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Lindelöf property and the iterated continuous function spaces

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Abstract. We give an example of a compact space X whose iterated continuous function spaces $C_{\rm p}(X)$, $C_{\rm p}C_{\rm p}(X),\ldots$ are Lindelöf, but X is not a Corson compactum. This solves a problem of Gul'ko (Problem 1052 in [11]). We also provide a theorem concerning the Lindelöf property in the function spaces $C_{\rm p}(X)$ on compact scattered spaces with the ω_1th derived set empty, improving some earlier results of Pol [12] in this direction.

1. Notation and terminology. Our terminology follows Arkhangel'skiĭ [1]. Given a topological space X, we denote by $C_{\rm p}(X)$ the space of real-valued continuous functions, equipped with the topology of pointwise convergence, and $C_{\rm p}C_{\rm p}(X), C_{\rm p}C_{\rm p}C_{\rm p}(X),\ldots$ are the iterated continuous function spaces.

We denote by $\mathcal{D}$ the discrete two-point space $\{0,1\}$ and $C_{\rm p}(X,\mathcal{D}) = \{f\in C_{\rm p}(X) : f(X)\subset \mathcal{D}\}$.

We denote by ω_1 the set of all countable ordinals.

A set $A\subset\omega_1$ is *stationary* if it intersects each closed set (in the order topology), unbounded in ω_1; we call A *bistationary* if both A and $\omega_1\setminus A$ are stationary. For information concerning stationary sets needed in this paper we refer the reader to Jech [7], or Fleissner [3].

A topological space X is $\aleph_0$-*monolithic* if for any countable subset $A\subset X$ the closure $\mathrm{cl}(A)$ has a countable network (see [1; Ch. II, §6]); a *network* for a space Y is a family of sets such that each open set is the union of some subfamily of this family.

A topological space X has *countable tightness* if for any x in $\mathrm{cl}(A)$ there is an at most countable subset $B\subset A$ with $x\in \mathrm{cl}(B)$ (see [1]).

A compact space X is a *Corson compactum* if X can be embedded in the subspace of the Tikhonov product $\mathbb{R}^{\Gamma}$ of the real line consisting of functions vanishing at all but countably many points in Γ (see [1], [9]).

2. Main results. Gul'ko proved in [5] (a more detailed exposition in [6]) that for any weakly compact set X in a Banach space (i.e. for any Eberlein compactum [1], [9]) the iterated function spaces $C_{\rm p}(X), C_{\rm p}C_{\rm p}(X), \ldots$ are Lindelöf. The author extended this result in [16] to the class of Corson compacta (essentially wider than the class of Eberlein compacta, see [1], [9]). Subsequently, Gul'ko conjectured that the Lindelöf property of all iterated continuous function spaces actually characterizes the class of Corson compacta (Problem 1052 in [11]).

We disprove this conjecture.

2.1. Theorem. *There exists a compact space X (with the third derived set empty) such that all iterated continuous function spaces $C_{\rm p}(X)$, $C_{\rm p}C_{\rm p}(X), \ldots$ are Lindelöf, but X is not a Corson compactum.*

Our space X is a compactum associated in a standard way with some special "ladder system" on the countable ordinals (see Section 4 for details). Similar function spaces were investigated earlier by Pol [13] and Ciesielski and Pol [2].

More precisely, we consider a class $\mathcal{S}$ of Lindelöf spaces, stable under the $C_{\rm p}$-operation, and we characterize those ladder systems on the countable ordinals which provide compacta X with $X \in \mathcal{S}$.

In the definition of $\mathcal{S}$ we follow the author's paper [16], where more general classes of spaces (extending some classes defined by Gul'ko in [5], [6]) were considered. We discuss the class $\mathcal{S}$ in Section 3.

