

**Weakly normal ideals on $\mathcal{P}_\kappa\lambda$
and the singular cardinal hypothesis**

by

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Abstract. In §1, we observe that a weakly normal ideal has a saturation property; we also show that the existence of certain precipitous ideals is sufficient for the existence of weakly normal ideals. In §2, generalizing Solovay's theorem concerning strongly compact cardinals, we show that $\lambda^{<\kappa}$ is decided if $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal and λ is regular or $\text{cf } \lambda \leq \kappa$. This is applied to solving the singular cardinal hypothesis.

0. Preliminaries. A strongly compact cardinal introduces certain regularities in the universe of set theory. For example, Solovay showed that the singular cardinal hypothesis holds above a compact cardinal.

If κ is λ -compact, $\mathcal{P}_\kappa\lambda$ carries a weakly normal fine ultrafilter. So, the existence of weakly normal ideals is a weaker hypothesis than the existence of strongly compact cardinals. In this paper, we use a weakly normal ideal to reprove those results of [14] for which Solovay used a strongly compact cardinal.

We would like to express our gratitude to Yo Matsubara for his helpful comments.

Our set theory is ZFC and much of notation is standard (see [4], [8], [15]). Throughout the paper κ is a regular uncountable cardinal and λ is a cardinal $\geq \kappa$. Unless specified otherwise, every ideal on $\mathcal{P}_\kappa\lambda$ is assumed to be κ -complete and fine. So, every ideal I contains the smallest ideal $I_{\kappa\lambda} = \{X \subset \mathcal{P}_\kappa\lambda : X \text{ is not unbounded}\}$. Set $I^+ = \mathcal{P}(\mathcal{P}_\kappa\lambda) - I$ and let I^* be the filter dual to I . The sets in I^+ and I^* are called *I-positive* and *I-measure one* respectively. $NS_{\kappa\lambda}$ is the ideal of nonstationary sets, and $SNS_{\kappa\lambda}$ is the ideal of strongly nonstationary sets. For each $x \in \mathcal{P}_\kappa\lambda$, \hat{x} is the set $\{y \in \mathcal{P}_\kappa\lambda : x \subset y\}$. If f is a function, $f''A$ is the image of A under f .

Research partially supported by Grant-in-Aid for Scientific Research (No. 01302006), Ministry of Education, Science and Culture of Japan.

1. Weakly normal ideals and saturated ideals. Weakly normal fine ultrafilters as well as weakly normal ideals defined below can be seen as a weak version of normal ultrafilters. On the other hand, the form of weak normality proposed by Mignone [10], which we call here “semi-weak normality”, is a weakening of normality of filters.

DEFINITION. An ideal I on $\mathcal{P}_\kappa\lambda$ is *weakly normal* if for every regressive function $f : \mathcal{P}_\kappa\lambda \rightarrow \lambda$, $\{x \in \mathcal{P}_\kappa\lambda : f(x) \leq \gamma\} \in I^*$ for some $\gamma < \lambda$. I is called *semi-weakly normal* if for all $X \in I^+$ and all regressive functions $f : X \rightarrow \lambda$, there is a $\gamma < \lambda$ such that $\{x \in X : f(x) \leq \gamma\} \in I^+$.

Our weak normality is a $\mathcal{P}_\kappa\lambda$ generalization of weak normality for filters on κ due to Kanamori [7]. It appears in the proof of Theorem 2.1.

We begin by showing that weak normality is a combination of semi-weak normality and a saturation property.

LEMMA 1.1. *I is weakly normal iff I is semi-weakly normal and there is no disjoint family of cf λ -many I -positive sets.*

Proof. Suppose that I is weakly normal. Let $X \in I^+$ and $f : X \rightarrow \lambda$ be regressive. We extend f to $g : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ that is also regressive. Using weak normality of I , we can find $Y \in I^*$ and $\gamma < \lambda$ so that $f(x) \leq \gamma$ for all $x \in Y$. Set $Z = X \cap Y$. Then $g|Z = f|Z$ and $Z \in I^+$. Thus I is semi-weakly normal.

