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An invariant of bi-Lipschitz maps

by

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Abstract. A new numerical invariant for the category of compact metric spaces and Lipschitz maps is introduced. This invariant takes a value less than or equal to 1 for compact metric spaces that are Lipschitz isomorphic to ultrametric ones. Furthermore, a theorem is provided which makes it possible to compute this invariant for a large class of spaces. In particular, by utilizing this invariant, it is shown that neither a fat Cantor set nor the set $\{0\} \cup \{1/n\}_{n\geq 1}$ is Lipschitz isomorphic to an ultrametric space.

1. Introduction. A metric space (M, d) is an *ultrametric space* if the metric satisfies a stronger form of the triangle inequality: for all $x, y, z \in M$,

$$d(x,y) \leq \max\{d(x,z), d(y,z)\}\,.$$

One special property of these spaces is the fact that any two closed balls in an ultrametric space are either disjoint or one is contained in the other. Consequently, every ultrametric space is totally disconnected. Indeed, it is well known that a non-empty compact perfect ultrametric space is a Cantor space, that is, a metric space homeomorphic to the Cantor set. Conversely, it is also well known that every totally disconnected compact metric space has a compatible ultrametric. In this paper we consider the problem of determining whether every Cantor space is Lipschitz isomorphic to an ultrametric one. We settle this issue by constructing a new numerical invariant for the category of compact metric spaces and Lipschitz maps and show that many Cantor spaces are not Lipschitz isomorphic to ultrametric spaces. We call this invariant the *logarithmic ratio*. A non-logarithmic version of this has been introduced by D. Sullivan [4] to study differentiable structures on fractal-like sets. The logarithmic ratio takes a value less than or equal to 1 for compact spaces which are Lipschitz isomorphic to ultrametric ones. We also prove a theorem enabling us to compute the logarithmic ratio for a large class of spaces. In particular, we use the logarithmic ratio to show that

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neither a fat Cantor set nor the set $\{0\} \cup \{1/n\}_{n\ge 1}$ is Lipschitz isomorphic to an ultrametric space.

It appears to us that most of the known invariants of the Lipschitz category are dimensions. The logarithmic ratio, however, is not obviously a dimension. In particular, the logarithmic ratio of a compact connected subset (containing more than one point) of a Euclidean space is infinite.

Compact metric spaces Lipschitz isomorphic to an ultrametric space have also been studied by G. Michon [1–3]. Making more precise an observation in [1], note first that if (M,d) is a metric space and if $\delta(x,y)$ is the infimum of the numbers $\varepsilon > 0$ for which there is a finite sequence $x = x_0, x_1, \ldots, x_k = y$ in M with $d(x_{i-1}, x_i) < \varepsilon$, then δ is an ultrapseudometric on M, $\delta \le d$, and $\varrho \le \delta$ for all ultrapseudometrics ϱ on M with $\varrho \le d$. Thus, if ϱ is an ultrametric on M Lipschitz equivalent to d, i.e., $a\varrho \le d \le b\varrho$ for some $0 < a \le b$, then $\delta \le d \le (b/a)\delta$ implying that δ, too, is an ultrametric Lipschitz equivalent to d.

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2. The invariant. Let (M,d) be a compact metric space and let $\mathcal{A}$ denote the family of all (finite) *clopen* partitions of M. That is, a family α is in $\mathcal{A}$ if and only if $\alpha = \{A_1, \ldots, A_n\}$ for some $n \ge 0$ with A_i non-void and clopen (i.e., both closed and open) in M for $1 \le i \le n$, and $M = \coprod_{i=1}^n A_i$. Note that $\mathcal{A} = \{\emptyset\}$ if $M = \emptyset$ and that $n = 0$ if $\alpha = \emptyset$. Then $\mathcal{A}$ is partially ordered by refinement; for $\alpha, \beta \in \mathcal{A}$ we write $\alpha \prec \beta$ if and only if β is a refinement of α. This means that for any $B \in \beta$ there is an $A \in \alpha$ with $B \subset A$. For $\alpha = \{A_1, \ldots, A_n\}$ and $\beta = \{B_1, \ldots, B_m\}$ in $\mathcal{A}$ their join $\alpha \vee \beta \in \mathcal{A}$ is given by

$$\alpha \vee \beta = \{A_i \cap B_j \mid 1 \le i \le n,\ 1 \le j \le m\} \setminus \{\emptyset\}\,.$$

Clearly $\alpha \prec \alpha \vee \beta$ and $\beta \prec \alpha \vee \beta$.

