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On generalized Peano and Peano derivatives

by

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Abstract. A function F is said to have a generalized Peano derivative at x if F is continuous in a neighborhood of x and if there exists a positive integer q such that a qth primitive of F in the neighborhood has the $(q+n)$th Peano derivative at x; in this case the latter is called the generalized nth Peano derivative of F at x and denoted by $F_{[n]}(x)$. We show that generalized Peano derivatives belong to the class $[\Delta']$. Also we show that they are path derivatives with a nonporous system of paths satisfying the I.I.C. condition as defined in [3]. This gives a new approach to studying generalized Peano and Peano derivatives since all their known properties can be obtained from the corresponding properties of path derivatives. Moreover, generalized Peano derivatives are selective derivatives.

Introduction. There have been, in recent years, numerous articles showing that certain generalized derivatives, which can often substitute for the ordinary derivative when the latter is not known to exist, share many properties of the ordinary derivative. For example, the nth Peano derivative has the Darboux property, is in the first class of Baire, possesses the Denjoy property and is in Zahorski's classes M_2 and M_3 and in Weil's class Z. Each Peano derivative can be represented in the form $f = g' + hk'$ where g, h and k are differentiable; the restriction of such a function f to a nowhere dense set can be extended so as to be a derivative on all of $\mathbb{R}$. Furthermore, the $(n-1)$th Peano derivative shares many properties of ordinary primitives: it is of generalized absolute continuity $[ACG]$, is differentiable on a dense open set and is determined by its values on a dense set.

In order to explain behavior of generalized derivatives some authors were looking for a framework within which most of these derivatives can be expressed. One such task was done by four authors, Agronsky, Biskner, Bruckner and Mařík [1]. They introduced a class $[\Delta']$ of functions which behave "almost" like derivatives. For every $f \in [\Delta']$ there is a derivative g' such that $f - g' = hk'$ where h and k are differentiable. Since the product hk'

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is a derivative when we impose some additional restriction on h or k, we can better understand why functions from this class behave like derivatives.

The other attempt in trying to better understand the behavior of generalized derivatives was done by Bruckner, O'Malley and Thomson [3]. The perspective they chose is to consider those derivatives for which the derivative of a function f at a point x can be viewed as

$$\lim_{y\in E_x,\, y\to x} \frac{f(y)-f(x)}{y-x} \tag{1}$$

for appropriate choices of sets E_x. One generalized derivative, then, differs from another only in the choice of the family $\{E_x : x \in \mathbb{R}\}$ of sets through which the difference quotient passes to its limit. This framework includes any generalized derivative for which the derivative at a point is a derived number of the function at that point. The authors showed that much of the information concerning the behavior of a generalized derivative is contained in the geometry of the collection $E = \{E_x : x \in \mathbb{R}\}$. They imposed certain conditions on the sets E_x related to "thickness" of the E_x and the way they intersect. Those functions f to which such system E of sets can be assigned are said to be *path differentiable* and the limit in (1) is called a *path derivative.*

The main goal of this paper is to show that every nth generalized Peano derivative belongs to the class $[\Delta']$ and that it is the path derivative of the $(n-1)$th generalized Peano derivative with respect to a system E of paths satisfying some thickness and intersection conditions introduced in [3]. This will give a new approach to studying generalized Peano and Peano derivatives. We will be able to get all their known properties from the corresponding properties of path derivatives. As a consequence of this we show that the nth generalized Peano derivative is a selective derivative of the $(n-1)$th one. In particular, Peano derivatives are selective derivatives. (See Definition 51.)

Preliminaries. The definition of the kth derivative of a real-valued function is iterative in nature and thus easily comprehended if one initially understands what a first derivative is. This nice feature can present a problem, however, because in order to find the kth derivative of a function f at a point x, one must know all the previous derivatives, not only at x, but at every point in the neighborhood of x. One type of generalized kth order differentiation, having Taylor's theorem as its motivation, attempts to skirt this drawback. This kind of differentiation is called Peano differentiation.

Definition 1. A function f is said to *have a k-th Peano derivative* at x if there exist numbers $f_1(x), \ldots, f_k(x)$ such that

$$f(x+h) = f(x) + hf_1(x) + \ldots + \frac{h^k}{k!} f_k(x) + \varepsilon_k(x,h) \tag{2}$$

where $\varepsilon_k(x,h) \to 0$ as $h \to 0$. The number $f_k(x)$ is called the kth *Peano derivative* of f at x.

The kth Peano derivative is a true generalization of the ordinary kth derivative although obviously there is no difference for $k = 1$. For properties of Peano derivatives see [4], [5], [10], [12], [13], [15]–[19].

In [7] Laczkovich generalized the notion of Peano derivatives. He introduced so-called absolute Peano derivatives.

DEFINITION 2. Let f be defined in a neighborhood of x. We say that the *absolute Peano derivative* of f at x exists and is A (in symbols $f^*(x) = A$) if there is a function g, a nonnegative integer k, and a $\delta > 0$ such that

$$\text{(i) } g_k = f \text{ on } (x-\delta, x+\delta) \quad \text{and} \quad \text{(ii) } g_{k+1}(x) = A\,.$$

Laczkovich showed that this concept is unambiguously defined, that if f^* exists on an interval it is a function of Baire class one, that it has the Darboux property, and if f^* is bounded above or below on an interval, then $f^* = f'$ on that interval. He also provided an example of an absolute Peano derivative on a compact interval that is not a Peano derivative of any order. Therefore absolute Peano derivatives are true generalization of Peano derivatives.

An even more general Peano derivative was introduced by M. Lee [9].

DEFINITION 3. Let F be a continuous function defined on $\mathbb{R}$, and let $n \in \mathbb{N}$. We say that F is *n times generalized Peano differentiable* at $x \in \mathbb{R}$ if there is a positive integer q, and coefficients $F_{[i]}(x)$, $i = 1, \ldots, n$, such that

$$\begin{aligned} F^{(-q)}(x+h) = {} & \sum_{j=0}^{q-1} h^j \frac{F^{(-q+j)}(x)}{j!} \\ & + \sum_{j=0}^{n} h^{q+j} \frac{F_{[j]}(x)}{(q+j)!} + h^{q+n} \varepsilon_{q+n}^{[q]}(x,h) \end{aligned} \tag{3}$$

where $\lim_{h\to 0} \varepsilon_{q+n}^{[q]}(x,h) = 0$.

Here $F_{[0]}(x) = F(x) = F^{(0)}(x)$ and $F^{(-j)}(x) = \int^x F^{(-j+1)}(t)\,dt$; i.e. $F^{(-j)}$ is an indefinite Riemann integral of the continuous function $F^{(-j+1)}$ for $j = 1, \ldots, q$.

Remark. Note that the definitions of $F_{[i]}(x)$, $i = 0, 1, \ldots, n$, and of $\varepsilon_{q+n}^{[q]}(x,h)$ do not depend on which q-fold indefinite Riemann integral $F^{(-q)}$ of the continuous function F is taken, because any two differ by a polynomial of degree less than q. The above definition is the same as the definition of the $(q+n)$th Peano derivative of $F^{(-q)}$ at x. Therefore by Lemma 7 of

[6], the coefficients $F_{[i]}(x)$, $i = 1, \ldots, n$, do not depend on q either. The coefficient $F_{[n]}(x)$ is called the nth *generalized Peano derivative* of F at x.