Next, assume that there exists a disjoint family $\{A_\alpha : \alpha < \text{cf } \lambda\}$ of I -positive sets. Let $\{\lambda_\alpha \mid \alpha < \text{cf } \lambda\}$ be a cofinal increasing sequence in λ . We may assume that $A_\alpha \subset \widehat{\{\lambda_\alpha\}}$ for any $\alpha < \text{cf } \lambda$. Define a regressive function $f : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ by $f''A_\alpha = \{\lambda_\alpha\}$. Since I is weakly normal, $B = \{x : f(x) \leq \gamma\} \in I^*$ for some $\gamma < \lambda$. Now pick a $\lambda_\alpha > \gamma$. By the definition of f , $A_\alpha \subset f^{-1}(\{\lambda_\alpha\})$ and $f^{-1}(\{\lambda_\alpha\}) \cap B = \emptyset$. This contradicts $A_\alpha \in I^+$.

Conversely, suppose that I is a semi-weakly normal ideal with no disjoint family of cf λ -many positive sets. If I is not weakly normal, there is a regressive function $f : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ such that $\{x : f(x) \geq \gamma\} \in I^+$ for any $\gamma < \lambda$. Since I is semi-weakly normal, we can find a $\gamma_0 < \lambda$ such that $A_0 = \{x : f(x) < \gamma_0\} \in I^+ - I^*$. Since $\mathcal{P}_\kappa\lambda - A_0 \in I^+$, we have a $\gamma_1 < \lambda$ so that $A_1 = \{x : \gamma_0 \leq f(x) < \gamma_1\} \in I^+ - I^*$. In the same way, for any $\alpha < \text{cf } \lambda$, we can define $\gamma_{\alpha+1} < \lambda$ such that $A_{\alpha+1} = \{x : \gamma_\alpha \leq f(x) < \gamma_{\alpha+1}\} \in I^+$. For α a limit ordinal less than $\text{cf } \lambda$, let $\eta_\alpha = \sup\{\gamma_\beta : \beta < \alpha\} < \lambda$. Since $\{x : \eta_\alpha \leq f(x)\}$ is I -positive, there is a γ_α so that $A_\alpha = \{x : \eta_\alpha \leq f(x) < \gamma_\alpha\} \in I^+ - I^*$.

Contrary to our hypothesis, we now have a pairwise disjoint family $\{A_\alpha : \alpha < \text{cf } \lambda\}$ of I -positive sets. ■

COROLLARY 1.2. *If $\text{cf } \lambda = \kappa$, then I is weakly normal iff it is semi-weakly normal and κ -saturated.*

COROLLARY 1.3. *Let $\text{cf } \lambda < \kappa$. Then I is weakly normal iff I is $\text{cf } \lambda$ -saturated.*

PROOF. It is easy to show that every ideal is semi-weakly normal if $\text{cf } \lambda < \kappa$. For more on semi-weak normality, see [10] and [11]. ■

1.2 and 1.3 show κ is large in some inner model if $\mathcal{P}_\kappa \lambda$ carries a weakly normal ideal provided that $\text{cf } \lambda \leq \kappa$. It will be shown in [3] that the existence of weakly normal ideals on $\mathcal{P}_\kappa \lambda$ is possible for κ with various degree of largeness.

Here we only state that some familiar ideals are not weakly normal.

COROLLARY 1.4. *None of $I_{\kappa\lambda}$, $SNS_{\kappa\lambda}$, $NS_{\kappa\lambda}$ is weakly normal.*

PROOF. It is known that $\mathcal{P}_\kappa \lambda$ is a disjoint union of λ stationary subsets (see [8] for example) and every extension of a weakly normal ideal is also weakly normal. ■

For normal ideals, easy observations suggest that:

COROLLARY 1.5. *Every $\text{cf } \lambda$ -saturated normal ideal is weakly normal.*

PROOF. Let $f : \mathcal{P}_\kappa \lambda \rightarrow \lambda$ be regressive and $A = \{\gamma < \lambda : f^{-1}(\{\gamma\}) \in I^+\}$. Since I is $\text{cf } \lambda$ -saturated, $|A| < \text{cf } \lambda$. Set $\delta = \sup A$. Then $\delta < \lambda$ and it is clear that $\{x : f(x) \leq \delta\} \in I^*$. ■

Conversely, saturated ideals produce weakly normal ideals under certain conditions. We already know some cases (1.3, 1.5). In fact, Corollary 1.8 below was proved in [2] using an analogue of Solovay's construction of incompressible functions (see [13]).

We use here a generic ultrapower which makes the proof much simpler.

DEFINITION. Let I and J be ideals.

- (1) $J \leq_{RK} I$ if $J = f_*(I) = \{X : f^{-1}(X) \in I\}$ for some $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$.
- (2) For $X \in I^+$, $I|X = \{Y \subset \mathcal{P}_\kappa \lambda : Y \cap X \in I\}$, which is also an ideal.
- (3) $I_\delta = f_*(I)$ for $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \delta$ such that $f(x) = x \cap \delta$.