Definition 2.1. Let $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{A}$ with $n = \mathrm{card}(\alpha)$. Define the *diameter* $\delta(\alpha) \in [0,\infty)$ of α by

$$\delta(\alpha) = \max\{\mathrm{diam}(A_i) \mid 1 \le i \le n\}$$

if $n \ge 1$ and by $\delta(\alpha) = 0$ if $n = 0$. Define the *gap* $\gamma(\alpha) \in [0,\infty)$ of α by

$$\gamma(\alpha) = \min\{d(x,y) \mid x \in A_i,\ y \in A_j,\ i \ne j\}$$

if $n > 1$ and by $\gamma(\alpha) = \mathrm{diam}(M)$ if $n \le 1$ (with the convention that $\mathrm{diam}(\emptyset) = 0$). Moreover, define the *logarithmic ratio* $R(\alpha) \in [0,\infty]$ of α by

$$R(\alpha) = \frac{\log \gamma(\alpha)}{\log \delta(\alpha)}$$

if $0 < \delta(\alpha) < 1$ and $(0 <)\ \gamma(\alpha) < 1$, by $R(\alpha) = 0$ if $\delta(\alpha) = 0$ and by $R(\alpha) = \infty$ if either $0 < \delta(\alpha) < 1$ and $\gamma(\alpha) \geq 1$ or $\delta(\alpha) \geq 1$.

Trivially, if $\alpha \prec \beta$, then $\delta(\beta) \leq \delta(\alpha)$. It follows from a straightforward argument that for any $\alpha, \beta \in \mathcal{A}$ we have

$$\gamma(\alpha \vee \beta) = \min\{\gamma(\alpha), \gamma(\beta)\}\,.$$

Hence, $\alpha \prec \beta$ implies $\gamma(\beta) \leq \gamma(\alpha)$. Also, if $\alpha, \beta \in \mathcal{A}$ and $\alpha \neq \beta$, then $\gamma(\alpha \vee \beta) \leq \max\{\delta(\alpha), \delta(\beta)\}$ (this estimate is sharp, i.e., equality can appear). Furthermore, if $\alpha, \beta \in \mathcal{A}$ with $\gamma(\alpha) \leq \gamma(\beta)$, then $R(\alpha \vee \beta) \leq R(\alpha)$.

LEMMA 2.2. *Suppose that M satisfies the following condition*:

(2.1) *For each $r > 0$ there is an $\alpha \in \mathcal{A}$ with $0 < \delta(\alpha) < r$.*

Then

$$\limsup_{r \to 0} \{\gamma(\alpha) \mid \alpha \in \mathcal{A},\ 0 < \delta(\alpha) < r\} = 0\,.$$

P r o o f. Deny. Then there exist $\varepsilon > 0$ and $\alpha, \beta \in \mathcal{A}$ such that $\alpha \neq \beta$ and $\gamma(\beta) \geq \gamma(\alpha) \geq \varepsilon > \max\{\delta(\alpha), \delta(\beta)\}$. This implies that $\varepsilon \leq \gamma(\alpha) = \gamma(\alpha \vee \beta) \leq \max\{\delta(\alpha), \delta(\beta)\} < \varepsilon$, a contradiction. ■

Note that M satisfies (2.1) if and only if M is totally disconnected but not discrete.

DEFINITION 2.3. Let (M, d) be a compact metric space and let $\mathcal{A}$ denote the family of all clopen partitions of M. Then the *logarithmic ratio* of M (with respect to the metric d) is defined by

$$R(M, d) = \liminf_{r \to 0} \{R(\alpha) \mid \alpha \in \mathcal{A} \text{ and } \delta(\alpha) < r\}\,,$$

with the convention that $\inf \emptyset = +\infty$.