M. Lee showed that every absolute Peano derivative on a compact interval is a generalized Peano derivative. He obtained many properties for generalized Peano derivatives using similar techniques to those used by various authors in establishing corresponding properties for Peano derivatives. As mentioned in the introduction we will show that we can get many of these properties for generalized Peano derivatives by showing that they belong to the class $[\Delta']$ and that they are path derivatives with the system of paths satisfying suitable properties. This is a new approach to studying both generalized Peano and Peano derivatives. Also, we will obtain some new properties for generalized Peano and Peano derivatives. The class $[\Delta']$ and path derivatives will be described briefly below. For more details the reader is referred to [1] and [3].

Definition 4. Let C be the family of all continuous functions on $\mathbb{R}$, Δ the family of all differentiable functions on $\mathbb{R}$ and Δ' the family of all derivatives on $\mathbb{R}$. If Γ is a family of functions defined on $\mathbb{R}$, then we denote by $[\Gamma]$ the family of all functions f on $\mathbb{R}$ with the following property: there exist $v_n \in \Gamma$ and closed sets A_n such that $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$ and $f = v_n$ on A_n for every $n = 1, 2, \ldots$

In [1] (Theorem 2) it is shown that the following four conditions are equivalent:

(i) There are g, h and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.

(ii) There are $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.

(iii) $f \in [\Delta']$.

(iv) There is a dense open set T and a function $k \in \Delta$ such that f is a derivative on T and $f = k'$ on $\mathbb{R} \setminus T$.

Definition 5. Let $x \in \mathbb{R}$. A *path leading to* x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is a point of accumulation of E_x. A *system of paths* is a collection $E = \{E_x : x \in \mathbb{R}\}$ such that each E_x is a path leading to x.

Definition 6. Let $f : \mathbb{R} \to \mathbb{R}$ and let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. If

$$\lim_{y \in E_x,\, y \to x} \frac{f(y) - f(x)}{y - x} = g(x)$$

is finite then we say that f is *E-differentiable at* x and write $f'_E(x) = g(x)$. If f is E-differentiable at every point x then we say simply that f is *E-differentiable*; we call f an *E-primitive* and g an *E-derivative*.

Definition 7. Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. Then E is said to be *bilateral at* x if x is a bilateral point of accumulation of E_x, and it is *nonporous at* x if E_x has left and right porosity 0 at x. (If E has any of these properties at each point, then we say that E has that property.)

The porosity of a set E at x from the right (left) is the value $\limsup_{r\to 0^+} l(x,r,E)/r$, where $l(x,r,E)$ denotes the length of the largest open interval contained in $(x,x+r)\cap(\mathbb{R}\setminus E)$ (resp. $(x-r,x)\cap(\mathbb{R}\setminus E)$). Porosity 0 at x means both right and left porosity 0. Note that a nonporous system is necessarily bilateral.

DEFINITION 8. Let $E=\{E_x : x\in\mathbb{R}\}$ be a system of paths. E will be said to satisfy one of the conditions listed below if there is a positive function δ on $\mathbb{R}$ so that whenever $0<y-x<\min\{\delta(x),\delta(y)\}$, the sets E_x and E_y intersect in the stated fashion:

(i) *intersection condition* (I.C.): $E_x\cap E_y\cap[x,y]\neq\emptyset$;

(ii) *internal intersection condition* (I.I.C.): $E_x\cap E_y\cap(x,y)\neq\emptyset$;

(iii) *external intersection condition* (E.I.C.):

$$E_x\cap E_y\cap(y,y-2x)\neq\emptyset \quad\text{and}\quad E_x\cap E_y\cap(2x-y,x)\neq\emptyset\,.$$

Generalized Peano derivatives and the class $[\Delta']$. From now on n will be a fixed positive integer, and F will be a continuous function defined on $\mathbb{R}$ so that $F_{[n]}$ exists on $\mathbb{R}$.

DEFINITION 9. For $q\in\mathbb{N}$, let A_q be the set of all $x\in\mathbb{R}$ so that (3) holds, and for $\varepsilon>0$, $\delta>0$ let

$$P_q=P_q(\varepsilon,\delta)=\{x\in A_q : |\varepsilon^{[q]}_{q+n}(x,h)|<\varepsilon \text{ for } |h|<\delta\}\,. \tag{4}$$

Note that if $x\in A_q$, then $x\in A_p$ for every $p\ge q$. Also $x\in A_q$ iff $F^{(-q)}$ has a $(q+n)$th Peano derivative at x with $(F^{(-q)})_{q+n}(x)=F_{[n]}(x)$.

LEMMA 10. *For* $q\le p$,

$$P_q(\varepsilon,\delta)\subset P_p\left(\varepsilon\frac{(q+n)!}{(p+n)!},\delta\right).$$

Proof. Let $x\in P_q(\varepsilon,\delta)$. Then $x\in A_q$ and for $t\in\mathbb{R}$,

$$\begin{aligned}F^{(-q)}(x+t)&=\sum_{j=0}^{q-1}t^j\frac{F^{(-q+j)}(x)}{j!}\\&\quad+\sum_{j=0}^{n}t^{q+j}\frac{F_{[j]}(x)}{(q+j)!}+t^{q+n}\varepsilon^{[q]}_{q+n}(x,t)\,.\end{aligned}\tag{5}$$

Integrating both sides of (5) from 0 to h we get

$$F^{(-q-1)}(x+h) - F^{(-q-1)}(x) = \sum_{j=0}^{q-1} h^{j+1}\frac{F^{(-q+j)}(x)}{(j+1)!} + \sum_{j=0}^{n} h^{q+1+j}\frac{F_{[j]}(x)}{(q+1+j)!} + \int_0^h t^{q+n}\varepsilon_{q+n}^{[q]}(x,t)\,dt\,.$$

Thus $x \in A_{q+1}$. By the remark after Definition 3, we have

$$h^{q+1+n}\varepsilon_{q+n+1}^{[q+1]}(x,h) = \int_0^h t^{q+n}\varepsilon_{q+n}^{[q]}(x,t)\,dt \tag{6}$$

and since $x \in P_q(\varepsilon,\delta)$, for $0 \neq |h| < \delta$ from (6) we have

$$|h|^{q+1+n}|\varepsilon_{q+n+1}^{[q+1]}(x,h)| < \int_0^{|h|} t^{q+n}\varepsilon\,dt = \varepsilon\frac{|h|^{q+1+n}}{q+1+n}\,.$$

Hence $|\varepsilon_{q+n+1}^{[q+1]}(x,h)| < \varepsilon/(q+n+1)$ whenever $|h| < \delta$. Therefore

$$P_q(\varepsilon,\delta) \subset P_{q+1}\left(\varepsilon\frac{(q+n)!}{(q+n+1)!},\delta\right).$$

The general result follows by induction. ■

Definition 11. For $x \in A_q$ and for $i = 1,\ldots,n$, define $\varepsilon_{q+i}^{[q]}(x,h)$ by

$$F^{(-q)}(x+h) = \sum_{j=0}^{q-1} h^j\frac{F^{(-q+j)}(x)}{j!} + \sum_{j=0}^{i} h^{q+j}\frac{F_{[j]}(x)}{(q+j)!} + h^{q+i}\varepsilon_{q+i}^{[q]}(x,h)\,. \tag{7}$$

Note that $\varepsilon_{q+i}^{[q]}(x,h)$ does not depend on which q-fold indefinite Riemann integral, $F^{(-q)}$, of F is taken.