PROPOSITION 1.6. *Suppose that I is a precipitous ideal on $\mathcal{P}_\kappa \lambda$. Then there is a semi-weakly normal ideal $J \leq_{RK} I|X$ for some $X \in I^+$.*

PROOF. Let G be a generic filter for P_I , the poset of I -positive subsets of $\mathcal{P}_\kappa \lambda$, and let $j : V \rightarrow M$ be the corresponding generic elementary embedding. Pick a name \underline{f} such that $1 \Vdash_{P_I} \underline{f}$ represents $\sup j'' \lambda$ in M . There are $X \in I^+$ and $f : X \rightarrow V$ with $X \Vdash_{P_I} \underline{f} = \check{f}$. Note that for every $\alpha < \lambda$, $\{x \in X : f(x) \leq \alpha\} \in I$.

Suppose $Y = \{x \in X : g(x) < f(x)\} \in I^+$. Since $Y \leq_{P_I} X$, $Y \Vdash_{P_I} \check{f}$ represents $\sup j'' \lambda$ and $[g]_G < [f]_G$. Thus $Y \Vdash_{P_I} \exists \alpha < \lambda ([g]_G < j(\alpha))$. So, $\{x \in Y : g(x) < \alpha\} \in I^+$ for some $\alpha < \lambda$.

Now if h is defined by $h(x) = x \cap f(x)$ for $x \in X$, the above observation shows that $J = h_*(I|X)$ is a semi-weakly normal ideal. ■

As a corollary, we get the next theorem.

THEOREM 1.7. *If I is a precipitous ideal on $\mathcal{P}_\kappa\lambda$ with no pairwise disjoint family of cf λ -many I -positive sets, then for any $X \in I^+$ we can find a $Y \in P(X) \cap I^+$ and a weakly normal ideal $J \leq_{RK} I|Y$.*

Proof. Use Lemma 1.1 and Proposition 1.6. ■

Yo Matsubara taught the author a simpler construction. Let $J = \{Y \subset \mathcal{P}_\kappa\lambda : 1 \Vdash_{\mathcal{P}_I} [\text{id}] \cap j''\lambda \in j(\mathcal{P}_\kappa\lambda - Y)\}$. Then J is weakly normal.

Recall that any countably complete ideal with the disjointing property is precipitous, and every κ -complete κ^+ -saturated ideal has the disjointing property. (See Foreman [5].)

COROLLARY 1.8. (i) *If $\text{cf } \lambda \geq \kappa$ and $\mathcal{P}_\kappa\lambda$ carries a κ -saturated ideal, then there exists a κ -saturated weakly normal ideal.*

(ii) *If $\text{cf } \lambda \geq \kappa^+$ and there is a κ^+ -saturated ideal on $\mathcal{P}_\kappa\lambda$, then there exists a weakly normal ideal on $\mathcal{P}_\kappa\lambda$.*

If κ is λ -compact, then it is δ -compact for all $\kappa \leq \delta < \lambda$. So, one can ask whether the existence of a weakly normal filter on $\mathcal{P}_\kappa\lambda$ assures the existence of one on $\mathcal{P}_\kappa\delta$ for any $\delta < \lambda$.

If I is a normal ideal on $\mathcal{P}_\kappa\lambda$, then I_δ is also normal. But the situation is not clear for weak normality. We can only prove:

THEOREM 1.9. (1) *If I is a weakly normal ideal on $\mathcal{P}_\kappa\lambda$ and $\text{cf } \lambda \leq \kappa$, then there is a weakly normal ideal for any $\kappa \leq \delta < \lambda$ such that $\text{cf } \delta \geq \kappa$.*

(2) *If there is a $\kappa^+(\kappa)$ -saturated ideal on $\mathcal{P}_\kappa\lambda$, then we have a weakly normal ideal on $\mathcal{P}_\kappa\delta$ for all $\delta < \lambda$ with $\text{cf } \delta \geq \kappa^+(\kappa)$.*

(3) *If $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal and $\text{cf } \lambda > \kappa$, then $\mathcal{P}_\kappa \text{cf } \lambda$ also bears a weakly normal ideal.*

(4) *If $\kappa < \delta < \lambda$, $\kappa < \text{cf } \delta = \text{cf } \lambda$ and $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal, then there exists a weakly normal ideal on $\mathcal{P}_\kappa\delta$.*

Proof. (1) and (2) are clear from 1.2, 1.3, 1.8, and the fact that I_δ is also $\kappa^+(\kappa)$ -saturated for any $\kappa^+(\kappa)$ -saturated ideal I on $\mathcal{P}_\kappa\lambda$.