Observe that $R(M, d)$ is defined and is in $[0, \infty]$ for every compact metric space M. Moreover, $R(M, d) = 0$ if M is discrete (or, equivalently, finite) and $R(M, d) = \infty$ if M is not totally disconnected.

The most important property of the logarithmic ratio is the fact that it is an invariant of the Lipschitz category. Specifically, we have the following result.

THEOREM 2.4. *Suppose (M_1, d_1) and (M_2, d_2) are Lipschitz isomorphic compact metric spaces. Then $R(M_1, d_1) = R(M_2, d_2)$.*

P r o o f. In view of the preceding remarks, we may as well assume that (M_1, d_1) is totally disconnected but not discrete. Suppose $f : (M_1, d_1) \to (M_2, d_2)$ is a Lipschitz isomorphism. Then there is a constant $C \geq 1$ such that

$$\frac{1}{C} d_1(x, y) \leq d_2(f(x), f(y)) \leq C d_1(x, y) \quad \text{for all } x, y \in M_1\,.$$

Let $\mathcal{A}_i$ denote the family of all clopen partitions of M_i for $i = 1, 2$. Then f induces a bijection $F : \mathcal{A}_1 \to \mathcal{A}_2$ by

$$F(\alpha) = \{f(A_1), \dots, f(A_n)\} \in \mathcal{A}_2$$

whenever $\alpha = \{A_1, \dots, A_n\} \in \mathcal{A}_1$. We will prove the theorem in four steps.

Step 1. Let $\alpha \in \mathcal{A}_1$; then $\delta(\alpha) > 0$. If u and v belong to the same member of α with $d_1(u, v) = \delta(\alpha)$, then $f(u)$ and $f(v)$ are in the same member of $F(\alpha)$ and therefore

$$\delta(\alpha) = d_1(u, v) \le C d_2(f(u), f(v)) \le C\delta(F(\alpha)) .$$

By symmetry, we conclude from this that

$$\frac{1}{C}\delta(\alpha) \le \delta(F(\alpha)) \le C\delta(\alpha) . \tag{2.2}$$

Step 2. Let $\alpha \in \mathcal{A}_1$ with $\alpha \neq \{M_1\}$. Suppose x and y belong to distinct members of α with $d_1(x, y) = \gamma(\alpha)$. Then $f(x)$ and $f(y)$ belong to distinct members of $F(\alpha)$ and so

$$\gamma(F(\alpha)) \le d_2(f(x), f(y)) \le C d_1(x, y) = C\gamma(\alpha) .$$

By symmetry, we conclude from this that

$$\frac{1}{C}\gamma(\alpha) \le \gamma(F(\alpha)) \le C\gamma(\alpha) . \tag{2.3}$$

Observe that (2.3) is also valid if $\alpha = \{M_1\}$.

Step 3. Let $\alpha \in \mathcal{A}_1$, let $\delta(\alpha) < 1/C$ and let $\gamma(\alpha) < 1/C$. From (2.2) and (2.3) we get the following:

$$0 < \log C^{-1} + \log \delta(\alpha)^{-1} \le \log \delta(F(\alpha))^{-1} \le \log C + \log \delta(\alpha)^{-1} ,$$
$$0 < \log C^{-1} + \log \gamma(\alpha)^{-1} \le \log \gamma(F(\alpha))^{-1} \le \log C + \log \gamma(\alpha)^{-1} .$$

Therefore,

$$R(F(\alpha)) = \frac{\log \gamma(F(\alpha))^{-1}}{\log \delta(F(\alpha))^{-1}} \ge \frac{\log C^{-1} + \log \gamma(\alpha)^{-1}}{\log C + \log \delta(\alpha)^{-1}} .$$

