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ω_1-Souslin trees under countable support iterations

by

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Abstract. We show the property "is proper and preserves every ω_1-Souslin tree" is preserved by countable support iteration.

Introduction. In [1], the forcing axiom SAD is introduced and its consistency is established by forcing. It is also shown that the forcing axiom does not imply the nonexistence of ω_1-Souslin trees by constructing a pair of an ω_1-Souslin tree and a notion of forcing in such a way that the ω_1-Souslin tree remains an ω_1-Souslin tree in the generic extensions via the forcing. In [2], a general theory of countable support iterations is developed and stronger versions of SAD are shown to be consistent.

We show the property "is proper and preserves every ω_1-Souslin tree" is preserved by countable support iteration. As an application we remark that countable support iterations for getting SAD preserve every ω_1-Souslin tree in the ground model.

0. Preliminaries

(0.0) Definition. A triple $(P, \leq, 1)$ is a *preorder* iff $\leq$ is a reflexive and transitive binary relation on P with a greatest element 1. The symbol $\dot{G}$ usually denotes the canonical P-name for a P-generic filter over the ground model V. For an element x in V, we usually use x itself instead of $\check{x}$ to denote its P-name. The preorder is *separative* iff for any $p, q \in P$, $q \Vdash_P$ "$p \in \dot{G}$" implies $q \leq p$. We consider separative preorders in this note and so a preorder is always a separative one. For a formula φ, we simply write $\Vdash_P$ "φ" instead of $1 \Vdash_P$ "φ". A subset D of P is *predense below* q in P iff $q \Vdash_P$ "$D \cap \dot{G} \neq \emptyset$".

For a set x, let $TC(x)$ denote the transitive closure of x. For a regular cardinal θ, let $H_\theta = \{x : |TC(x)| < \theta\}$. A countable subset N of H_θ is a *countable elementary substructure* of H_θ iff the structure $(N, \in)$ is an elementary substructure of $(H_\theta, \in)$. For a regular cardinal θ and a countable elementary substructure N of H_θ with $(P,\le,1)\in N$, a condition q in P is (P,N)-*generic* iff for any dense subset $D\in N$ of P, $D\cap N$ is predense below q. For a P-generic filter G over V and a P-name τ, $\tau[G]$ denotes the interpretation of τ by G. But $\{\tau[G]\mid \tau$ is a P-name and $\tau\in N\}$ is denoted by $N[G]$, which is a countable elementary substructure of $H_\theta^{V[G]}$. Let $((P_\alpha,\le_\alpha,1_\alpha)_{\alpha\le\nu},(\dot Q_\alpha,\dot\le_\alpha,\dot 1_\alpha)_{\alpha<\nu})$ be a countable support iteration. For $p\in P_\alpha$, we denote $\{\beta<\alpha\mid p(\beta)\ne\dot 1_\beta\}$ by $\mathrm{supp}(p)$ and so $|\mathrm{supp}(p)|\le\omega$. For $p\in P_\alpha$ and $\beta\le\alpha$, $p\lceil\beta$ denotes the initial segment of p decided by β and $[\beta,\alpha)$ denotes the interval $\{\gamma\mid\beta\le\gamma<\alpha\}$. For a P_α-generic filter G_α over V, $G_\alpha\lceil\beta=\{p\lceil\beta\mid p\in G_\alpha\}$, which is a P_β-generic filter over V. For an ω_1-Souslin tree T and $\delta<\omega_1$, T_δ denotes the δth level of T and $T\lceil\delta=\bigcup\{T_\alpha\mid\alpha<\delta\}$.

The following is from [2] with minor modifications.

(0.1) Definition. A preorder $(P,\le,1)$ is *proper* iff for all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with $(P,\le,1)\in N$, we have $\forall p\in P\cap N\,\exists q\le p$ q is (P,N)-generic.