(3) Let $\kappa < \delta = \text{cf } \lambda < \lambda$, let $\{\lambda_\alpha \mid \alpha < \delta\}$ be a cofinal normal sequence in λ , and $K_\alpha = [\lambda_\alpha, \lambda_{\alpha+1})$. If $f(\beta) =$ the unique ordinal α such that $\beta \in K_\alpha$, then f is a mapping from λ onto δ , and $g : \mathcal{P}_\kappa\lambda \rightarrow \mathcal{P}_\kappa\delta$ defined by $g(x) = f''x$ is also onto. For a weakly normal ideal I on $\mathcal{P}_\kappa\lambda$, define J by

$$X \in J \quad \text{iff} \quad X \subset \mathcal{P}_\kappa\delta \text{ and } g^{-1}(X) \in I.$$

Then J is a κ -complete proper ideal. For $\alpha < \delta$,

$$\begin{aligned} g^{-1}(\{x : \alpha \notin x\}) &= \{x \in \mathcal{P}_\kappa \lambda : f(\beta) \neq \alpha \text{ for all } \beta \in x\} \\ &= \{x : x \cap K_\alpha = \emptyset\} \in I. \end{aligned}$$

So, J is fine.

To see that J is weakly normal, let $h : \mathcal{P}_\kappa \delta \rightarrow \delta$ be regressive. We have $h \circ g(x) \in g(x)$ for all $x \in \mathcal{P}_\kappa \lambda$ and $g(x) = f''x$. Thus $h \circ g(x) = f(\gamma_x)$ for some $\gamma_x \in x$. Using weak normality of I , we can find a $\gamma < \lambda$ such that $X = \{x : \gamma_x \leq \gamma\} \in I^*$. By our definition, f is increasing. Hence $f(\gamma_x) \leq f(\gamma)$ for all $x \in X$, which means that $\{x \in \mathcal{P}_\kappa \delta : h(x) \leq f(\gamma)\} \in J^*$.

(4) Set $\eta = \text{cf } \lambda$, let $\{\lambda_\alpha \mid \alpha < \eta\}$ and $\{\delta_\alpha \mid \alpha < \eta\}$ be cofinal normal sequences of cardinals in λ and δ respectively such that $\lambda_0 \geq \delta$, and let $K_\alpha = [\lambda_\alpha, \lambda_{\alpha+1})$ and $L_\alpha = [\delta_\alpha, \delta_{\alpha+1})$ for each $\alpha < \eta$.

Define $f : \lambda \rightarrow \delta$ and $g : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \delta$ so that $f''K_\alpha = L_\alpha$ and $g(x) = f''x$. Then g is surjective and $J = g_*(I)$ is weakly normal if I is weakly normal. ■

For the existence of weakly normal ideals, we give another construction in 2.7 and 2.10.

2. $\lambda^{<\kappa}$ and the singular cardinal hypothesis. Solovay [14], using fine ultrafilters, proved that the size of $\mathcal{P}_\kappa \lambda$ is decided if κ is λ -compact. Here we show that the existence of weakly normal filters is enough to get his results in several cases; we also consider the singular cardinal hypothesis.

THEOREM 2.1. *If λ is regular and there is a weakly normal filter U on $\mathcal{P}_\kappa \lambda$, then $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$.*

We just follow Solovay's argument. For the reader's convenience, we present the complete proof.

A minor observation on weakly normal filters is needed.

LEMMA 2.2. $\{x : \text{cf}(\sup x) < \kappa\} \in U$ for every weakly normal filter U .

Proof. We only have to show that $\{x : \sup x \in x\}$ has U -measure 0. Then $\{x : x \text{ is cofinal in } \sup x\} \in U$ and the lemma is proved.

Suppose that $\{x : \sup x \in x\} \in U^+$. Since U is semi-weakly normal, there is a $\gamma < \lambda$ such that $\{x : \sup x \leq \gamma\} \in U^+$. Now $\{x : x \subset \gamma + 1\} \in I^+$, contrary to U being fine. ■

We define a filter D on λ by

$$X \in D \quad \text{iff} \quad X \subset \lambda \text{ and } \{x : \sup x \in X\} \in U.$$

LEMMA 2.3. D is a κ -complete weakly normal filter on λ and $\{\alpha : \text{cf } \alpha < \kappa\} \in D$.