Step 4. Now suppose that $\beta \in \mathcal{A}_2$. Since F is a bijection, there is an $\alpha \in \mathcal{A}_1$ such that $\beta = F(\alpha)$. Hence, $\delta(\beta) \ge \delta(\alpha)/C$ and if $\delta(\beta) < r$, then $\delta(\alpha) < Cr$. Therefore,

$$\begin{aligned} R(M_2, d_2) &= \liminf_{r \to 0} \{R(\beta) \mid \beta \in \mathcal{A}_2 \text{ and } \delta(\beta) < r\} \\ &\ge \liminf_{r \to 0} \{R(F(\alpha)) \mid \alpha \in \mathcal{A}_1 \text{ and } \delta(\alpha) < Cr\} \\ &= \liminf_{r \to 0} \{R(F(\alpha)) \mid \alpha \in \mathcal{A}_1 \text{ and } \delta(\alpha) < r\} \\ &\ge \liminf_{r \to 0} \left\{ \frac{\log \gamma(\alpha) + \log C}{\log \delta(\alpha) + \log C^{-1}} \,\middle|\, \alpha \in \mathcal{A}_1 \text{ and } \delta(\alpha) < r \right\} \\ &= R(M_1, d_1) , \end{aligned}$$

where the last inequality and equality follow from Lemma 2.2. By symmetry, $R(M_1, d_1) \geq R(M_2, d_2)$ and so $R(M_1, d_1) = R(M_2, d_2)$. ■

We remark that the same argument shows that $R(M_1, d_1) = R(M_2, d_2)$ if there is a homeomorphism $f : (M_1, d_1) \to (M_2, d_2)$ such that

$$C_1 d_1(x, y)^\varepsilon \leq d_2(f(x), f(y)) \leq C_2 d_1(x, y)^\varepsilon$$

for some positive constants ε, C_1 and C_2.

In general, it is not very easy to compute the logarithmic ratio. But, if there is a suitably nice sequence $\{\alpha_n\}_{n\geq 1} \subset \mathcal{A}$ of clopen partitions of M, then the limit along this sequence is equal to the true limit. In order to be more precise, we have to make the following definitions.

Definition 2.5. Two points $x, y \in M$ are said to be *associated endpoints* of (M, d) if there is a clopen subset $A \subset M$ such that $x \in A$, $y \in M \setminus A$ and for the partition $\alpha = \{A, M \setminus A\}$ we have $\gamma(\alpha) = d(x, y)$.

Clearly, if $\beta \in \mathcal{A} \setminus \{\{M\}\}$ with $\gamma(\beta) = d(x, y)$ for some $x, y \in M$, then there are $B_1, B_2 \in \beta$, $B_1 \neq B_2$, such that $x \in B_1$ and $y \in B_2$. Therefore, for the partition $\alpha = \{B_1, M \setminus B_1\}$ we have $x \in B_1$ and $y \in B_2 \subset M \setminus B_1$ with $\gamma(\alpha) = d(x, y)$. Hence, x and y are associated endpoints of M.

Definition 2.6. Let A be a clopen subset of (M, d). Then the *largest gap* in A is defined as

$$\Gamma(A) = \sup\{d(x, y) \mid x, y \in A \text{ and } x, y \text{ are associated endpoints of } A\},$$

with the convention that $\sup \emptyset = 0$.

Theorem 2.7. *Suppose there is a sequence $\{\alpha_n\}_{n\geq 1} \subset \mathcal{A}$ such that*

(a) $\delta(\alpha_n) > 0$ *and $\delta(\alpha_n)$ converges monotonically to zero as n tends to infinity*, *and*

(b) *there is a constant $C \geq 1$ such that for all $n \geq 1$ there is an $A \in \alpha_n$ with* $\operatorname{diam}(A) = \delta(\alpha_n)$ *and* $\Gamma(A) \leq C\gamma(\alpha_{n+1})$.

Then $R(M, d) = \liminf_{n\to\infty} R(\alpha_n)$.