The following formula follows directly from Definition 11.

Formula 12. *Let $x \in A_q$. Then for $i \in \mathbb{N}$ with $2 \le i \le n$ we have*

$$\varepsilon_{q+i-1}^{[q]}(x,t) = t\frac{F_{[i]}(x)}{(q+i)!} + t\varepsilon_{q+i}^{[q]}(x,t)\,.$$

Definition 13. For any function f defined on $\mathbb{R}$ the *Riemann difference* $\Delta_t^m f(x)$ at a point x of order m is defined by

$$\Delta_t^m f(x) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt)\,.$$

The relations among the Δ_t^m are given by the following simple assertion.

LEMMA 14. *For any function f defined on $\mathbb{R}$, for any $m \in \mathbb{N}$ and $t \in \mathbb{R}$ we have*

$$\Delta_t^{m+1} f(x) = \Delta_t^m f(x+t) - \Delta_t^m f(x)\,.$$

LEMMA 15. *Let $x \in A_q$. Then for each $i = 1, \ldots, n$,*

$$\Delta_t^{q+m} F^{(-q)}(x) = \begin{cases} t^{q+m} F_{[m]}(x) + t^{q+m} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+m} \varepsilon_{q+m}^{[q]}(x, jt) & \text{if } m = i, \\ t^{q+i} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) & \text{if } m > i. \end{cases}$$

Proof.

$$\begin{aligned} \Delta_t^{q+m} F^{(-q)}(x) &= \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} F^{(-q)}(x+jt) \\ &= \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} \biggl(\sum_{l=0}^{q-1} (jt)^l \frac{F^{(-q+l)}(x)}{l!} \\ &\quad + \sum_{l=0}^{i} (jt)^{q+l} \frac{F_{[l]}(x)}{(q+l)!} + (jt)^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) \biggr) \\ &= \sum_{l=0}^{q-1} t^l \frac{F^{(-q+l)}(x)}{l!} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^l \\ &\quad + \sum_{l=0}^{i} t^{q+l} \frac{F_{[l]}(x)}{(q+l)!} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+l} \\ &\quad + t^{q+i} \sum_{j=0}^{q+m} (-1)^{q+m-j} \binom{q+m}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) \end{aligned}$$

and by Lemma 1 of [6] this is equal to the desired expression. ∎

The next formula is very useful in the proof of Theorem 17 below.

FORMULA 16. *Suppose $x, x+t \in A_q$ and $i \in \mathbb{N}$ with $1 \le i \le n$. Then*

$$\begin{aligned} F_{[i]}(x+t) - F_{[i]}(x) = &\sum_{j=0}^{q+i+1} (-1)^{q+i+1-j} \binom{q+i+1}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) \\ &+ \sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) \end{aligned}$$

$$- \sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x+t, jt) .$$

Proof. By Lemma 14,

$$\Delta_t^{q+i+1} F^{(-q)}(x) = \Delta_t^{q+i} F^{(-q)}(x+t) - \Delta_t^{q+i} F^{(-q)}(x) . \tag{8}$$

Applying Lemma 15 with $m = i+1$ to the left hand side and with $m = i$ to the right hand side of (8) we get

$$t^{q+i} \sum_{j=0}^{q+i+1} (-1)^{q+i+1-j} \binom{q+i+1}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt)$$

$$= t^{q+i} F_{[i]}(x+t) + t^{q+i} \sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x+t, jt)$$

$$- t^{q+i} F_{[i]}(x) - t^{q+i} \sum_{j=0}^{q+i} (-1)^{q+i-j} \binom{q+i}{j} j^{q+i} \varepsilon_{q+i}^{[q]}(x, jt) .$$

Dividing both sides by t^{q+i} gives the above formula. ■

THEOREM 17. *For any interval* $[a,b]$, $F_{[n]}$ *is bounded on* $\overline{P}_q(\varepsilon,\delta) \cap [a,b]$.

Proof. Let $x, y \in P_q(\varepsilon,\delta) \cap [a,b]$ so that for $t = y - x$ we have $|t| < \delta/(q+n+1)$, and let $B = \sum_{j=1}^{q+n+1} \binom{q+n+1}{j} j^{q+n}$. Then the right hand side of Formula 16 applied with $i = n$ is bounded by $3B\varepsilon$. It follows that $F_{[n]}$ is bounded on $P_q(\varepsilon,\delta) \cap [a,b]$. From Formula 12 (applied with $i = n$) it follows that for $|h| < \delta$, $|\varepsilon_{q+n-1}^{[q]}(\,\cdot\,,h)|$ is bounded on $P_q(\varepsilon,\delta) \cap [a,b]$. Now from Formula 16 (applied with $i = n-1$) we see that $F_{[n-1]}$ is bounded on $P_q(\varepsilon,\delta) \cap [a,b]$, and again going back to Formula 12 (applied with $i = n-1$) we see that for $|h| < \delta$, $|\varepsilon_{q+n-2}^{[q]}(\,\cdot\,,h)|$ is bounded on $P_q(\varepsilon,\delta) \cap [a,b]$. Continuing we can deduce that there is a constant C so that $|F_{[i]}(x)| \le C$ for $1 \le i \le n$, for $x \in P_q(\varepsilon,\delta) \cap [a,b]$.

Let $x \in \overline{P}_q$, and let $\{x_m\}$ be a sequence in $P_q(\varepsilon,\delta)$ such that $\lim_{m\to\infty} x_m = x$. Choose $p \ge q$ such that $x \in A_p$. Let $[a,b]$ be such that $\{x_m\} \subset P_q(\varepsilon,\delta) \cap [a,b]$. From the first part of the proof we see that for $1 \le i \le n$, $F_{[i]}$ is bounded on $P_q(\varepsilon,\delta) \cap [a,b]$. Therefore we can choose a subsequence $\{x_{m_j}\}$ converging to x such that $\{F_{[i]}(x_{m_j})\}$ converges for each $1 \le i \le n$. Let these sequences converge to $G_i(x)$, $i = 1, \ldots, n$, respectively.