Let $((P_\alpha,\le_\alpha,1_\alpha)_{\alpha\le\nu},(\dot Q_\alpha,\dot\le_\alpha,\dot 1_\alpha)_{\alpha<\nu})$ be a countable support iteration such that for all $\alpha<\nu$, $\Vdash_{P_\alpha}$ "$(\dot Q_\alpha,\dot\le_\alpha,\dot 1_\alpha)$ is proper". Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $(P_\nu,\le_\nu,1_\nu)\in N$.

(0.2) Iteration Lemma for Proper. *Let $\beta\le\alpha\le\nu$, $\beta\in N$ and $\alpha\in N$. Then for any $x\in P_\beta$ and any P_β-name τ, if x is (P_β,N)-generic and $x\Vdash_{P_\beta}$ "$\tau\in P_\alpha\cap N$ and $\tau\lceil\beta\in\dot G_\beta$", then there is an $x^*\in P_\alpha$ such that $x^*\lceil\beta=x$, x^* is (P_α,N)-generic, $x^*\Vdash_{P_\alpha}$ "$\tau[\dot G_\alpha\lceil\beta]\in\dot G_\alpha$" and* $\mathrm{supp}(x^*)\cap[\beta,\alpha)\subseteq N$.

In particular, for any $x\in P_\beta$ and any $p\in P_\alpha\cap N$, if x is (P_β,N)-generic and $x\le_\beta p\lceil\beta$, then there is an $x^*\in P_\alpha$ such that $x^*\lceil\beta=x$, x^* is (P_α,N)-generic, $x^*\le_\alpha p$ and $\mathrm{supp}(x^*)\cap[\beta,\alpha)\subseteq N$.

1. Preserving ω_1-Souslin trees. For the rest of this note a Souslin tree means an ω_1-Souslin tree.

(1.1) Proposition. *Let $(P,\le,1)$ be a proper preorder and $(T,<_T)$ be a Souslin tree. The following are equivalent.*

(1) $\Vdash_P$ "*$(T,<_T)$ remains a Souslin tree*".

(2) *For all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with $(P,\le,1)$, $(T,<_T)\in N$, if q is (P,N)-generic and $t\in T$, then (q,t) is $(P\times T,N)$-generic.*

(3) *For all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with $(P,\le,1),(T,<_T)\in N$, let $\delta=N\cap\omega_1$. Then $\forall p\in P\cap N\,\exists q\le p\,\forall t\in T_\delta$ (q,t) is $(P\times T,N)$-generic.*

Proof. (1) implies (2): As $\Vdash_P$"T has the c.c.c.", we know $\Vdash_P$"$\forall t\in T$ t is $(T,N[\dot G_P])$-generic". For any $(q,t)\in P\times T$, (q,t) is $(P\times T,N)$-generic iff q is (P,N)-generic and $q\Vdash_P$"t is $(T,N[\dot G_P])$-generic". So for any $(q,t)\in P\times T$, if q is (P,N)-generic, then (q,t) is $(P\times T,N)$-generic.

(2) implies (3): By assumption $(P,\le,1)$ is proper. So for all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with $(P,\le,1),(T,<_T)\in N$, given $p\in P\cap N$ there is a $q\le p$ such that q is (P,N)-generic. Now by (2) for any $t\in T_\delta$, (q,t) is $(P\times T,N)$-generic.

(3) implies (1): Suppose $\Vdash_P$"$\dot A$ is a maximal antichain of T" and $p\in P$. Let $B=\{(x,s)\in P\times T\mid x\Vdash_P\text{“}\check s\in\dot A\text{”}\}$. Then B is a predense subset of $P\times T$. Fix a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $p,B,(P,\le,1),(T,<_T)\in N$. By (3), we have a $q\le p$ such that for all $t\in T_\delta$, (q,t) is $(P\times T,N)$-generic. So $B\cap N$ is predense below (q,t) for all $t\in T_\delta$. We conclude $q\Vdash_P$"$\forall t\in T_\delta\,\exists s<_T t$ $s\in\dot A$". Hence $q\Vdash_P$"$\dot A\subseteq T\lceil\delta$". ∎