Proof. We may assume that $\delta(\alpha_n) < 1$ and $\gamma(\alpha_n) < 1/C$ for each n. Clearly there exists a sequence $\{\beta_n\}_{n\geq 1} \subset \mathcal{A}$ such that $\delta(\beta_n)$ converges monotonically to 0 and

$$R(M, d) = \lim_{n\to\infty} R(\beta_n).$$

We may also assume that $\delta(\beta_1) < \delta(\alpha_1)$. Then for each $k \geq 1$ we may choose $n(k) \geq 1$ such that

$$\delta(\alpha_{n(k)+1}) \leq \delta(\beta_k) < \delta(\alpha_{n(k)}).$$

Furthermore, there is an $A \in \alpha_{n(k)}$ such that $\operatorname{diam}(A) = \delta(\alpha_{n(k)})$ and $\Gamma(A) \leq C\gamma(\alpha_{n(k)+1})$; so $\delta(\beta_k) < \operatorname{diam}(A)$. Thus, there is a $B \in \beta_k$ such that $\emptyset \neq A \cap B \neq A$. Because both A and B are clopen, $\{A \cap B, A \setminus B\}$ is a clopen partition of A. Therefore, there are associated endpoints $x \in A \cap B$ and $y \in A \setminus B$ of A. But $y \in B'$ for some $B' \in \beta_k$, $B' \neq B$. Hence,

$$\gamma(\beta_k) \leq \operatorname{dist}(B, B') \leq d(x, y)\,.$$

On the other hand, $d(x, y) \leq \Gamma(A)$ by definition; so we have

$$\gamma(\beta_k) \leq C\gamma(\alpha_{n(k)+1}) < 1\,.$$

Therefore, for each $k \geq 1$

$$R(\beta_k) = \frac{\log \gamma(\beta_k)}{\log \delta(\beta_k)} \geq \frac{\log(C\gamma(\alpha_{n(k)+1}))}{\log \delta(\alpha_{n(k)+1})} = R(\alpha_{n(k)+1}) + \frac{\log C}{\log \delta(\alpha_{n(k)+1})}\,.$$

Hence,

$$\liminf_{k\to\infty} R(\beta_k) \geq \liminf_{k\to\infty} R(\alpha_{n(k)+1})$$

and thus $R(M, d) = \liminf_{n\to\infty} R(\alpha_n)$. ∎

We use this result to compute the logarithmic ratio of two familiar spaces.

Corollary 2.8. *Let M be the standard Cantor subset of the interval $[0, 1]$. Then $R(M, |\cdot|) = 1$.*

Proof. Let $\{\alpha_n\}_{n\geq 1} \subset \mathcal{A}$ be the sequence of standard partitions of M with $\operatorname{card}(\alpha_n) = 2^n$. Then $\delta(\alpha_n) = 1/3^n$ and $\gamma(\alpha_n) = 1/3^n$ and so $R(\alpha_n) = 1$ for all $n \geq 1$. Furthermore, for each $A \in \alpha_n$ we have $\Gamma(A) = 1/3^{n+1} = \gamma(\alpha_{n+1})$ for all $n \geq 1$. Thus, by Theorem 2.7 we have $R(M, |\cdot|) = \liminf_{n\to\infty} R(\alpha_n) = 1$. ∎

More interesting, however, is the case when M is a fat Cantor subset of the interval $[0, 1]$. Specifically, M is constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the nth step has length $s3^{-n}$ with $0 < s < 1$. It is a standard exercise to show that the Lebesgue measure of M is equal to $1 - s$.

Corollary 2.9. *Let M be a fat Cantor set with $0<s<1$. Then $R(M, |\cdot|) = \log 3/\log 2$.*

Proof. Again let $\{\alpha_n\}_{n\geq 1} \subset \mathcal{A}$ be the sequence of standard partitions of M. Then $\gamma(\alpha_n) = s3^{-n}$ and

$$\delta(\alpha_n) = \frac{1}{2^n} - \frac{s}{2^{n+1}} \sum_{i=1}^{n} \left(\frac{2}{3}\right)^i = \frac{1-s}{2^n} + \frac{s}{3^n}\,, \qquad n \geq 1\,.$$

Hence,

$$R(\alpha_n) = \frac{\log(s/3^n)}{\log[(1-s)/2^n + s/3^n]}\,, \qquad n \geq 1\,.$$