Let $|h| < \delta$, and, as we may, suppose that $|h + x - x_{m_j}| < \delta$ for every $j \in \mathbb{N}$. Since $x_{m_j} \in P_q(\varepsilon,\delta)$, by Lemma 10 we have

$$|\varepsilon_{p+n}^{[p]}(x_{m_j}, h + x - x_{m_j})| < \varepsilon \frac{(q+n)!}{(p+n)!} .$$

Thus we may also suppose that the sequence $\varepsilon_{p+n}^{[p]}(x_{m_j}, h + x - x_{m_j})$ converges. Denote its limit by $E(h)$. Now letting $j \to \infty$ in the formula

$$\begin{aligned}F^{(-p)}(x+h) &= F^{(-p)}(x_{m_j}) + (h + x - x_{m_j})F^{(-p+1)}(x_{m_j}) + \ldots \\ &+ \frac{(h+x-x_{m_j})^p}{p!}F_{[0]}(x_{m_j}) + \ldots + \frac{(h+x-x_{m_j})^{p+n-1}}{(p+n-1)!}F_{[n-1]}(x_{m_j}) \\ &+ (h+x-x_{m_j})^{p+n}\left(\frac{F_{[n]}(x_{m_j})}{(p+n)!} + \varepsilon_{p+n}^{[p]}(x_{m_j}, h+x-x_{m_j})\right)\end{aligned}$$

we get

$$(9)\qquad \begin{aligned}F^{(-p)}(x+h) = \sum_{j=0}^{p-1}\frac{h^j}{j!}F^{(-p+j)}(x) + \frac{h^p}{p!}F_{[0]}(x) + \frac{h^{p+1}}{(p+1)!}G_1(x) + \ldots \\ + \frac{h^{p+n-1}}{(p+n-1)!}G_{n-1}(x) + h^{p+n}\left(\frac{G_n(x)}{(p+n)!} + E(h)\right).\end{aligned}$$

Since $G_n(x)/(p+n)! + E(h)$ is bounded, by the uniqueness of Peano derivatives from (9) we have $G_i(x) = F_{[i]}(x)$ for $1 \le i \le n-1$ and

$$(10)\qquad \frac{F_{[n]}(x)}{(p+n)!} + \varepsilon_{p+n}^{[p]}(x,h) = \frac{G_n(x)}{(p+n)!} + E(h).$$

Since $|E(h)| \le \varepsilon(q+n)!/(p+n)!$, from (10) we have

$$(11)\qquad \begin{aligned}\left|\frac{F_{[n]}(x) - G_n(x)}{(p+n)!}\right| &= |E(h) - \varepsilon_{p+n}^{[p]}(x,h)| \\ &\le \varepsilon\frac{(q+n)!}{(p+n)!} + |\varepsilon_{p+n}^{[p]}(x,h)|.\end{aligned}$$

The left hand side of (11) does not depend on h so letting $h \to 0$ in the right hand side of (11) we get

$$(12)\qquad \left|\frac{F_{[n]}(x) - G_n(x)}{(p+n)!}\right| \le \varepsilon\frac{(q+n)!}{(p+n)!}.$$

Finally, from the first part of the proof we know that there is a constant C so that $|F_{[n]}(y)| \le C$ for $y \in P_q(\varepsilon,\delta) \cap [a,b]$. Since $\lim_{j\to\infty} F_{[n]}(x_{m_j}) = G_n(x)$, $|G_n(x)| \le C$. Hence by (12), $|F_{[n]}(x)| \le \varepsilon(q+n)! + C$. Note that the bound on $F_{[n]}(x)$ does not depend on the choice of p. ∎

If x_1 and x are two different points in A_q, then since $F_{[n]}(y) = (F^{(-q)})_{q+n}(y)$ for $y \in A_q$, we have a formula for generalized Peano derivatives similar to that in Theorem 1 of [6]. Since the formula is the crux of the proof of the main theorem of this section, Theorem 19, we state it as a theorem.

THEOREM 18. *Let $x, x_1 \in A_q$ such that $x \neq x_1$, and $t \neq 0$. Then*

$$\frac{F_{[n-1]}(x_1) - F_{[n-1]}(x)}{x_1 - x} - F_{[n]}(x) = \frac{t}{x_1 - x}\frac{q+n-1}{2}F_{[n]}(x)$$
$$+ \sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}\frac{(x_1 - x + jt)^{q+n}}{t^{q+n-1}(x_1 - x)}\varepsilon_{q+n}^{[q]}(x, x_1 - x + jt)$$
$$-\frac{t}{x_1 - x}\bigg\{\frac{q+n-1}{2}F_{[n]}(x_1)$$
$$+ \sum_{j=0}^{q+n-1}(-1)^{q+n-1-j}\binom{q+n-1}{j}j^{q+n}\varepsilon_{q+n}^{[q]}(x_1, jt)\bigg\}.$$

Finally, we are ready to prove the main result of this section.

THEOREM 19. *Suppose F is n times generalized Peano differentiable at each $x \in \mathbb{R}$. Then $F_{[n-1]}$ is differentiable on $\overline{P}_q = \overline{P}_q(\varepsilon, \delta)$ relative to $\overline{P}_q$ with*

$$F_{[n-1]}|'_{\overline{P}_q}(x) = F_{[n]}(x).$$

Proof. Let $x \in \overline{P}_q$. There is a $p \geq q$ so that $x \in A_p$. Let $1 > \varepsilon_1 > 0$ be given. There is an η with $0 < \eta < \delta$ such that $|\varepsilon_{p+n}^{[p]}(x, h)| < \varepsilon_1$ whenever $|h| < \eta$. Let $\{x_m\}$ be a sequence in P_q converging to x so that $|x_m - x| < \eta/(p+n)$. By Theorem 17, there is a constant C so that $|F_{[n]}(x_m)| \leq C$, for every $m \in \mathbb{N}$. Let $t = (x_m - x)\varepsilon_1^{1/(p+n)}$. Then $|jt| < \delta$ and $|x_m - x + jt| < \eta$, for $j = 0, 1, \ldots, q + n - 1$. Therefore

$$|\varepsilon_{p+n}^{[p]}(x, x_m - x + jt)| < \varepsilon_1 \tag{13}$$

and by Lemma 10,

$$|\varepsilon_{p+n}^{[p]}(x_m, jt)| < \varepsilon \quad \text{for } j = 0, 1, \ldots, p + n - 1 \text{ and } m \in \mathbb{N}. \tag{14}$$

Since $x_m \in A_p$, the formula of Theorem 18 gives

$$\left|\frac{F_{[n-1]}(x_m) - F_{[n-1]}(x)}{x_m - x} - F_{[n]}(x)\right| \leq \varepsilon_1^{1/(p+n)}\frac{p+n-1}{2}|F_{[n]}(x)|$$
$$+ \sum_{j=0}^{p+n-1}\binom{p+n-1}{j}\frac{(1 + j\varepsilon_1^{1/(p+n)})^{p+n}}{\varepsilon_1^{(p+n-1)/(p+n)}}|\varepsilon_{p+n}^{[p]}(x, x_m - x + jt)|$$
$$+ \varepsilon_1^{1/(p+n)}\bigg|\frac{p+n-1}{2}F_{[n]}(x_m)$$
$$+ \sum_{j=0}^{p+n-1}(-1)^{p+n-1-j}\binom{p+n-1}{j}j^{p+n}\varepsilon_{p+n}^{[p]}(x_m, jt)\bigg|.$$