Proof. It is clear that D is a κ -complete filter. For any $\alpha < \lambda$, $\{x : \sup x \geq \alpha\}$ is a member of U , hence $\{\beta : \beta \geq \alpha\}$ is in D . So D is uniform.

Suppose that $f : \lambda \rightarrow \lambda$ is regressive. Define $g : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ by $g(x) = f(\sup x)$. Then $g(x) < \sup x$ for every $x \in \mathcal{P}_\kappa\lambda$. Pick a $\gamma < \lambda$ such that $A = \{x : g(x) \leq \gamma\} \in U$. Then $B = \{\sup x : x \in A\} \in D$ and $f(\alpha) \leq \gamma$ for any $\alpha \in B$. This says that D is weakly normal.

By the previous lemma, $\{x : \text{cf}(\sup x) < \kappa\} \in U$. This obviously yields $\{\alpha < \lambda : \text{cf } \alpha < \kappa\} \in D$. ■

Let A_α be a cofinal subset of α whose cardinality is less than κ if $\text{cf } \alpha < \kappa$, and $A_\alpha = 0$ otherwise.

Since D is uniform, $X_\eta = \{\alpha : A_\alpha - (\eta+1) \neq \emptyset\} \in D$ for every $\eta < \lambda$. By the weak normality of D , there is an $\eta' < \lambda$ such that $\{\alpha : A_\alpha \cap [\eta, \eta') \neq \emptyset\} \in D$. With this in mind, we can define inductively a sequence $\{\eta_\xi \mid \xi < \lambda\} \subset \lambda$ as follows:

$$\begin{aligned} \eta_0 &= 0, \\ \eta_\xi &= \sup\{\eta_\beta : \beta < \xi\} \quad \text{for } \xi \text{ a limit ordinal,} \\ \eta_{\xi+1} &\text{ is chosen so that } \{\alpha : A_\alpha \cap [\eta_\xi, \eta_{\xi+1}) \neq \emptyset\} \in D. \end{aligned}$$

Let $I_\xi = [\eta_\xi, \eta_{\xi+1})$ and $\mathcal{M}_\alpha = \{\xi < \lambda : I_\xi \cap A_\alpha \neq \emptyset\}$. Since I_ξ 's are disjoint and $|A_\alpha| < \kappa$, we have $|\mathcal{M}_\alpha| < \kappa$ for every $\alpha < \lambda$. Moreover, for each $\xi < \lambda$, $\{\alpha : A_\alpha \cap I_\xi \neq \emptyset\} = \{\alpha : \xi \in \mathcal{M}_\alpha\} \in D$.

Let $\{x_\zeta : \zeta < \delta\}$ enumerate $x \in \mathcal{P}_\kappa\lambda$. Since D is κ -complete and $|\delta| < \kappa$ and $\{\alpha : x_\zeta \in \mathcal{M}_\alpha\} \in D$ for all $\zeta < \delta$, we have $\{\alpha : x \subset \mathcal{M}_\alpha\} \in D$. Hence $\mathcal{P}_\kappa\lambda = \bigcup\{\mathcal{P}(\mathcal{M}_\alpha) : \alpha < \lambda\}$. Now we have got $\lambda^{<\kappa} = |\mathcal{P}_\kappa\lambda| = \lambda \cdot 2^{<\kappa}$. The proof of Theorem 2.1 is complete. ■

Thus, as seen in [9], the following seems to be the most natural generalization of Solovay's theorem: if λ is regular and there is a precipitous λ -saturated ideal on $\mathcal{P}_\kappa\lambda$ then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.

COROLLARY 2.4. *If $\mathcal{P}_\kappa\lambda$ carries a λ -saturated normal ideal with $\text{cf } \lambda \geq \kappa$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.*

Proof. In case λ is regular, we can use the above theorem and Corollary 1.5. Suppose that $\text{cf } \lambda = \delta$, $\kappa \leq \delta < \lambda$, and I is a normal λ -saturated ideal on $\mathcal{P}_\kappa\lambda$. Then I is in fact η -saturated for some regular cardinal $\eta < \lambda$. For each regular cardinal ϱ between η and λ , I_ϱ is also normal η -saturated, hence weakly normal by 1.5. So, $\varrho^{<\kappa} = 2^{<\kappa} \cdot \varrho$.