2.2. Remark. Some closely related questions concerning the iterated function spaces $C_{\rm p}(X), C_{\rm p}C_{\rm p}(X), \ldots$ were investigated by Sipachova [15] and Okunev [10]. In particular, Sipachova proved that if X is an Eberlein compactum, then each of $C_{\rm p}(X), C_{\rm p}C_{\rm p}(X), \ldots$ is a Lindelöf Σ-space ($\equiv$ countably determined space, see [9]), and Okunev extended this result to the class of Gul'ko compacta (see definition in [9]). The result of Okunev is definitive in this direction: Gul'ko compacta X are characterized by the property that $C_{\rm p}(X)$ is a Lindelöf Σ-space.

Our next result provides information about the Lindelöf property in the function spaces $C_{\rm p}(X)$ on compacta whose ω_1th derived set is empty. Some results in this direction were obtained by Pol [12].

Before stating the result let us recall that, given a topological space E, the *G_δ-modification* of E is the set E equipped with the topology generated by all countable intersections of open sets in E.

2.3. Theorem. *Let X be a compact scattered space with the ω_1th derived set empty. Then the following properties are equivalent.*

(a) $C_{\rm p}(X)$ *is Lindelöf*,

(b) *the G_δ-modification of $C_{\rm p}(X, \mathcal{D})$ is Lindelöf*,

(c) *X is $\aleph_0$-monolithic.*

Let us recall that X being scattered, the weak topology on the function space coincides on norm-bounded sets with the pointwise topology (see [14]). Hence, in Theorem 2.3 we could as well consider the Banach function space $C(X)$ equipped with the weak topology.

2.4. Remark. Assuming Martin's Axiom and the negation of the Continuum Hypothesis, the implication (a)⇒(c) was proved by Reznichenko for arbitrary compact Hausdorff spaces (see [17], [1; Ch. IV, §8]).

3. A stable class $\mathcal{S}$ of Lindelöf spaces. In [16], for any infinite cardinal τ, the author introduced two classes of topological spaces, $\mathcal{D}(\tau)$ and $\mathcal{L}(\tau)$, which are in duality with respect to the C_{p}-operation. The classes $\mathcal{D}(\tau)$ and $\mathcal{L}(\tau)$ extended some earlier constructions of Gul'ko [5], [6], and the ideas developed by Gul'ko in these papers were basic in investigation of properties of the classes.

The class $\mathcal{S}$ we consider is contained in the intersection $\mathcal{D}(\aleph_0) \cap \mathcal{L}(\aleph_0)$. The properties of $\mathcal{S}$ we need could be derived from the general results in [16]. However, in the case of $\mathcal{S}$, the proofs can be made more direct and clear, and we decided to include them for the reader's convenience.

3.1. Definition. The class $\mathcal{S}$ consists of all topological spaces X satisfying the following two conditions:

(a) for every $n \in \mathbb{N}$, the space X^n is Lindelöf and has countable tightness,

(b) if $F_n \subset X^n$ for $n = 1, 2, \ldots$ is a sequence of closed subsets, then there exists a mapping $r : X \to X$ such that $r(X)$ has a countable network and $(r \times \ldots \times r)(F_n) \subset F_n$ for every $n \in \mathbb{N}$.

3.2. Theorem. *If $X \in \mathcal{S}$ then $C_{\mathrm{p}}(X) \in \mathcal{S}$.*

Proof. Let $X_m = X \oplus \ldots \oplus X$ be the discrete union of m copies of X.

Claim 1. *If $F_{nm} \subset (X_m)^n$ are closed sets, $n, m \in \mathbb{N}$, then there exists a mapping $r : X \to X$ such that $r(X)$ has a countable network and $(r_m \times \ldots \ldots \times r_m)(F_{nm}) \subset F_{nm}$, where the $r_m : X_m \to X_m$ are induced by r.*

Indeed, consider $(X_m)^n$ as a finite discrete union of clopen subspaces X_{nmk}, each homeomorphic to X^n, and set $F_{nmk} = F_{nm} \cap X_{nmk}$. Let $\pi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\pi(n, m, k) \geq n$ for every n, m, k in $\mathbb{N}$. Then we can consider each F_{nmk} as a closed subset of $X^{\pi(n,m,k)}$. Now Claim 1 follows from condition (b) of Definition 3.1.