(1.2) Lemma. *Let $((P_\alpha,\le_\alpha,1_\alpha)_{\alpha\le\nu},(\dot Q_\alpha,\dot\le_\alpha,\dot 1_\alpha)_{\alpha<\nu})$ be a countable support iteration and $(T,<_T)$ be a Souslin tree. If ν is a limit ordinal and for all $\alpha<\nu$, $\Vdash_{P_\alpha}$"$(T,<_T)$ remains a Souslin tree and $(\dot Q_\alpha,\dot\le_\alpha,\dot 1_\alpha)$ is proper", then $\Vdash_{P_\nu}$"$(T,<_T)$ remains a Souslin tree".*

Proof. Suppose $p\in P_\nu$ and $\Vdash_{P_\nu}$"$\dot A$ is a maximal antichain of T". Let $B=\{(x,s)\in P_\nu\times T\mid x\Vdash\text{“}\check s\in\dot A\text{”}\}$. Fix a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $p,(P_\nu,\le_\nu,1_\nu),(T,<_T),B\in N$. Fix $\langle\alpha_n\mid n<\omega\rangle$ such that $\alpha_0=0$, $\alpha_n\in\nu\cap N$ and $\alpha_n<\alpha_{n+1}$ for all $n<\omega$ and $\sup\{\alpha_n\mid n<\omega\}=\sup(\nu\cap N)$. Let $\delta=N\cap\omega_1<\omega_1$ and $\langle t_n\mid n<\omega\rangle$ enumerate T_δ. We construct $\langle\dot x_n\mid n<\omega\rangle$ and $\langle q_n\mid n<\omega\rangle$ such that for all $n<\omega$

(1) $\dot x_0$ is the P_0-name $\check p$.

(2) $q_0=\emptyset\in P_0$.

(3) $\dot x_n$ is a P_{α_n}-name.

(4) q_n is (P_{α_n},N)-generic.

(5) $q_n\Vdash_{P_{\alpha_n}}$"$\dot x_n\in P_\nu\cap N$ and $\dot x_n\lceil\alpha_n\in\dot G_{\alpha_n}$".

(6) $q_{n+1}\lceil\alpha_n=q_n$.

(7) $q_{n+1}\Vdash_{P_{\alpha_{n+1}}}$"$\dot x_{n+1}\le_\nu\dot x_n[\dot G_{\alpha_{n+1}}\lceil\alpha_n]$ and $\exists s<_T t_n$ $(\dot x_{n+1},s)\in\check B$".

The construction is by recursion on $n<\omega$. For $n=0$, let $\dot x_0,q_0$ be as specified. Now suppose we have $\dot x_n$ and q_n. Since (4) and (5) hold, we have a $q_{n+1} \in P_{\alpha_{n+1}}$ such that $q_{n+1}\lceil\alpha_n = q_n$, q_{n+1} is $(P_{\alpha_{n+1}}, N)$-generic and $q_{n+1} \Vdash_{P_{\alpha_{n+1}}}$ "$\dot{x}_n[\dot{G}_{\alpha_{n+1}}\lceil\alpha_n]\lceil\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$" by iteration lemma (0.2) for proper. Since $\Vdash_{P_{\alpha_{n+1}}}$ "$(T, <_T)$ remains a Souslin tree", we know (q_{n+1}, t_n) is $(P_{\alpha_{n+1}} \times T, N)$-generic by Proposition (1.1).