By (14) and (13) together with Theorem 17 the above is

$$\le \varepsilon_1^{1/(p+n)}\frac{p+n-1}{2}|F_{[n]}(x)| + \sum_{j=0}^{p+n-1}\binom{p+n-1}{j}(1+j\varepsilon_1^{1/(p+n)})^{p+n}\varepsilon_1^{1/(p+n)}$$
$$+\varepsilon_1^{1/(p+n)}\Big\{\frac{p+n-1}{2}C + \sum_{j=0}^{p+n-1}\binom{p+n-1}{j}j^{p+n}\varepsilon\Big\}.$$

Since ε_1 was arbitrary we have

$$\frac{F_{[n-1]}(x_m)-F_{[n-1]}(x)}{x_m-x} - F_{[n]}(x)\to 0 \qquad \text{as } x_m\in P_q, x_m\to x\,.$$

Now for the general case let $\{x_m\}$ be a sequence in $\overline{P}_q$ such that $x_m\to x$. Let $y_m\in P_q$ be such that $|y_m-x_m|\le \frac{1}{m}|x_m-x|$ and that

$$\left|\frac{F_{[n-1]}(y_m)-F_{[n-1]}(x_m)}{y_m-x_m} - F_{[n]}(x_m)\right|\le 1\,. \tag{15}$$

By what was just proved, there is such a sequence y_m. By Theorem 17, there is a constant C such that $|F_{[n]}(x_m)|\le C$ for every $m\in\mathbb{N}$. This and (15) give

$$\left|\frac{F_{[n-1]}(y_m)-F_{[n-1]}(x_m)}{y_m-x_m}\right|\le C+1\,. \tag{16}$$

Now

$$\frac{F_{[n-1]}(x_m)-F_{[n-1]}(x)}{x_m-x} - F_{[n]}(x) = \frac{F_{[n-1]}(x_m)-F_{[n-1]}(y_m)}{x_m-y_m}\,\frac{x_m-y_m}{x_m-x}$$
$$+\Big\{\frac{F_{[n-1]}(y_m)-F_{[n-1]}(x)}{y_m-x} - F_{[n]}(x)\Big\}\frac{y_m-x}{x_m-x} - F_{[n]}(x)\frac{x_m-y_m}{x_m-x}\,.$$

So by (16),

$$\left|\frac{F_{[n-1]}(x_m)-F_{[n-1]}(x)}{x_m-x} - F_{[n]}(x)\right| \tag{17}$$
$$\le (C+1)\frac{1}{m} + \left|\frac{F_{[n-1]}(y_m)-F_{[n-1]}(x)}{y_m-x} - F_{[n]}(x)\right|\Big(1+\frac{1}{m}\Big) + C\frac{1}{m}\,.$$

Finally, since $x_m\to x$, $y_m\to x$. But $y_m\in P_q$, and hence by the first part

$$\frac{F_{[n-1]}(y_m)-F_{[n-1]}(x)}{y_m-x} - F_{[n]}(x)\to 0\,. \tag{18}$$

Therefore by (17) and (18),

$$\frac{F_{[n-1]}(x_m)-F_{[n-1]}(x)}{x_m-x} - F_{[n]}(x)\to 0 \qquad \text{as } x_m\in \overline{P}_q,\ x_m\to x\,.$$

This completes the proof. ■

Lemma 20. *For each* $\varepsilon > 0$, $\bigcup_{q=0}^{\infty} \bigcup_{m=1}^{\infty} P_q(\varepsilon, 1/m) = \mathbb{R}$.

Proof. The assertion follows from Definition 9. ∎

Corollary 21. $F_{[n]} \in [\Delta']$.

Proof. The corollary follows directly from Theorem 19, Lemma 20 and the fact that for any function g defined on a closed set P that is differentiable with respect to P, there is a function G differentiable on $\mathbb{R}$ so that $G|_P = g$ and $G' = g'$. (See Mařík [14].) ∎

Corollary 22. $F_{[n]}$ *is a Baire* 1 *function.*

Proof. The assertion is an immediate consequence of the previous corollary. ∎

That $F_{[n]}$ is a Baire 1 function was proved in [9]. Corollary 22 gives another proof of this assertion. Note that we have proved even more than Corollary 21, namely that the nth generalized Peano derivative is a composite derivative of the $(n-1)$th generalized Peano derivative.

Definition 23. Let f be a function defined on $\mathbb{R}$. If there exist a function g and closed sets A_n, $n = 1, 2, \ldots$, such that $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$ and $g|'_{A_n}(x) = f(x)$ for $x \in A_n$, then we say that f is a *composite derivative* of g.

Corollary 24. $F_{[n]}$ *is a composite derivative of* $F_{[n-1]}$.

Corollary 25. $F_{[n]}$ *is the approximate derivative of* $F_{[n-1]}$ *a.e.*

Proof. The assertion follows from Corollary 24 and the fact that every composite derivative is the approximate derivative of its primitive a.e. ∎

Generalized Peano and path derivatives. In this section we will prove that the nth generalized Peano derivative is a path derivative of the $(n-1)$th generalized Peano derivative with a nonporous system of paths that satisfies the I.C. condition. In particular, the same holds for Peano derivatives, which gives a positive answer to the questions posed in [3].

First we will assume inductively that certain properties for generalized Peano derivatives hold. Later using induction we will get rid of these assumptions. This will give us a new approach to studying generalized Peano derivatives.

Theorem 26. *Let* $l \in \mathbb{N}$ *with* $l \le n-1$. *Assume that for each function* g *defined on a closed interval* I *and having an* l*-th generalized Peano derivative*, $g_{[l]}$, *on* I, $g_{[l]}$ *is a Darboux function and if* $g_{[l]} \ge 0$ *on* I, *then* $g_{[l]} = g^{(l)}$ *on* I. *Suppose* $F_{[n]}$ *exists on* $\mathbb{R}$. *Then there is a bilateral nonporous system* $E = \{E_x : x \in \mathbb{R}\}$ *of paths satisfying the I.C. condition such that* $F_{[n]}$ *is the* E*-derivative of* $F_{[n-1]}$.

To prove Theorem 26, we need some lemmas and the theorem below, due to Mařík (see [13]).

THEOREM 27. *Let $k \in \mathbb{N}$, $x \in \mathbb{R}$. Suppose that a function f has a k-th Peano derivative at x. Define*

$$P(y) = \sum_{i=0}^{k} (y-x)^i \frac{f_i(x)}{i!} \qquad (y \in \mathbb{R}).$$

Let $\varepsilon > 0, \eta > 0$. Then there is a $\delta > 0$ with the following property: If I is a subinterval of $(x-\delta, x+\delta)$ and j an integer with $0 < j \le k$, and if either $f^{(j)} \le P^{(j)}$ on I or $f^{(j)} \ge P^{(j)}$ on I, then

$$m\{y \in I : |f^{(j)}(y) - P^{(j)}(y)| \ge \varepsilon |y-x|^{k-j}\} \le \eta(mI + d(x, I))$$

(*here m denotes the Lebesgue measure and $d(x,I)$ denotes the distance from x to I*).

