FUNDAMENTA
MATHEMATICAE
142 (1993)

# Imposing pseudocompact group topologies on Abelian groups

by

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*Dedicated to Professor K. H. Hofmann*
*on the occasion of his 60th birthday*

**Abstract.** The least cardinal $\lambda$ such that some (equivalently: every) compact group with weight $\alpha$ admits a dense, pseudocompact subgroup of cardinality $\lambda$ is denoted by $\mathrm{m}(\alpha)$. Clearly, $\mathrm{m}(\alpha) \leq 2^\alpha$. We show:

THEOREM 3.3. *Among groups of cardinality* $\gamma$, *the group* $\oplus_\gamma \mathbb{Q}$ *serves as a "test space" for the availability of a pseudocompact group topology in this sense*: *If* $\mathrm{m}(\alpha) \leq \gamma \leq 2^\alpha$ *then* $\oplus_\gamma \mathbb{Q}$ *admits a* (*necessarily connected*) *pseudocompact group topology of weight* $\alpha \geq \omega$ (*and also a pseudocompact group topology of weight* $\log \gamma$).

THEOREM 4.12. *Let* $G$ *be Abelian with* $|G| = \gamma$. *If either* $\mathrm{m}(\alpha) \leq \alpha$ *and* $\mathrm{m}(\alpha) \leq r_0(G) \leq \gamma \leq 2^\alpha$, *or* $\alpha > \omega$ *and* $\alpha^\omega \leq r_0(G) \leq 2^\alpha$, *then* $G$ *admits a pseudocompact group topology of weight* $\alpha$.

THEOREM 4.15. *Every connected*, *pseudocompact Abelian group* $G$ *with* $\mathrm{w}G = \alpha \geq \omega$ *satisfies* $r_0(G) \geq \mathrm{m}(\alpha)$.

THEOREM 5.2(b). *If* $G$ *is divisible Abelian with* $2^{r_0(G)} \leq \gamma$, *then* $G$ *admits at most* $2^\gamma$*-many pseudocompact group topologies.*

THEOREM 6.2. *Let* $\beta = \alpha^\omega$ *or* $\beta = 2^\alpha$ *with* $\beta \geq \alpha$, *and let* $\beta \leq \gamma < \kappa \leq 2^\beta$. *Then both* $\oplus_\gamma \mathbb{Q}$ *and the free Abelian group on* $\gamma$*-many generators admit exactly* $2^\kappa$*-many pseudocompact group topologies of weight* $\kappa$. *Of these*, *some* $\kappa^+$*-many form a chain and some* $2^\kappa$*-many form an anti-chain.*

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1991 *Mathematics Subject Classification*: Primary 54A05; Secondary 20K45, 22C05.

*Key words and phrases*: pseudocompact group; $G_\delta$-dense subgroup; singular cardinals hypothesis; torsion-free rank; connected topological group; 0-dimensional group; divisible hull; chain; anti-chain.

The first-named co-author gratefully acknowledges generous hospitality and partial support received from the Universities of Bremen and Hannover, and support from a "study visit" grant from the Deutscher Akademischer Austauschdienst (DAAD) of the Federal Republic of Germany (summer, 1991).

**1. Introduction.** This work contributes to these two questions: (a) Which Abelian groups admit a pseudocompact group topology? (b) When (Abelian) $G$ admits such a topology, for what cardinal numbers $\alpha$ can such a topology $\mathcal{T}$ be chosen so that the weight $\mathrm{w}(G,\mathcal{T})$ of $\langle G,\mathcal{T}\rangle$ is equal to $\alpha$?

We consider and we construct only topological groups satisfying the Hausdorff separation axiom, so our topological groups $\langle G,\mathcal{T}\rangle$ all satisfy $|G| \le 2^{\mathrm{w}(G,\mathcal{T})}$.