Now in order to get a $P_{\alpha_{n+1}}$-name $\dot{x}_{n+1}$, let us fix an arbitrary $P_{\alpha_{n+1}}$-generic filter $G_{\alpha_{n+1}}$ over V with $q_{n+1} \in G_{\alpha_{n+1}}$. Let $G_{\alpha_n} = G_{\alpha_{n+1}}\lceil\alpha_n$. We know G_{α_n} is a P_{α_n}-generic filter over V with $q_n \in G_{\alpha_n}$. Let $x_n = \dot{x}_n[G_{\alpha_n}]$. Then $x_n \in P_\nu \cap N$ and $x_n\lceil\alpha_{n+1} \in G_{\alpha_{n+1}}$. Let $D = \{(a, s) \in P_{\alpha_{n+1}} \times T \mid a$ and $x_n\lceil\alpha_{n+1}$ are incompatible in $P_{\alpha_{n+1}}\} \cup \{(a, s) \in P_{\alpha_{n+1}} \times T \mid \exists x \in P_\nu (x \leq_\nu x_n, (x, s) \in B$ and $x\lceil\alpha_{n+1} = a)\}$. Then D is a predense subset of $P_{\alpha_{n+1}} \times T$ and $D \in N$. Hence $D \cap N$ is predense below (q_{n+1}, t_n). For convenience sake, let us fix a T-generic filter G_T over $V[G_{\alpha_{n+1}}]$ with $t_n \in G_T$. Then there is an $(a, s) \in D \cap N \cap (G_{\alpha_{n+1}} \times G_T)$. Since $a \in G_{\alpha_{n+1}}$ and $x_n\lceil\alpha_{n+1} \in G_{\alpha_{n+1}}$, there must be an $x \in P_\nu$ such that $x \leq_\nu x_n$, $(x, s) \in B$ and $x\lceil\alpha_{n+1} = a$. Since $(P_\nu, \leq_\nu, 1_\nu), x_n, s, B, \alpha_{n+1}$ and a are all in N, we may assume $x \in N$. Since $s \in N \cap G_T$ and $t_n \in G_T$, we have $s <_T t_n$. Let $\dot{x}_{n+1}$ be a $P_{\alpha_{n+1}}$-name of this x. This completes the construction.

Let $q = \bigcup\{q_n \mid n < \omega\}^\frown 1_\nu\lceil[\sup(\nu \cap N), \nu)$. Then $q \in P_\nu$. We claim $q \Vdash_{P_\nu}$ "$\forall n < \omega\, \exists s \in \dot{A}\ s <_T t_n$" and so $q \Vdash$ "$\dot{A} \subseteq T\lceil\delta$". To see this, let G_ν be an arbitrary P_ν-generic filter over V with $q \in G_\nu$. Put $G_{\alpha_n} = G_\nu\lceil\alpha_n$ and $x_n = \dot{x}_n[G_{\alpha_n}]$ for each $n < \omega$.

Since $q_n \in G_{\alpha_n}$ holds for all $n < \omega$, we have

(8) $\quad x_0 = p.$

(9) $\quad x_n \in P_\nu \cap N$ and $x_n\lceil\alpha_n \in G_{\alpha_n}$.

(10) $\quad x_{n+1} \leq_\nu x_n$ and $\exists s <_T t_n\ (x_{n+1}, s) \in B$.

Since $x_n \in P_\nu \cap N$, we know $\mathrm{supp}(x_n) \subseteq \nu \cap N$ for all $n < \omega$. We conclude $x_n \in G_\nu$ for all $n < \omega$. Therefore for all $n < \omega$ there is an $s \in \dot{A}[G_\nu]$ with $s <_T t_n$. Since G_ν is an arbitrary P_ν-generic filter over V with $q \in G_\nu$, we have $q \leq_\nu p$. ∎

(1.3) THEOREM. *Let* $((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha\leq\nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha<\nu})$ *be a countable support iteration of arbitrary length* ν. *If for all* $\alpha < \nu$, $\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ *is proper and preserves every* ω_1*-Souslin tree*", *then* $(P_\nu, \leq_\nu, 1_\nu)$ *is proper and preserves every* ω_1*-Souslin tree.*

Proof. Immediate from Lemma (1.2). ∎

(1.4) Remark. Since the preorders which appear in the forcing axiom SAD are proper and preserve every ω_1-Souslin tree, countable support iterations for getting SAD preserve every ω_1-Souslin tree in L.

References

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