LEMMA 28. *Suppose the assumptions of Theorem 26 hold. Let $x \in A_q$. Then for every $\varepsilon > 0$ and $\eta > 0$ there is a $\delta > 0$ such that if I is a closed subinterval of $(x-\delta, x+\delta)$, x is not in I and*

$$\left| \frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y-x} - F_{[n]}(x) \right| \ge \varepsilon \tag{19}$$

for all $y \in I$ then $mI \le \eta d(x, I)$.

P r o o f. Let I be an interval such that (19) holds, and let

$$g(y) = F(y) - y^{n-1} \frac{F_{[n-1]}(x)}{(n-1)!} - (y-x)^n \frac{F_{[n]}(x)}{n!}.$$

Then g has an $(n-1)$th generalized Peano derivative and $g_{[n-1]}(y) = F_{[n-1]}(y) - F_{[n-1]}(x) - (y-x)F_{[n]}(x)$. So by assumptions $g_{[n-1]}$ is Darboux. By the assumption of Lemma 28, $|g_{[n-1]}(y)| \ge \varepsilon|y-x|$ on I. If $y > x$ then $|g_{[n-1]}(y)| > 0$ on I and since it is Darboux we have either $g_{[n-1]}(y) > 0$ or $-g_{[n-1]}(y) > 0$ on I. Hence by the assumptions, $g_{[n-1]}$ is the $(n-1)$th ordinary derivative of g on I. If $y < x$ by the same argument we find that $g_{[n-1]}$ is the $(n-1)$th ordinary derivative of g on I. Therefore F is $n-1$ times ordinarily differentiable on I and by the uniqueness of generalized Peano derivatives $F^{(n-1)} = F_{[n-1]}$ on I. Now we can apply Theorem 27, with $k = n+q$, $j = n-1+q$, $f = F^{(-q)}$ and with η replaced by $\eta_1 = \eta/(1+\eta)$, which gives δ so that if $I \subset (x-\delta, x+\delta)$ then $mI \le \eta_1(mI + d(x,I))$. Therefore $mI \le \eta d(x,I)$. ■

The next lemma is Lemma 3.6.1 of [3].

LEMMA 29. *Let G be a function that has the following property at a point x: there is a number λ so that for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that, whenever $0 < y-x < \delta$, there must be a number z with $0 < y-z < \varepsilon(y-x)$*

and $|G(z) - \lambda| < \varepsilon$*. Then there is a path* E_x *leading to* x *and nonporous on the right at* x *such that*

$$\lim_{y \in E_x,\, y \to x} G(y) = \lambda\,.$$

LEMMA 30. *Under the assumptions of Theorem* 26, *for each* $x \in I$ *there is a path* E_x *leading to* x *and nonporous at* x *so that*

$$\lim_{y \in E_x,\, y \to x} \frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y - x} = F_{[n]}(x)\,.$$

Proof. For fixed x, by Lemma 28 (applied with $\eta = \varepsilon/(1-\varepsilon)$ for $\varepsilon < 1$ and $\eta = \varepsilon$ otherwise) the function G defined by

$$G(y) = \frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y - x}$$

satisfies the conditions of Lemma 29 where $\lambda = F_{[n]}(x)$. The rest follows from Lemma 29, together with a left-hand version of that lemma. ∎

LEMMA 31. *Let* $m \le l$ *be positive integers and* $\varepsilon > 0$*. Then*

$$P_m(\varepsilon, 1/m) \subset P_l(\varepsilon, 1/l)\,.$$

Proof. By Lemma 10, $P_m(\varepsilon, 1/m) \subset P_l(\varepsilon, 1/m)$, and by Definition 9, $P_l(\varepsilon, 1/m) \subset P_l(\varepsilon, 1/l)$. ∎

Now we are ready to prove Theorem 26.

Proof of Theorem 26. For each $x \in \mathbb{R}$ let E'_x be a path satisfying the assertion of Lemma 30. We define the system $E = \{E_x : x \in \mathbb{R}\}$ of paths as follows:

For $x \in \mathbb{R}$ let $E_x = E'_x \cup P_m(1, 1/m)$ where m is a positive integer such that $x \in P_m(1, 1/m)$. That E is nonporous (therefore bilateral) follows directly from Lemma 30. Also Lemma 30 and Theorem 19 assure that $F_{[n-1]}$ is E-differentiable with $F_{[n-1]}|'_E(x) = F_{[n]}(x)$ for every $x \in \mathbb{R}$. It remains to prove that E satisfies the intersection condition I.C. We will prove that for any two different points x and y, $E_x \cap E_y \cap [x, y] \ne \emptyset$, which is stronger than the I.C. condition.

Let $x \in P_m(1, 1/m)$ and $y \in P_l(1, 1/l)$ be any two different points. If $m \le l$ then, by Lemma 31, $P_m(1, 1/m) \subset P_l(1, 1/l)$ and hence $x \in E_y$. Similarly if $m \ge l$ then $y \in E_x$.

Therefore $E_x \cap E_y \cap [x, y] \ne \emptyset$, hence E satisfies the I.C. condition. This completes the proof of the theorem. ∎

We can weaken the assumptions of Theorem 26. In order to do that we list some theorems from [3] about path derivatives.

THEOREM 32. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths that is bilateral and satisfies the intersection condition. If f is an exact E-derivative and is Baire* 1, *then f has the Darboux property.*

Proof. This is Theorem 6.4 of [3]. ∎

THEOREM 33. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths that is bilateral and satisfies the intersection condition. If $\underline{f}'_E(x) \geq 0$ in $[a,b]$ then f is nondecreasing on $[a,b]$.*

Proof. See 4.7.1 of [3]. ∎

THEOREM 34. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and suppose f is monotonic. If E is nonporous at a point x, then*

$$\underline{F}'_E(x) = \underline{F}'(x) \quad \text{and} \quad \overline{F}'_E(x) = \overline{F}'(x)\,.$$

Proof. See Theorem 4.4.3 of [3]. ∎

THEOREM 35. *Let F be a continuous function defined on $\mathbb{R}$ so that $F_{[n]}$ exists on $\mathbb{R}$. Then there is a bilateral nonporous system $E = \{E_x : x \in \mathbb{R}\}$ of paths satisfying the I.C. condition such that $F_{[n]}$ is an E-derivative of $F_{[n-1]}$.*

Proof. The proof is by induction on n. For $n = 0$ there is nothing to prove. Let $1 \leq l \leq n-1$, and let a function g defined on some closed interval I have an lth generalized Peano derivative on I. Suppose the assertion of the theorem is true for every $1 \leq j \leq n-1$, and every function h defined on some closed interval J so that $h_{[j]}$ exists on J. (Note that we can restrict ourselves to closed subintervals because we can always extend h to $\mathbb{R}$ so that $h_{[j]}$ exists on $\mathbb{R}$. For example if $J = [a,b]$, then we can define $h(y) = \sum_{i=0}^{j}(y-x)^i f_i(a)/i!$ for $y \in (-\infty, a)$ and $h(y) = \sum_{i=0}^{j}(y-x)^i f_i(b)/i!$ for $y \in (b,\infty)$.) By Corollary 22, $g_{[l]}$ is a Baire 1 function. By the induction hypothesis and Theorem 32, $g_{[l]}$ is a Darboux function. Suppose that $g_{[l]} \geq 0$ on I. Again by the induction hypothesis but now using Theorem 33, $g_{[l-1]}$ is nondecreasing on I. By Theorem 34, $g'_{[l-1]} = g_{[l]}$ on I. Also there is an α such that $g_{[l-1]} - \alpha \geq 0$ on I. Let $h(x) = g(x) - \alpha x^{l-1}/(l-1)!$. Then $h_{[l-1]} = g_{[l-1]} - \alpha$ and hence $h_{[l-1]} \geq 0$ on I. Proceeding as before yields $h'_{[l-2]} = h_{[l-1]}$ on I. This implies $g'_{[l-2]} = g_{[l-1]}$ on I. Continuing in this fashion one can deduce that $g^{(l)}$ exists on I. Now we can apply Theorem 26. ∎