Since $\lambda^{<\kappa} = \sup\{\varrho^{<\kappa} : \varrho \text{ is a regular cardinal } < \lambda\}$, we get $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$. ■

By a similar argument, we get

PROPOSITION 2.5. *If $\lambda > \text{cf } \lambda = \kappa$ and there is a weakly normal ideal on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.*

Proof. Let I be a weakly normal ideal on $\mathcal{P}_\kappa\lambda$. Then I is κ -saturated and I_ϱ is also κ -saturated for all regular ϱ between κ and λ . Hence we can find a weakly normal ideal on $\mathcal{P}_{\kappa\varrho}$ by Corollary 1.8, and $\varrho^{<\kappa} = 2^{<\kappa} \cdot \varrho$. ■

COROLLARY 2.6. *If there is a κ^+ -saturated ideal on $\mathcal{P}_\kappa\lambda$ and λ is a limit cardinal with $\text{cf } \lambda \geq \kappa$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.*

The assumption of normality in 2.4 may be necessary to produce weakly normal ideals on $\mathcal{P}_{\kappa\varrho}$ for $\varrho < \lambda$. The author was not able to get a weakly normal ideal on $\mathcal{P}_{\kappa\varrho}$ from one on $\mathcal{P}_\kappa\lambda$ although we have a weakly normal ideal on \mathcal{P}_κ of λ as seen in 1.9.

If κ is λ -compact and $\text{cf } \lambda < \kappa$, Solovay's theorem says that $\lambda^{<\kappa} = \lambda^+$. We propose a generalization with a somewhat complicated proof. Note that it is easier when κ is inaccessible.

THEOREM 2.7. *Assume that $\mathcal{P}_\kappa\lambda$ bears a weakly normal filter and $\text{cf } \lambda < \kappa$. Then $\lambda^{<\kappa} = (\lambda^+)^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.*

Proof. Without loss of generality we may assume that $2^{<\kappa} < \lambda$.

Note that $\lambda^{<\kappa} \geq \lambda^+$. Let $\{x_\alpha : \alpha < \lambda^{<\kappa}\}$ be an enumeration of $\mathcal{P}_\kappa\lambda$ and U a weakly normal filter on $\mathcal{P}_\kappa\lambda$. For each $x \in \mathcal{P}_\kappa\lambda$ we define $f(x) = \{\alpha < \lambda^+ : x_\alpha \subset x\}$. Thus, $f(x) \subset \lambda^+$ and $|f(x)| \leq |\mathcal{P}(x)| = 2^{|x|}$. Let δ be the least cardinal such that $2^\alpha < \delta$ for every $\alpha < \kappa$. Since $\text{cf } \lambda < \kappa \leq \text{cf } \delta$ and we have assumed that $2^{<\kappa} < \lambda$, we obtain $\delta < \lambda$.

Now f is a function from $\mathcal{P}_\kappa\lambda$ into $\mathcal{P}_\delta\lambda^+$. In the following, we also use $\text{cf } \delta \geq \kappa$.

Let F be defined by

$$X \in F \quad \text{iff} \quad X \subset \mathcal{P}_\delta\lambda^+ \text{ and } f^{-1}(X) \in U.$$

LEMMA 2.8. *F is a κ -complete filter with the following properties:*

- (i) $\{x : \alpha \in x\} \in F$ for all $\alpha < \lambda^+$.
- (ii) F is $\text{cf } \lambda$ -saturated.

Proof. (i) For every $\alpha < \lambda^+$, $\{x \in \mathcal{P}_\kappa\lambda : x_\alpha \subset x\} \in U$, and $\alpha \in f(x)$ if $x_\alpha \subset x$.

(ii) is clear since U is $\text{cf } \lambda$ -saturated. ■

Now we apply Theorem 1.7. Since F is a κ -complete κ -saturated fine filter on $\mathcal{P}_\delta\lambda^+$, F is precipitous. We have a κ -complete weakly normal ideal I on $\mathcal{P}_\delta\lambda^+$ and a κ -complete uniform weakly normal filter D on λ^+ such that $\{\alpha < \lambda^+ : \text{cf } \alpha < \delta\} \in D$ as in the proof of Theorem 2.1. Then we get $\{M_\alpha : \alpha < \lambda^+\}$ such that $|M_\alpha| < \delta$ for all $\alpha < \lambda^+$, and $\mathcal{P}_\kappa\lambda^+ = \bigcup \{\mathcal{P}_\kappa(M_\alpha) : \alpha < \lambda^+\}$.