For any $x = (x_1, \ldots, x_n) \in X^n$ and an n-tuple $I = (I_1, \ldots, I_n)$ of open intervals in the real line with rational ends, let

$$W(x, I) = \{f \in C_{\mathrm{p}}(X) : f(x_i) \in I_i \text{ for } i = 1, \ldots, n\}.$$

If $f \in C_{\mathrm{p}}(X)$, $x \in \mathrm{cl}(A)$ and $f(x_i) \in I_i$ for every i then there exists $y \in A$ with $f(y_i) \in I_i$ for all i and therefore we have

Claim 2. *Let $n \in \mathbb{N}$, $A \subset X^n$ and I be a fixed n-tuple as above. Then $W(x, I) \subset \bigcup\{W(y, I) : y \in A\}$ for any $x \in \mathrm{cl}(A)$.*

Given $n \in \mathbb{N}$, denote by $\mathcal{B}_n$ the countable family of all n-tuples $I = (I_1, \ldots, I_n)$ of open intervals in the real line with rational ends.

We pass to the proof that $C_{\mathrm{p}}(X)$ meets the conditions of Definition 3.1.

Let $H_m \subset (C_{\mathrm{p}}(X))^m$ be a sequence of closed subsets. Observe that $(C_{\mathrm{p}}(X))^m$ can be identified with $C_{\mathrm{p}}(X_m)$, where X_m was defined at the beginning of the proof. For any n, m in $\mathbb{N}$ and $I \in \mathcal{B}_n$ denote by $\mathcal{U}_{nm}(I)$ the family of all sets $W(x, I)$ with $x \in (X_m)^n$, disjoint from H_n. Let

$$F_{nm}(I) = \{x \in (X_m)^n : W(x, I) \in \mathcal{U}_{nm}(I)\}.$$

By Claim 2, each $F_{nm}(I)$ is closed in $(X_m)^n$. Therefore Claim 1 provides us with a mapping $r : X \to X$ such that $r(X)$ has a countable network and $(r_m \times \ldots \times r_m)(F_{nm}(I)) \subset F_{nm}(I)$ for any n, m in $\mathbb{N}$ and $I \in \mathcal{B}_n$.

Let $r^* : C_{\mathrm{p}}(X) \to C_{\mathrm{p}}(X)$ be the dual mapping $f \to r^*(f) = f \circ r$, for $f \in C_{\mathrm{p}}(X)$. Notice that $r^* \times \ldots \times r^* = r^*_m$.

We show that $r^*_m(H_m) \subset H_m$ for every $m \in \mathbb{N}$. Suppose on the contrary that there exists $f \in H_m$ with $r^*_m(f) \notin H_m$. Then there exists an open set U in $C_{\mathrm{p}}(X_m)$ containing $r^*_m(f)$ and disjoint from H_m. We can assume that $U = W(x, I)$ for some $n \in \mathbb{N}$, $x \in (X_m)^n$ and $I \in \mathcal{B}_n$. Since $W(x, I) \cap H_m = \emptyset$, we have $x \in F_{nm}(I)$, and therefore $(r_m \times \ldots \times r_m)(x) \in F_{nm}(I)$. It follows that $W((r_m \times \ldots \times r_m)(x), I) \cap H_m = \emptyset$. But $r^*_m(f) \in W(x, I)$ if, and only if, $f \in W((r_m \times \ldots \times r_m)(x), I)$. Therefore $f \notin H_m$, contradicting our choice of f. This proves that $r^*_m(H_m) \subset H_m$.

Since $C_{\mathrm{p}}(r_m(X))$ has a countable network (because $r_m(X)$ has one, see [1]) and we can identify $r^*_m(C_{\mathrm{p}}(X))$ with $C_{\mathrm{p}}(r_m(X))$, we have verified condition (b) of Definition 3.1 for $C_{\mathrm{p}}(X)$.