1.1. Notation and Definitions. For $G$ a group and $A \subseteq G$, $\langle A\rangle$ denotes the subgroup of $G$ generated by $A$.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{T}$ and $\mathbb{R}$ denote as usual the set of integers, the set of rational numbers, the circle group, and the set of real numbers, respectively, in each case with the usual algebraic properties. For $p \in \mathbb{P}$ (the set of primes) we write

$$\mathbb{Z}(p) = \{\zeta \in \mathbb{T} : \zeta^p = 1\} \quad \text{and} \quad \mathbb{Z}(p^\infty) = \{e^{2\pi i k/p^n} : k \in \mathbb{Z},\ 0 \le n \in \mathbb{Z}\}.$$

Given an Abelian group $G$, as in [HR] and [Fu] we denote by $r_0(G)$ and $r_p(G)$ (for $p \in \mathbb{P}$) the torsion-free rank and the $p$-rank of $G$, respectively; the rank of $G$ is the cardinal number $r(G) = r_0(G) + \sum\{r_p(G) : p \in \mathbb{P}\}$.

The torsion subgroup of an Abelian group $G$ is denoted by $\mathrm{tor}(G)$.

We write $G \approx H$ to indicate that the groups $G$ and $H$ are algebraically isomorphic.

We denote by $FA(\gamma)$ the free Abelian group of rank $\gamma$.

The symbol $\Sigma_{\mathbf{a}}$ denotes the compact, connected, Abelian, metrizable solenoid group which is the dual group of the discrete group of rational numbers: $\Sigma_{\mathbf{a}} = \widehat{\mathbb{Q}_{\mathrm{d}}}$. For realizations of $\Sigma_{\mathbf{a}}$ and a development of its properties, the reader may consult [HR](10.13, 25.3, 25.26(c)).

The symbol $\Delta_p$ ($p \in \mathbb{P}$) denotes the Abelian group of $p$-adic integers in its usual (compact, 0-dimensional, metrizable) topology. As with every compact, 0-dimensional group, the topology of $\Delta_p$ is *linear* in the sense that the open subgroups of $\Delta_p$ form a basis at the identity (cf. [HR](7.7, 10.2ff.)).

As usual, a topological space $X$ is said to be *pseudocompact* if every locally finite family of open subsets of $X$ is finite. It is easy to show on the basis of this definition (cf. [CRos2](1.1)) that every pseudocompact group $G = \langle G,\mathcal{T}\rangle$ is totally bounded—that is, for every non-empty open $U \subseteq G$ there is a finite $F \subseteq G$ such that $G = FU$. It is a theorem of Weil [We] that the totally bounded groups are exactly the subgroups of compact groups. (Further, if $G$ is a group and $K$ is a compact group containing $G$ then the group $\mathrm{cl}_K G$ is, up to an isomorphism-and-homeomorphism fixing $G$ pointwise, the only compact group in which $G$ is dense; we denote this compact group by $\overline{G}$ and we call it the *Weil completion* of $G$.) There is, then, as we search for pseudocompact groups, no loss of generality in restricting attention to dense subgroups of compact groups. The following results will be useful.

1.2. Theorem. *Let $K$ be a compact group with* $\mathrm{w}K = \alpha \geq \omega$. *Then*

(a) $|K| = 2^\alpha$;
(b) $\mathrm{d}(K) = \log \alpha$; *and*
(c) *every dense subgroup $G$ of $K$ satisfies* $\mathrm{w}G = \alpha$. ■

[Theorem 1.2 is well-known. A proof is given, with appropriate references to the literature, in [C].]

1.3. Theorem [CRos2]. *For a compact group $K$ and a dense subgroup $G$ of $K$, the following statements are equivalent.*

(a) *$G$ is $G_\delta$-dense in $K$*;
(b) *$G$ is pseudocompact*;
(c) $K = \beta G$. ■

1.4. Remark. For practical purposes in this paper the equivalence (a)⇔(b) of Theorem 1.3 may be adopted as the definition of pseudocompactness (in the context of topological groups); our aim is to construct $G_\delta$-dense subgroups of compact groups and we will not be concerned explicitly with locally finite open families.