Hence $(\lambda^+)^{<\kappa} \leq \lambda^+ \cdot \delta^{<\kappa}$.

LEMMA 2.9. $\delta^{<\kappa} = \delta$.

Proof. $\delta = 2^{<\kappa}$ or $(2^{<\kappa})^+$. If $\delta = 2^{<\kappa}$, then $\delta^{<\kappa} = (2^{<\kappa})^{<\kappa} = 2^{<\kappa} = \delta$. Otherwise

$$\delta^{<\kappa} = ((2^{<\kappa})^+)^{<\kappa} = (2^{<\kappa})^{<\kappa} \cdot (2^{<\kappa})^+ = 2^{<\kappa} \cdot (2^{<\kappa})^+ = (2^{<\kappa})^+ = \delta. \blacksquare$$

Now $\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \delta^{<\kappa} \cdot \lambda^+ = \delta \cdot \lambda^+ = \lambda^+$. The proof of Theorem 2.7 is complete. \blacksquare

OPEN QUESTION. Can one compute $\lambda^{<\kappa}$ if $\kappa < \text{cf } \lambda < \lambda$ and there exists a weakly normal filter on $\mathcal{P}_\kappa \lambda$?

Finally, we consider, normal λ^+ -saturated ideals. Before stating the theorem, we need a definition and a lemma.

DEFINITION. Let $\kappa \leq \mu \leq \nu$. An ideal I on $\mathcal{P}_\kappa \nu$ is μ -normal if I is closed under diagonal unions of $< \mu$ -sequences, that is, if $\{X_\alpha : \alpha < \eta < \mu\} \subset I$, then $\nabla\{X_\alpha : \alpha < \eta\} = \{x \in \mathcal{P}_\kappa \nu : \exists \alpha \in x (x \in X_\alpha)\} \in I$.

Let $\eta(\mu)$ be the least cardinal $\geq \mu$. (μ is not necessarily a cardinal.)

LEMMA 2.10. Assume that $\kappa \leq \text{cf } \delta \leq \delta \leq \text{cf } \nu$ and $\eta(\mu) \leq \text{cf } \nu$. Every κ -complete, fine, $\eta(\mu)$ -saturated, μ -normal ideal on $\mathcal{P}_\delta \nu$ is precipitous. Hence, if such an ideal exists, then there is a weakly normal ideal on $\mathcal{P}_\delta \nu$.

Proof. It suffices to show such an ideal I has the disjointing property. Let $\{X_\alpha : \alpha < \gamma\}$ be an almost disjoint family. We may assume $\gamma \leq \mu$ and $X_\alpha \subset \widehat{\{\alpha\}}$ for all $\alpha < \gamma$. Set $Y_\alpha = X_\alpha - \nabla\{X_\xi \cap X_\alpha : \xi < \alpha\}$. Since $\alpha < \mu$ and I is μ -normal, Y_α is also in I^+ . It is routine to show that $\{Y_\alpha : \alpha < \gamma\}$ is a pairwise disjoint family and $(Y_\alpha - X_\alpha) \cup (X_\alpha - Y_\alpha) \in I$. \blacksquare

LEMMA 2.11. Suppose that $\lambda < \lambda^{<\kappa}$ and δ is the least cardinal such that $\delta > 2^\alpha$ for all $\alpha < \kappa$. If there is a normal λ^+ -saturated ideal on $\mathcal{P}_\kappa \lambda$, then there is a κ -complete, $(\lambda + 1)$ -normal, λ^+ -saturated, fine ideal on $\mathcal{P}_\delta \lambda^+$.

Proof. Let $\{x_\alpha : \lambda \leq \alpha < \lambda^{<\kappa}\}$ be an enumeration of $\mathcal{P}_\kappa \lambda$ and $f(x) = x \cup \{\alpha < \lambda^+ : x_\alpha \subset x\}$ for $x \in \mathcal{P}_\kappa \lambda$. By our assumption $|f(x)| \leq 2^{|x|} < \delta$. Hence $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\delta \lambda^+$.

Suppose that I is a normal λ^+ -saturated ideal on $\mathcal{P}_\kappa \lambda$ and define J by $X \in J$ iff $X \subset \mathcal{P}_\delta \lambda^+$ and $f^{-1}(X) \in I$. Since $f^{-1}(\mathcal{P}_\delta \lambda^+) = \mathcal{P}_\kappa \lambda$, J is a proper κ -complete λ^+ -saturated ideal on $\mathcal{P}_\delta \lambda^+$.