We now check (a), i.e. we prove that each finite product $(C_{\mathrm{p}}(X))^m = C_{\mathrm{p}}(X_m)$ is Lindelöf.

Fix $m \in \mathbb{N}$, and let $\mathcal{U}$ be an open cover of $C_{\mathrm{p}}(X_m)$ consisting of basic neighbourhoods $W(x, I)$ defined above. For each n, and $I \in \mathcal{B}_n$, let

$$\mathcal{U}_n(I) = \{W(x, I) : x \in (X_m)^n \text{ and } W(x, I) \in \mathcal{U}\},$$
$$A_n(I) = \{x \in (X_m)^n : W(x, I) \in \mathcal{U}_n(I)\}.$$

Applying Claim 1 to $\mathrm{cl}(A_n(I))$ we get a mapping $r : X \to X$ whose image $r(X)$ has a countable network and $S_n(I) = (r_m \times \ldots \times r_m)(\mathrm{cl}(A_n(I))) \subset \mathrm{cl}(A_n(I))$ for each n. The sets $S_n(I)$ have countable networks, and therefore there are countable sets $B_n(I)$ dense in $S_n(I)$. Since $(X_m)^n$ has countable tightness, each $x \in B_n(I)$ is in the closure of a countable subset of $A_n(I)$, and therefore, by Claim 2, each $W(x,I)$ with $x \in B_n(I)$ is covered by a countable subfamily of $\mathcal{U}_n(I)$.

Therefore, to complete the proof we verify that the countable collection

$$\mathcal{W} = \{W(x,I) : x \in B_n(I),\ n \in \mathbb{N},\ I \in B_n\}$$

covers $C_{\rm p}(X_m)$.

Let $f \in C_{\rm p}(X_m)$. There exists $W(x,I) \in \mathcal{U}$ with $r_m^*(f) \in W(x,I)$, i.e. $f \in W((r_m \times \ldots \times r_m)(x), I)$. There exists n such that $x \in A_n(I)$ and then $(r_m \times \ldots \times r_m)(x) \in S_n(I)$. Since $B_n(I)$ is dense in $S_n(I)$, applying Claim 2 again, we conclude that $W((r_m \times \ldots \times r_m)(x), I)$ is covered by elements $W(y,I)$ with $y \in B_n(I)$, and therefore f is in the union of the family $\mathcal{W}$.

Finally, the countable tightness of the finite powers $(C_{\rm p}(X_m))^n$ follows from the Arkhangel'skiĭ–Pytkeev theorem [1; Ch. II, §1], since all finite powers $(X_m)^n$ are Lindelöf.

3.3. Remark. It was proved in [16] that all Corson compacta are in $\mathcal{S}$. In particular, all iterated function spaces $C_{\rm p}(X), C_{\rm p}C_{\rm p}(X), \ldots$ for X a Corson compactum are Lindelöf.

This is closely related to the well-known results of Gul'ko [1], [4–6] concerning retraction systems on Corson compacta.

4. The compacta X_A associated with ladder systems on ω_1 and the proof of Theorem 2.1. Given a countable limit ordinal α, a *ladder* on α is a set $S_\alpha = \{\alpha(1), \alpha(2), \ldots\}$ of isolated ordinals from α such that $\alpha(1) < \alpha(2) < \ldots$ and $\alpha = \sup_i \alpha(i)$. Let A be a set of countable limit ordinals. An *A-ladder system* on ω_1 is a collection $\langle S_\alpha : \alpha \in A\rangle$, where S_α is a ladder on α. With each A-ladder system on ω_1 we associate a compact space X_A in the following standard way: we give the set ω_1 a locally compact topology by making the points in $\omega_1 \setminus A$ isolated and taking as a base of neighbourhoods of a point $\alpha \in A$ the sets $\{\alpha\} \cup (S_\alpha \setminus F)$, where F is a finite set, and we let X_A be the one-point compactification of this space, ω_1 being the point at infinity.