Also, for each $n \geq 1$ and each $A \in \alpha_n$ we have

$$\Gamma(A) \geq \frac{s}{3^{n+1}} = \gamma(\alpha_{n+1}) .$$

On the other hand, two points x and y of A with $x < y$ are associated endpoints of A if and only if $(x, y) \cap A = \emptyset$. If x, y are such endpoints of A, then they are in the same member of α_m for the last time at some level $m \geq n$ and so in different members A_{m+1}, A'_{m+1} of α_{m+1}. Hence, x is the right hand endpoint of A_{m+1} and y is the left hand endpoint of A'_{m+1}. This means that

$$y - x = \frac{s}{3^{m+1}} \leq \frac{s}{3^{n+1}}$$

and so

$$\Gamma(A) \leq \frac{s}{3^{n+1}} = \gamma(\alpha_{n+1}) .$$

By combining the two inequalities, we see that for each $A \in \alpha_n$ and for each $n \geq 1$,

$$\Gamma(A) = \frac{s}{3^{n+1}} = \gamma(\alpha_{n+1}) .$$

Therefore,

$$R(M, |\cdot|) = \liminf_{n\to\infty} R(\alpha_n) = \log 3/\log 2. \ ■$$

Since the logarithmic ratio is an invariant of the Lipschitz category, we see immediately that the Cantor ternary set and a fat Cantor set are not Lipschitz isomorphic. Observe that if M is the Cantor ternary set, we may define an ultrametric d on M by setting $d(x, y) = 1/3^n$ if x and y are in the same member of α_n but in different members of α_{n+1} (with $\alpha_0 = \{M\}$). Then $|x - y| \leq d(x, y) \leq 3|x - y|$ and so $(M, |\cdot|)$ is Lipschitz isomorphic to (M, d). This raises the question whether a fat Cantor set is Lipschitz isomorphic to an ultrametric space. The following theorem provides the answer.

THEOREM 2.10. *Let (M, d) be a compact ultrametric space. Then $R(M, d) \leq 1$.*

Proof. If M is discrete, we have $R(M, d) = 0$. Now assume that M is not discrete. Let $\{r_n\}_{n\geq 1} \subset (0, \mathrm{diam}(M))$ converge to zero, and for each $n \geq 1$ let α_n be the cover of M by closed r_n-balls. Then $\alpha_n \in \mathcal{A}$ and $0 < \delta(\alpha_n) \leq r_n < \gamma(\alpha_n)$. Furthermore, for sufficiently large n we have $\gamma(\alpha_n) < 1$, which implies $R(\alpha_n) < 1$. Therefore, by definition,

$$R(M, d) \leq \liminf_{n\to\infty} R(\alpha_n) \leq 1. \ ■$$

As an immediate corollary of this theorem we get:

COROLLARY 2.11. *A fat Cantor set is not Lipschitz isomorphic to an ultrametric space.*

There is also a partial converse to the above theorem.

Proposition 2.12. *Let (M, d) be a compact metric space and suppose that there is a sequence of clopen partitions $\{\alpha_n\}_{n\ge 1} \subset \mathcal{A}$ of (M, d) such that*

(a) $\alpha_n \prec \alpha_{n+1}$, $n \ge 1$;

(b) $0 < \delta(\alpha_n) \to 0$ *and there is a constant* $C \ge 1$ *such that* $\delta(\alpha_n)/\delta(\alpha_{n+1}) \le C$, $n \ge 1$;

(c) $\limsup_{n\to\infty} R(\alpha_n) \le 1$.

Then there is a compatible ultrametric ϱ for M with the following property: *For each $\varepsilon > 0$ there is a constant $K > 0$ such that*

$$K\varrho(x, y)^{1+\varepsilon} \le d(x, y) \le \varrho(x, y) \quad \textit{for all } x, y \in M\,.$$

Proof. Observe that conditions (b) and (c) imply that $R(M, d) \le 1$. Set $\alpha_0 = \{M\}$ and define an ultrametric ϱ for the set M by setting $\varrho(x, y) = \delta(\alpha_n)$ whenever x and y are in the same member of α_n but in distinct members of α_{n+1}. For these x, y and n, we then have

$$\gamma(\alpha_{n+1}) \le d(x, y) \le \delta(\alpha_n) = \varrho(x, y)\,.$$

Thus, $d \le \varrho$, and it is easy to see that the metric ϱ is also continuous and hence compatible.