Given a pseudocompact group $G$, let $\mathrm{m}(G)$ be the least cardinal number with this property: there is a dense, pseudocompact subgroup $H$ of $G$ such that $|H| = \mathrm{m}(G)$. It is shown in [CRob] that compact groups $K$ and $K'$ with $\mathrm{w}K = \mathrm{w}K'$ satisfy $\mathrm{m}(K) = \mathrm{m}(K')$; accordingly for $\alpha \geq \omega$ we may choose any compact group $K$ such that $\mathrm{w}(K) = \alpha$ and define $\mathrm{m}(\alpha)$ by $\mathrm{m}(\alpha) = \mathrm{m}(K)$. The following result contains most of the information known about the cardinal numbers $\mathrm{m}(\alpha)$.

1.5. Theorem ([CEG], [CRob]). *Let $\alpha \geq \omega$. Then*

(a) $\log \alpha \leq \mathrm{m}(\alpha) \leq (\log \alpha)^\omega$;
(b) $\mathrm{m}(\alpha) \geq \mathbf{c}$; *and*
(c) $\mathrm{cf}(\mathrm{m}(\alpha)) > \omega$. ■

1.6. Discussion. (a) In [CRob] the authors define

$$\mathrm{m}(\alpha) = \mathrm{m}(\{0,1\}^\alpha) = \mathrm{d}(\mathrm{P}(\{0,1\}^\alpha)),$$

where as usual for a topological space $X = \langle X, \mathcal{T} \rangle$ the symbol $\mathrm{P}(X)$ denotes the set $X$ with the smallest topology in which each $\mathcal{T}$-$G_\delta$-set is open.

(b) The following simple result is useful when dealing with small cardinals.

Theorem. *Let $\alpha$ be an infinite cardinal. Then* $\mathrm{m}(\alpha) = \mathbf{c}$ *if and only if* $\alpha \leq 2^{\mathbf{c}}$.

Proof. For $\omega \le \alpha \le 2^{\mathbf{c}}$ we have

$$\mathbf{c} \le \mathrm{m}(\alpha) \le (\log \alpha)^{\omega} \le (\log 2^{\mathbf{c}})^{\omega} \le \mathbf{c}^{\omega} = \mathbf{c}$$

using 1.5(a) and 1.5(b), while if $\alpha > 2^{\mathbf{c}}$ then again from 1.5(a) follows

$$\mathrm{m}(\alpha) \ge \log \alpha \ge \log((2^{\mathbf{c}})^{+}) > \mathbf{c}\,. \ \blacksquare$$

(c) Clearly if $G$ is $G_\delta$-dense in $K = \overline{G}$ and $G \subseteq H \subseteq K$, then $H$ also is $G_\delta$-dense in $K$. Together with 1.2(a) and 1.2(c) this shows:

Theorem. *Given infinite cardinal numbers $\alpha$ and $\gamma$, there is a pseudocompact group $G$ with $|G| = \gamma$ and $\mathrm{w}G = \alpha$ if and only if $\mathrm{m}(\alpha) \le \gamma \le 2^{\alpha}$; further, every compact group $K$ with $\mathrm{w}K = \alpha$ contains densely such a group $G$.*

(d) We denote by (M) and by (†) the following two statements.

(M) $\mathrm{m}(\alpha) = (\log \alpha)^{\omega}$ for all $\alpha \ge \omega$.

(†) If $\kappa \ge \mathbf{c}$ and $\mathrm{cf}(\kappa) > \omega$, then $\kappa^{\omega} = \kappa$.