4.1. Proposition. *X_A is a Corson compactum if and only if A is a nonstationary set in ω_1.*

Proof. Suppose A is a nonstationary set in ω_1. Then there exists a closed (in the order topology) unbounded set B in ω_1 with $A \cap B = \emptyset$. Without loss of generality we may assume that $1 \in B$. Then for every $\alpha \in \omega_1$ there exists $\beta(\alpha) \in B$ such that $\beta(\alpha) \le \alpha < \beta(\alpha)^+$, where $\beta(\alpha)^+$ is the ordinal in B next to $\beta(\alpha)$.

Let $\alpha \in A$ and S_α be a ladder on α. For any $m \in \mathbb{N}$ set $S_\alpha^m = \{\alpha(n) : n \ge m\}$. Define

$$\mathcal{U} = \{\{\alpha\} : \alpha \notin A\} \cup \{(\beta(\alpha), \beta(\alpha)^+) \cap S_\alpha^m : \alpha \in A,\ m \in \mathbb{N}\}\,.$$

It is easy to check that $\mathcal{U}$ is a point-countable family of clopen sets such that for any pair of distinct points in X_A there is a set in $\mathcal{U}$ containing exactly one of them. By a Rosenthal-type characterization theorem [1], [9] we conclude that X_A is a Corson compactum.

Now let A be a stationary set in ω_1. Assume on the contrary that X_A is a Corson compactum and therefore it has a point-countable family $\mathcal{U}$ consisting of open F_σ-sets such that given two points in X, some element in $\mathcal{U}$ contains exactly one of them. For every $\alpha \in A$ fix $U_\alpha \in \mathcal{U}$ such that $\alpha \in U_\alpha$. We may assume that the sets U_α are distinct for distinct ordinals α. For each $\alpha \in A$ there exists $h(\alpha) \in U_\alpha$ such that $h(\alpha) < \alpha$, since α is nonisolated in X_A. By the pressing-down lemma we have $h(\alpha) = \alpha_0$ for all α in a stationary subset A' of A. The point α_0 belongs to every U_α with $\alpha \in A'$, hence $\mathcal{U}$ is not point-countable.

4.2. Proposition. *X_A is in the class $\mathcal{S}$ if and only if $\omega_1 \setminus A$ is a stationary set in ω_1.*

Proof. Since X_A is a compactum with countable tightness each finite power of X_A has countable tightness (by Malykhin's theorem [8]) and the Lindelöf property. Hence we need only verify condition (b) of Definition 3.1 for X_A.

Let $\omega_1 \setminus A$ be a stationary set in ω_1 and $F_n \subset (X_A)^n$ be a sequence of closed subsets.

Let $L(\omega_1)$ be the set $\omega_1 \cup \{\omega_1\}$ with the topology where all points in ω_1 are isolated and the neighbourhoods of ω_1 contain all but countably many points in $L(\omega_1)$. The space $L(\omega_1)$ is the G_δ-modification of X_A, i.e. open sets in $L(\omega_1)$ are countable intersections of open sets in X_A. In particular, each F_n is closed in $L(\omega_1)^n$.

The mapping $r_\beta : L(\omega_1) \to L(\omega_1)$ defined by

$$r_\beta(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \beta\,, \\ \omega_1 & \text{otherwise,} \end{cases}$$

is a continuous retraction. By Gul'ko's theorem [1; IV.3.12] the set

$$C_n = \{\beta \in \omega_1 : (r_\beta \times \ldots \times r_\beta)(F_n) \subset F_n\}$$

is closed and unbounded in ω_1 for every $n \in \mathbb{N}$. Hence there exists $\beta \in (\bigcap_n C_n) \setminus A$, $\omega_1 \setminus A$ being stationary. Notice that the same r_β considered as a mapping from X_A into itself is continuous if, and only if, $\beta \notin A$. Furthermore, the image of r_β is countable, so $r = r_\beta$ satisfies 3.1(b). This proves the "if" part of the proposition.

To check the reverse implication, assume that there exists a closed unbounded set C in ω_1 contained in A. Without loss of generality we can assume that $1 \in C$.

