde{x})$.

(b) By (3.12), $W^{\mathrm{s}}(\widetilde{x})$ contains no nondegenerate subcontinuum. Since $W^{\mathrm{s}}(\widetilde{x})$ is an F_σ-set in (G, f), $W^{\mathrm{s}}(\widetilde{x}) = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed. Since $\dim F_i = 0$ for each i, by the sum theorem of dimension theory we see that $\dim W^{\mathrm{s}}(\widetilde{x}) = 0$.

(3.13) Remark. Of course (3.10) is not true for general expansive homeomorphisms. Consider for example an arbitrary Anosov diffeomorphism. Even in the 1-dimensional case (3.10) is not true. Let $g : G \to G$ be the map as in (a) of (2.3). Let $Y = ((G, g), \widetilde{0}) \vee ((G, g)', \widetilde{0}')$ be the one-point union of $((G,g),\widetilde{0})$ and $((G,g)',\widetilde{0}')$, where $((G,g)',\widetilde{0}')$ is a copy of $((G,g),\widetilde{0})$. Define a homeomorphism $f:Y\to Y$ by

$$f(y)=\begin{cases}\widetilde{g}(y) & \text{if } y\in (G,g),\\ \widetilde{g}^{-1}(y) & \text{if } y\in (G,g)'.\end{cases}$$

Then f is an expansive homeomorphism and for both $\sigma=\mathrm{s}$ and u, $W^{\sigma}(\widetilde{0})$ is not equal to the arc component $A(\widetilde{0})$.

Also, (3.10) is not true in the case that the shift map $\widetilde{f}:(G,f)\to(G,f)$ of f is continuum-wise expansive, where $f:G\to G$ is a map of a graph G. In fact, there is a map $f:I\to I$ of the unit interval I such that $\widetilde{f}:(I,f)\to(I,f)$ is a continuum-wise expansive homeomorphism and (I,f) is a pseudo-arc (= hereditarily indecomposable arc-like continuum) (see [16, (2.3)]). Since (I,f) contains no arc and $W^{\mathrm{u}}(\widetilde{x})$ contains a nondegenerate subcontinuum of (I,f) for each $\widetilde{x}\in(I,f)$ (see the proof of [15, (3.2)]), $W^{\mathrm{u}}(\widetilde{x})$ is not equal to the arc component $A(\widetilde{x})=\{\widetilde{x}\}$.

In connection with (3.10), we have the following questions.

Question 1. In the situation of (3.10), under what assumptions does $X=(G,f)$ admit a closed neighborhood base $\{B_n\}_{n=1}^{\infty}$ such that each B_n is the product of an arc in W^{u} and a Cantor set in W^{s}? Is the condition that X is σ-mixed sufficient? For Williams' mixing expanding maps on 1-dimensional branched manifolds, the answer is positive [27].

Question 2. Does "positive pseudo-expansiveness" imply "pseudo-expanding" in a metric giving the same topology as the original metric (cf. [3])?

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