(The former notation is taken from [CRob], the latter from [vD].) It is known [CEG], [CRob] that the singular cardinals hypothesis (here abbreviated SCH) implies (M). Since SCH is equivalent to (†) (cf. [J](§8)), it is natural to inquire whether (M)⇒(†). Assuming the consistency relative to ZFC of suitable large cardinal axioms this implication cannot be proved, since Masaveu [M] has recently shown that (M) holds in certain of the models of Magidor [M1], [M2] where SCH fails. It remains an open question, raised in [CEG] and [CRob], whether (M) is a theorem of ZFC.

(e) The cardinal-valued "function" $\alpha \to \mathrm{m}(\alpha)$ is monotone in the sense that if $\omega \le \alpha \le \beta$ then $\mathrm{m}(\alpha) \le \mathrm{m}(\beta)$. Indeed, let $G$ be a dense subgroup of $\mathrm{P}(\{0,1\}^{\beta})$ with $|G| = \mathrm{m}(\beta)$ and let $h$ be a continuous homomorphism from $\{0,1\}^{\beta}$ onto $\{0,1\}^{\alpha}$. Then $h$ remains continuous as a function from $\mathrm{P}(\{0,1\}^{\beta})$ onto $\mathrm{P}(\{0,1\}^{\alpha})$, so the group $h[G]$ is dense in $\mathrm{P}(\{0,1\}^{\alpha})$ and we have

$$\mathrm{m}(\alpha) = \mathrm{d}(\mathrm{P}(\{0,1\}^{\alpha})) \le |h[G]| \le |G| = \mathrm{m}(\beta)\,. \ \blacksquare$$

(f) We show that, as to the relation between $\alpha$ and $\mathrm{m}(\alpha)$, anything can happen. (Here as usual we say that $\alpha$ is a *strong limit cardinal*, and we write $\alpha \in \mathrm{SLC}$, if every $\beta < \alpha$ satisfies $2^{\beta} < \alpha$.)

Theorem. *For every cardinal $\beta$ there are cardinals $\alpha_0, \alpha_1$, and $\alpha_2$ with each $\alpha_i > \beta$ such that*

(0) $\mathrm{m}(\alpha_0) < \alpha_0$;
(1) $\mathrm{m}(\alpha_1) = \alpha_1$;
(2) $\mathrm{m}(\alpha_2) > \alpha_2$.

Proof. (0) According to Theorem 1.5(a) it is enough to choose $\alpha = \alpha_0 > \beta$ so that $(\log\alpha)^\omega < \alpha$. For example, take $\alpha = (2^\beta)^+$ or $\alpha = 2^{2^\beta}$ or $\alpha = 2^{(\beta^\omega)}$.

(1) Let $\alpha = \alpha_1 > \beta$ satisfy $\alpha \in \mathrm{SLC}$ and $\mathrm{cf}(\alpha) > \omega$. Then

$$\alpha^\omega = \sum_{\gamma<\alpha} \gamma^\omega \le \sum_{\gamma<\alpha} 2^\gamma \le \sum_{\gamma<\alpha} \alpha = \alpha \le \alpha^\omega\,,$$

and from $\alpha = \log\,\alpha$ and 1.5(a) follows

$$\alpha = \log\alpha \le \mathrm{m}(\alpha) \le (\log\alpha)^\omega = \alpha^\omega = \alpha\,.$$

(2) Let $\alpha = \alpha_2 > \beta$ satisfy $\alpha \in \mathrm{SLC}$ and $\mathrm{cf}(\alpha) > \omega$. Then $\alpha = \log\alpha \le \mathrm{m}(\alpha)$ with $\mathrm{cf}(\alpha) = \omega$, and since $\mathrm{cf}(\mathrm{m}(\alpha)) > \omega$ by 1.5(b), we have $\alpha < \mathrm{m}(\alpha)$. ∎

**2. Acknowledgements and abstracts.** We thank Stevo Todorčević and Thomas Jech for helpful conversations concerning the equivalence (†)⇔SCH cited above in 1.6(d). We also gratefully acknowledge several conversations with Jan van Mill (summer of 1987) on topics closely related to the content of this paper. In particular, the ideas which later grew to the present Lemmas 4.1 and 4.4 first surfaced in these conversations.