For any α, let α^+ be the successor of α in C and let $\beta_1(\alpha), \beta_2(\alpha), \ldots$ enumerate the points in the interval $[\alpha, \alpha^+)$.

Define a sequence of closed subsets of X^2 by

$$F_0 = \{(1,1)\} \quad \text{and} \quad F_n = \mathrm{cl}(\{(\alpha, \beta_n(\alpha)) : \alpha \in C\}), \quad \text{for } n \in \mathbb{N}.$$

Since $C \subset A$, every infinite sequence in C converges in X_A to ω_1. It follows that $F_n \cap (\{\alpha\} \times X_A) = \{(\alpha, \beta_n(\alpha))\}$ for $\alpha \in C$.

Assume that $r : X_A \to X_A$ is continuous and $(r \times r)(F_n) \subset F_n$ for every $n = 1, 2, \ldots$ We show that r is the identity map. We have $r(1) = 1$ because $(r \times r)(F_0) \subset F_0$. Suppose that $r(\gamma) = \gamma$ for all $\gamma < \beta$. If $\beta \in C$ then β is an accumulation point of $[1, \beta)$, and therefore $r(\beta) = \beta$. Let $\beta \notin C$. Then $\beta = \beta_n(\alpha)$ for some $\alpha \in C$ and $n \in \mathbb{N}$. Observe that $\alpha < \beta$ and $r(\alpha) = \alpha$. Hence $(r \times r)(F_n) \subset F_n$ implies $r(\beta_n(\alpha)) = \beta_n(\alpha)$. This completes the inductive proof of the "only if" part of the proposition.

4.3. COROLLARY. *If A is a bistationary set in ω_1 then all iterated function spaces $C_{\mathrm{p}}(X_A)$, $C_{\mathrm{p}}C_{\mathrm{p}}(X_A), \ldots$ are Lindelöf, but X_A is not a Corson compactum.*

5. Proof of Theorem 2.3. For any Y closed in X the restriction operator $R(f) = f|_Y$ maps continuously $C_{\mathrm{p}}(X)$ onto $C_{\mathrm{p}}(Y)$; in particular, if $C_{\mathrm{p}}(X)$ is Lindelöf, so is $C_{\mathrm{p}}(Y)$. By a result of Pol [12], we infer that if $C_{\mathrm{p}}(X)$ is Lindelöf and Y is separable, then Y is metrizable. This gives the implication (a)$\Rightarrow$(c).

The implication (b)$\Rightarrow$(a) was proved in [13].

To show (c)$\Rightarrow$(b), let X be a compact $\aleph_0$-monolithic scattered space. Denote by $X^{(\alpha)}$ the αth derived set of X (see [14]). Since $X^{(\omega_1)} = \emptyset$, there is a countable ordinal α such that $X^{(\alpha)}$ is finite; we then call α the *height* of X. We can restrict ourselves to the case where $X^{(\alpha)}$ is a singleton $\{x^*\}$. We can also concentrate on the space

$$C^0_{\mathrm{p}}(X, \mathcal{D}) = \{f \in C_{\mathrm{p}}(X, \mathcal{D}) : f(x^*) = 0\}$$

instead of $C_{\mathrm{p}}(X, \mathcal{D})$, the latter being the union of two disjoint closed copies of the former.

Now observe that a base of the G_δ-modification of $C^0_{\mathrm{p}}(X, \mathcal{D})$ consists of the sets

$$U(A, \varphi) = \{f \in C^0(X, \mathcal{D}) : f|_A = \varphi\},$$

where A is a countable subset of X and $\varphi : A \to \mathcal{D}$ is a function. We prove that $C^0(X, \mathcal{D})$ is a Lindelöf space in the G_δ-topology by induction on the height of X.

The case $\alpha = 0$ is evident.