Now using the properties of path derivatives we get the following corollaries:

COROLLARY 36. *$F_{[n]}$ is Darboux.*

Proof. The assertion follows directly from Theorems 32, 26, and Corollary 22. ∎

The assertion of the next corollary follows directly from the properties of functions that are Baire 1 and Darboux. (See [2].)

COROLLARY 37. *Let F be a continuous function on $\mathbb{R}$ such that $F_{[n]}$ exists.*

(1) *For each x, there exist sequences $x_m \nearrow x$ and $y_m \searrow x$ such that $F_{[n]}(x) = \lim_{m\to\infty} F_{[n]}(x_m) = \lim_{m\to\infty} F_{[n]}(y_m)$.*

(2) *For each x,*

$$F_{[n]}(x) \in [\liminf_{z\to x^-} F_{[n]}(z), \limsup_{z\to x^-} F_{[n]}(z)] \cap [\liminf_{z\to x^+} F_{[n]}(z), \limsup_{z\to x^+} F_{[n]}(z)]\,.$$

(3) *For each real number a, the sets $\{F_{[n]} \le a\}$ and $\{F_{[n]} \ge a\}$ have compact components.*

(4) *The graph of $F_{[n]}$ is connected.*

(5) *$F_{[n]}$ has a perfect road at each point.*

(6) *Each of $\{F_{[n]} < a\}$ and $\{F_{[n]} > a\}$ is bilaterally c-dense in itself.*

(7) *Each of $\{F_{[n]} < a\}$ and $\{F_{[n]} > a\}$ is bilaterally dense in itself.*

DEFINITION 38. Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and f a function on $\mathbb{R}$. We say that f *has the monotonicity property relative to E* if for any interval $[a,b]$ the conditions $f'_E(x)$ exists a.e. in $[a,b]$ and $f'_E(x) \ge \alpha$ (resp. $f'_E(x) \le \alpha$) a.e. in $[a,b]$ imply that the function $f(x) - \alpha x$ (resp. $\alpha x - f(x)$) is nondecreasing on $[a,b]$.

THEOREM 39. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and let f be a function. If E is bilateral and satisfies the intersection condition, and f is E-differentiable, then f has the monotonicity property relative to E.*

Proof. See Theorem 6.6.1 of [3]. ■

COROLLARY 40. *Let $[a,b]$ be an interval, and α be any constant. If $F_{[n]} \ge \alpha$ (or $F_{[n]} \le \alpha$) then*

(a) *$F_{[n-1]}(x) - \alpha x$ (resp. $\alpha x - F_{[n-1]}(x)$) is nondecreasing and continuous on $[a,b]$,*

(b) *$F^{(n)}$ exists and $F^{(n)} = F_{[n]}$ on $[a,b]$.*

Proof. The assertion follows directly from Theorems 35, 39 and 34. ■

DEFINITION 41. Let f be a function defined on $\mathbb{R}$. If for any interval (a,b), $f^{-1}(a,b) \ne \emptyset$ implies $m\{x : f(x) \in (a,b)\} > 0$ then we say that a function f has the *Denjoy property* (here m denotes the Lebesgue measure on $\mathbb{R}$).

THEOREM 42. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and let f be an E-differentiable function that has the monotonicity property relative to E. If f'_E is Darboux Baire 1, then f'_E has the Denjoy property.*

Proof. This is Theorem 6.7 of [3]. ■

COROLLARY 43. *$F_{[n]}$ has the Denjoy property.*

Proof. The assertion follows directly from Theorems 35, 39, 42 and Corollary 36. ∎

THEOREM 44. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a nonporous system of paths satisfying the intersection condition. Suppose that f is an E-differentiable function with f'_E Baire 1. Then if f'_E attains the values M and $-M$ on an interval I_0, then there is a subinterval I of I_0 on which f is differentiable and f' attains both values M and $-M$.*

Proof. This is Theorem 8.1 of [3]. ∎

An immediate consequence of Theorems 44 and 35 is the following corollary.

COROLLARY 45. *Suppose $F_{[n]}(x)$ exists for all x in I_0 and let $M \geq 0$. If $F_{[n]}$ attains both M and $-M$ on I_0, then there is a subinterval I of I_0 on which $F_{[n]} = F^{(n)}$ and $F^{(n)}$ attains both M and $-M$ on I.*

Generalized Peano and selective derivatives. In this section we show that every generalized Peano derivative $F_{[n]}$ is a selective derivative of $F_{[n-1]}$.

DEFINITION 46. Let P_y be a set containing y so that y is a bilateral point of accumulation of P_y,

$$\lim_{z \in P_y, z \to y} \frac{F_{[n-1]}(z) - F_{[n-1]}(y)}{z - y} = F_{[n]}(y)$$

and

$$\left| \frac{F_{[n-1]}(z) - F_{[n-1]}(y)}{z - y} - F_{[n]}(y) \right| \leq 1 \quad \text{for every } z \in P_y\,.$$

Theorem 35 assures the existence of P_y.

To define the system $\{E_x : x \in \mathbb{R}\}$ of paths with respect to which a given nth generalized Peano derivative, $F_{[n]}$, is the path derivative of $F_{[n-1]}$, we begin with some notation.

For $x, y \in \mathbb{R}$ let $\delta(x, y) = \min\{1, |y - x|/3\}$. For $x \in \mathbb{R}$ and $M \in \mathbb{N}$ let

$$\begin{aligned} R_x = \bigcup \{P_y \cap [y, y + \delta(x,y)^2) : y \in \overline{P}_M(1, 1/M) \text{ and } y \text{ is right isolated} \\ \text{from } \overline{P}_N(1, 1/N) \text{ for } N \in \mathbb{N}\}\,, \\ L_x = \bigcup \{P_y \cap (y - \delta(x,y)^2, y] : y \in \overline{P}_M(1, 1/M) \text{ and } y \text{ is left isolated} \\ \text{from } \overline{P}_N(1, 1/N) \text{ for } N \in \mathbb{N}\}\,. \end{aligned}$$

Definition 47. Let $x \in \mathbb{R}$. If there is an $M_x \in \mathbb{N}$ such that x is a bilateral point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, then let

$$E_x = \overline{P}_{M_x}(1, 1/M_x) \cup R_x \cup L_x\,.$$

If x is a right isolated point of $\overline{P}_M(1, 1/M)$ for every positive constant M but there is an M_x so that x is a left point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, or if x is a left isolated point of $\overline{P}_M(1, 1/M)$ for every positive constant M but there is an M_x so that x is a right point of accumulation of $\overline{P}_{M_x}(1, 1/M_x)$, let

$$E_x = \overline{P}_{M_x}(1, 1/M_x) \cup P_x \cup R_x \cup L_x\,.$$

Finally, if x is an isolated point of $\overline{P}_M(1, 1/M)$ for every positive constant M then let $M_x = 1$ and let

$$E_x = \overline{P}_{M_x}(1, 1/M_x) \cup P_x \cup R_x \cup L_x\,.$$

Definition 48. Let E be the system of paths $\{E_x : x \in \mathbb{R}\}$.