Let $\varepsilon > 0$. By (b) and (c), there is an $m \ge 1$ such that if $n > m$, then $\delta(\alpha_n) < 1$, $\gamma(\alpha_n) < 1$ and

$$\frac{\log\gamma(\alpha_n)}{\log\delta(\alpha_n)} \le 1 + \varepsilon\,,$$

which gives $\delta(\alpha_n)^{1+\varepsilon} \le \gamma(\alpha_n)$. Hence, if x, y and n are as in the definition of ϱ and if $n \ge m$, then

$$\delta(\alpha_{n+1})^{1+\varepsilon} \le \gamma(\alpha_{n+1}) \le d(x, y)\,.$$

But $\varrho(x, y) = \delta(\alpha_n) \le C\delta(\alpha_{n+1})$ by hypothesis. Then, for any two points x and y in M which are in the same member of α_m, we get $\varrho(x, y)^{1+\varepsilon} \le C^{1+\varepsilon} d(x, y)$. On the other hand, if x and y are in different members of α_m, we have

$$\varrho(x, y)^{1+\varepsilon} \le \frac{\delta(\alpha_0)^{1+\varepsilon}}{\gamma(\alpha_m)} d(x, y)\,.$$

Now let $K = \min\{C^{-(1+\varepsilon)}, \gamma(\alpha_m)\delta(\alpha_0)^{-(1+\varepsilon)}\}$. Then $K\varrho(x, y)^{1+\varepsilon} \le d(x, y)$ for all $x, y \in M$. ∎

We observe that the above proof establishes the following proposition as well.

Proposition 2.13. *Let (M, d) be a compact metric space and suppose that there is a sequence of clopen partitions $\{\alpha_n\}_{n\ge 1} \subset \mathcal{A}$ of (M, d) satisfying conditions* (a) *and* (b) *of Proposition* 2.12 *such that* $(R(M, d) \le)$

$\limsup_{n\to\infty} R(\alpha_n) < 1$. *Then* (M, d) *is Lipschitz isomorphic to an ultrametric space.*

There is a quick way to decide that a compact metric space is not Lipschitz isomorphic to an ultrametric space. Let $M = \{1/k\}_{k\geq 1} \cup \{0\} \subset \mathbb{R}^1$. Let $\{\alpha_n\}_{n>1}$ be the sequence of clopen partitions of M defined by

$$\alpha_n = \left\{ M \setminus \left\{\frac{1}{k}\right\}_{k=1}^{n-1}, \{1\}, \left\{\frac{1}{2}\right\}, \ldots, \left\{\frac{1}{n-1}\right\} \right\}.$$

Then for each $n > 1$ we have $\delta(\alpha_n) = 1/n$ and $\gamma(\alpha_n) = 1/(n-1) - 1/n$. Moreover, $\alpha_n \prec \alpha_{n+1}$ for all $n > 1$ and for the set $A \in \alpha_n$ with $\operatorname{diam}(A) = \delta(\alpha_n)$ we have

$$\Gamma(A) = \frac{1}{n} - \frac{1}{n+1} = \gamma(\alpha_{n+1})\,.$$

Therefore,

$$R(M, |\cdot|) = \liminf_{n\to\infty} R(\alpha_n) = \liminf_{n\to\infty} \frac{\log[1/(n-1) - 1/n]}{\log(1/n)} = 2\,.$$

Consequently, $(M, |\cdot|)$ is not Lipschitz isomorphic to an ultrametric space. Indeed, this argument proves the following proposition.

PROPOSITION 2.14. *Let* (M, d) *be a compact metric space. Suppose there is a sequence* $\{x_n\}_{n\geq 1}$ *in* M *converging to some* $x \in M$ *such that the subspace* $N = \{x_n\}_{n\geq 1} \cup \{x\}$ *is Lipschitz isomorphic to the space* $(\{1/n\}_{n\geq 1} \cup \{0\}, |\cdot|)$. *Then* (M, d) *is not Lipschitz isomorphic to an ultrametric space.*

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