Let α be a limit ordinal. Then there exists a sequence $\alpha_n < \alpha$ converging to α, and we have $C^0(X,\mathcal{D}) = \bigcup_n E(\alpha_n)$, where

$$E(\alpha_n) = \{f \in C^0(X,\mathcal{D}) : f|_{X^{(\alpha)}} \equiv 0\}\,.$$

Let Z_n be the factor space obtained by collapsing the set $X^{(\alpha)}$ to a point z_n^*. Clearly, $E(\alpha_n)$ is homeomorphic to

$$C^0(Z_n,\mathcal{D}) = \{f \in C^0(Z,\mathcal{D}) : f(z_n^*) = 0\}\,.$$

Furthermore, $Z_n^{(\alpha)} = \emptyset$ and Z_n is $\aleph_0$-monolithic as a continuous image of the $\aleph_0$-monolithic space X. By the inductive assumption, each $E(\alpha_n)$, and therefore $C^0_{\mathrm{p}}(X,\mathcal{D})$, is Lindelöf.

Let $\alpha = \beta + 1$. The derived set $X^{(\beta)}$ is homeomorphic to the one-point compactification of some discrete space Γ. Write $X^{(\beta)} = \{x^*\} \cup \{x_\gamma : \gamma \in \Gamma\}$ and let

$$C_n = \{f \in C^0(X,\mathcal{D}) : |\{\gamma \in \Gamma : f(x_\gamma) = 1\}| \le n\}, \qquad n = 1, 2, \ldots,$$
$$E_\gamma = \{f \in C_1 : f(x_\gamma) = 1\}\,.$$

Obviously, $C^0(X,\mathcal{D}) = \bigcup_n C_n$, hence it is sufficient to prove that C_n is Lindelöf for every n. The case $n = 0$ is similar to the case just considered and by the inductive assumption E_γ is Lindelöf for every $\gamma \in \Gamma$. Next, note that C_n is the continuous image of $(C_1)^n$ under the mapping $(f_1,\ldots,f_n) \to \max(f_1,\ldots,f_n)$. So it remains to prove that C_1 is a Lindelöf space in the G_δ-topology.

Let $\mathcal{U}$ be an open cover of C_1. We have already noticed that the subspace C_0 of C_1 is Lindelöf; let $\mathcal{U}' = \{V_n : n \in \mathbb{N}\}$ be a subfamily of $\mathcal{U}$ covering C_0. We may assume that $V_n = U(A_n, \varphi_n)$ for $n \in \mathbb{N}$. As every E_γ is Lindelöf for $\gamma \in \Gamma$ we need only prove that $\bigcup \mathcal{U}'$ covers all E_γ except possibly for a countable set. Suppose on the contrary that there are $f_\gamma \in E_\gamma$ with $f_\gamma \notin \bigcup_n V_n$ for $\gamma \in \Gamma' \subset \Gamma$, Γ' being uncountable. Let $A = \bigcup_n A_n$ and let $\gamma \in \Gamma'$ be such that $x_\gamma \notin \mathrm{cl}(A)$. Then there is a clopen neighbourhood W of x_γ such that $W \cap \mathrm{cl}(A) = \emptyset$. Let $g_\gamma = f_\gamma \chi_{X\setminus W}$, where $\chi_{X\setminus W}$ is the characteristic function of the set $X \setminus W$. Since $g_\gamma \in C_0$, we have $g_\gamma \in V_n$ for some n. But by the assumption g_γ coincides with f_γ on A_n and hence $f_\gamma \in U(A_n, \varphi_n) = V_n$. This contradiction proves the claim.

6. Open problems. Is there a compact space X such that $C_{\mathrm{p}}(X)$ is Lindelöf but $C_{\mathrm{p}}C_{\mathrm{p}}(X)$ is not? The same question can be posed for X being scattered $\aleph_0$-monolithic.

The answer is unknown even for X_A from Section 4 with A an arbitrary set.

Gul'ko [4] proved that if X is the Σ-product of the real lines then $C_{\mathrm{p}}(X)$ is Lindelöf. The uncountable Σ-product X is not Lindelöf and can be embedded as a closed subspace in $C_p C_p(X)$. Therefore the compactness requirement in the above question is essential.

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