If $\alpha < \lambda$, then $\{x \in \mathcal{P}_\kappa \lambda : \alpha \in x\} \in I^*$. For $\lambda \leq \alpha < \lambda^+$, $\{x : x_\alpha \subset x\}$ is also in I^* . Hence $\{x : \alpha \in f(x)\} \in I^*$ for all $\alpha < \lambda^+$, which shows J is fine.

Suppose that $\{X_\alpha : \alpha < \lambda\} \subset J$ and $X = \nabla\{X_\alpha : \alpha < \lambda\}$. Then

$$\begin{aligned} f^{-1}(X) &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in f(x)\} \\ &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in f(x) \cap \lambda\} \\ &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in x\} = \nabla\{f^{-1}(X_\alpha) : \alpha < \lambda\} \in I. \end{aligned}$$

Thus J is $(\lambda + 1)$ -normal. \blacksquare

THEOREM 2.12. *If $\mathcal{P}_\kappa\lambda$ carries a normal λ^+ -saturated ideal, then $\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.*

Proof. Without loss of generality, we may assume that $2^{<\kappa} < \lambda < \lambda^{<\kappa}$. By Lemma 2.11, there is a λ^+ -saturated κ -complete $(\lambda+1)$ -normal ideal on $\mathcal{P}_\delta\lambda^+$ with $\kappa \leq \text{cf } \delta \leq \delta \leq \lambda^+$. Using Lemma 2.10, we conclude that there is a κ -complete weakly normal ideal on $\mathcal{P}_\delta\lambda^+$.

Note that $\delta^{<\kappa} = \delta$. As an easy application of the $\mathcal{P}_\kappa\lambda^+$ case, we have $(\lambda^+)^{<\kappa} = \lambda^+ \cdot \delta^{<\kappa} = \lambda^+$. Hence $\lambda^{<\kappa} \leq 2^{<\kappa} \cdot \lambda^+$. ■

COROLLARY 2.13. *If $\text{cf } \lambda < \kappa$ and there is a normal λ -saturated ideal on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.*

Note that Matsubara [8] already proved a somewhat stronger form of our theorem: If $\mathcal{P}_\kappa\lambda$ carries a normal λ^+ -saturated ideal and GCH holds below κ , then $2^\lambda = \lambda^+$. Furthermore, if this ideal is λ -saturated, then $2^{<\lambda} = \lambda$.

With these results, we consider the singular cardinal hypothesis (SCH): if $2^{\text{cf } \tau} < \tau$, then $\tau^{\text{cf } \tau} = \tau^+$.

Solovay's result [14] is: SCH holds above a strongly compact cardinal. We prove SCH holds in some interval under the existence of weakly normal ideals on $\mathcal{P}_\kappa\lambda$.

THEOREM 2.14. (i) *If $\mathcal{P}_\kappa\lambda$ carries a normal η -saturated ideal and $\eta < \lambda$, then SCH holds between $2^{<\kappa} \cdot \eta$ and λ .*

(ii) *If there is a κ^+ -saturated ideal on $\mathcal{P}_\kappa\lambda$, then SCH holds between $2^{<\kappa}$ and λ .*

(iii) *If $\text{cf } \lambda \leq \kappa$ and there exists a weakly normal ideal on $\mathcal{P}_\kappa\lambda$, then SCH holds between $2^{<\kappa}$ and λ .*

Proof. In any case, by Silver's results [12], we only have to know that $\delta^{<\kappa} = \delta$ for every regular δ in each interval.

(i) As we have already seen in Corollary 2.4, there is a normal δ -saturated ideal on $\mathcal{P}_\kappa\delta$. So, Theorem 2.1 and Corollary 1.5 work.

(ii) $\mathcal{P}_\kappa\delta$ also carries a κ^+ -saturated ideal, and hence, by Corollary 1.8, a weakly normal ideal as well.

(iii) Here, every weakly normal ideal is $\text{cf } \lambda$ -saturated and $\text{cf } \lambda \leq \kappa$. Thus, this is contained in (ii). ■

Remark. It can also be shown that the combinatorial principle E_λ^η fails for every regular $\eta < \kappa$ if there is a weakly normal filter on $\mathcal{P}_\kappa\lambda$ and λ is regular.

Another weakening of strong compactness which implies the failure of E_λ^η has been found in Johnson [6].

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Received 5 November 1991