Lemma 49. *Let $n \in \mathbb{N}$ and let F be a function defined on $\mathbb{R}$ such that $F_{[n]}(x)$ exists at every $x \in \mathbb{R}$. Then E is bilateral and satisfies the I.I.C. condition.*

Proof. Clearly E is bilateral. We will prove a stronger condition than I.I.C.: for any two points x and y, $E_x \cap E_y \cap (x, y) \neq \emptyset$. Let $x < y$ be any two points. Suppose $M_x \le M_y$. If x is a right point of accumulation of $\overline{P}_{M_x}(1, 1/M_x) \subset \overline{P}_{M_y}(1, 1/M_y)$, then $E_x \cap E_y \cap (x, y) \neq \emptyset$.

If x is a right isolated point of $\overline{P}_{M_x}(1, 1/M_x)$, then by choice of M_x, x is a right isolated point of $\overline{P}_M(1, 1/M)$ for every $M \in \mathbb{N}$ and $x \in \overline{P}_{M_y}(1, 1/M_y)$. Thus

$$\emptyset \neq P_x \cap [x, x + \delta(x, y)^2) \cap (x, y) \subset E_x \cap E_y \cap (x, y)\,.$$

If $M_x > M_y$ and if y is a left point of accumulation of $\overline{P}_{M_y}(1, 1/M_y) \subset \overline{P}_{M_x}(1, 1/M_x)$ then $E_x \cap E_y \cap (x, y) \neq \emptyset$.

If y is a left isolated point of $\overline{P}_{M_y}(1, 1/M_y)$, then by a similar argument $E_x \cap E_y \cap (x, y) \neq \emptyset$. Therefore E satisfies the I.I.C. condition. ∎

Theorem 50. *Let F be a continuous function defined on $\mathbb{R}$ so that $F_{[n]}$ exists at every point $x \in \mathbb{R}$. Then $F_{[n-1]}$ is E-differentiable with $F'_{[n-1]E}(x) = F_{[n]}(x)$.*

Proof. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then there is an η with $\varepsilon > \eta > 0$ such that

$$\left|\frac{F_{[n-1]}(y) - F_{[n-1]}(x)}{y - x} - F_{[n]}(x)\right| < \varepsilon \tag{20}$$

whenever $|y - x| < \eta$ where $y \in \overline{P}_{M_x}(1, 1/M_x)$ or $y \in P_x$. Let $z \in E_x$ be such that $|z - x| < \eta/2$. If $z \in P_y$ for some $y \in \overline{P}_{M_x}(1, 1/M_x)$ such that y is an isolated point of $\overline{P}_N(1,1/N)$ from either left or right, and for every positive constant N, then $\eta/2 > |z-x| \ge |x-y|-|y-z| \ge 2\delta(x,y)-\delta(x,y) = \delta(x,y)$. Therefore $|y-x| \le |y-z| + |x-z| < \delta(x,y)+\eta/2 < \eta$. Hence by (20),

$$\left|\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x} - F_{[n]}(x)\right| < \varepsilon\,. \tag{21}$$

Thus

$$\begin{aligned}
&\left|\frac{F_{[n-1]}(z)-F_{[n-1]}(x)}{z-x} - F_{[n]}(x)\right| \\
&\qquad = \left|\left(\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x} - F_{[n]}(x)\right)\frac{y-x}{z-x}\right. \\
&\qquad\quad + \left(\frac{F_{[n-1]}(z)-F_{[n-1]}(y)}{z-y} - F_{[n]}(y)\right)\frac{z-y}{z-x} \\
&\qquad\quad \left. + \frac{z-y}{z-x}(F_{[n]}(y)-F_{[n]}(x))\right| \\
&\qquad \le \left|\frac{F_{[n-1]}(y)-F_{[n-1]}(x)}{y-x} - F_{[n]}(x)\right|\left|1-\frac{z-y}{z-x}\right| \\
&\qquad\quad + \left|\frac{F_{[n-1]}(z)-F_{[n-1]}(y)}{z-y} - F_{[n]}(y)\right|\left|\frac{z-y}{z-x}\right| \\
&\qquad\quad + \left|\frac{z-y}{z-x}\right|(|F_{[n]}(x)|+|F_{[n]}(y)|)\,.
\end{aligned}$$

By (21), Theorem 17 and the relationship between x, y and z, the above is

$$\begin{aligned}
&\le \varepsilon\left(1+\frac{\delta(x,y)^2}{\delta(x,y)}\right) + 1\cdot\frac{\delta(x,y)^2}{\delta(x,y)} + \frac{\delta(x,y)^2}{\delta(x,y)}4M \\
&\le 2\varepsilon + \delta(x,y)(1+4M) \le 2\varepsilon + \frac{\varepsilon}{2}(1+4M)
\end{aligned}$$

where M is a constant coming from Theorem 17. Since ε was arbitrary we conclude that $F'_{[n-1]E}(x)$ exists and equals $F_{[n]}(x)$. ∎

Definition 51. If for a given function F there is a function p of two variables, called a *selection*, satisfying $p(x,y) = p(y,x)$ and $p(x,y) \in (x,y)$, so that

$$\lim_{y\to x}\frac{F(p(x,y))-F(x)}{p(x,y)-x} \tag{22}$$

exists, we say that F is *selectively differentiable* at x; the limit in (22) is then called a *selective derivative* of F at x and denoted by $F'_p(x)$.

Corollary 52. *Let F be a continuous function defined on $\mathbb{R}$ so that $F_{[n]}$ exists at every $x \in \mathbb{R}$. Then $F_{[n]}$ is a selective derivative of $F_{[n-1]}$.*

Proof. Define a selection $p(x,y)$ as follows: If $x < y$, let $p(x,y) = z$, where z is any point in $E_x \cap E_y \cap (x,y)$; if $x = y$, let $p(x,x) = x$. Then for fixed x_0 we have

$$\lim_{y\to x_0} \frac{F_{[n-1]}(p(x_0,y)) - F_{[n-1]}(x_0)}{p(x_0,y) - x_0} = \lim_{z\to x_0} \frac{F_{[n-1]}(z) - F_{[n-1]}(x_0)}{z - x_0}\,.$$

Since $z \in E_x$ we see that the above limit exists and equals $F_{[n]}(x_0)$. ■

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