We thank the referee for several helpful suggestions, both expository and substantive.

We announced some of the results of this paper in our abstracts [CRe2] and [CRe3].

**3. Constraints on $|G|$ and $\mathrm{w}G$ for pseudocompact groups $G$.** To begin this section we restate and reprove succinctly a collection of theorems from [vD].

For an infinite cardinal $\alpha$ we denote by $\mathcal{G}(\alpha)$ the class of groups which admit a pseudocompact group topology of weight $\alpha$.

3.1. Theorem [vD]. (a) *If there is $G \in \mathcal{G}(\alpha)$ such that $|G| = \gamma$, then $2^\omega \le \gamma \le 2^\alpha$ and*

(1) $\log\alpha \le \gamma$, *and*
(2) $\log\alpha < \gamma$ *if* $\mathrm{cf}(\gamma) = \omega$.

(b) *Assume* (†). *If $2^\omega \le \gamma \le 2^\alpha$, and if*

(1) $\log\alpha \le \gamma$, *and*
(2) $\log\alpha < \gamma$ *if* $\mathrm{cf}(\gamma) = \omega$,

*then there is $G \in \mathcal{G}(\alpha)$ such that $|G| = \gamma$.*

Proof. (a) That $2^\omega \le \gamma \le 2^\alpha$ follows from 1.5(b) and the second paragraph of our Introduction, while 1.5 (together with $\gamma = |G| \ge \mathrm{m}(\alpha)$) gives (1) and (2).

(b) It is enough to show $\gamma \geq \mathrm{m}(\alpha)$. If $\mathrm{cf}(\gamma) > \omega$ then $\gamma = \gamma^\omega \geq (\log\alpha)^\omega \geq \mathrm{m}(\alpha)$ from (†), (1) and 1.5(a), while if $\mathrm{cf}(\gamma) = \omega$ and $\log\alpha < \gamma$ then from (†) applied to $(\log\alpha)^+$ follows $\gamma \geq (\log\alpha)^+ = ((\log\alpha)^+)^\omega \geq (\log\alpha)^\omega \geq \mathrm{m}(\alpha)$. ∎

3.2. Remarks. (a) With $\alpha$ and $\gamma$ as in 3.1, not every group $G$ with $|G| = \gamma$ satisfies $G \in \mathcal{G}(\alpha)$. For a very wide class of examples, let $G$ be a group of the form $G = H \oplus S$, where $H$ and $S$ are chosen as follows. (1) $|H| = \gamma$; (2) $\omega \leq |S| < 2^\omega$; (3) for suitable $n$ the function $\phi : G \to G$ given by $\phi(n) = x^n$ satisfies $|\phi[S]| \geq \omega$, and $\phi(x) = 1$ for all $x \in H$. (For a specific example with $n = 2$ take $H = \bigoplus_\gamma \mathbb{Z}(2)$ and $S = \bigoplus_\omega \mathbb{Z}(3)$ or $S = \mathbb{Q}$.) Since the continuous image of a pseudocompact space is pseudocompact and the map $x \to x^n$ is continuous with respect to every topological group topology on $G$, it follows from 1.5(b), together with the relations $\phi[G] = \phi[S]$ and $\omega \leq |\phi[S]| < 2^\omega$, that $G$ supports no pseudocompact group topology.

(b) A reading of [DS](p. 85) may yield the impression that van Douwen [vD] has proved the equivalence of 3.1 above in ZFC. This impression is incorrect: van Douwen raises explicitly the question "whether (†) is needed for" this result.

(c) If some group $G$ with $|G| = \gamma$ satisfies $G \in \mathcal{G}(\alpha)$—that is, if $\mathrm{m}(\alpha) \leq \gamma \leq 2^\alpha$—then also some $G'$ with $|G'| = \gamma$ satisfies $G' \in \mathcal{G}(\log\gamma)$. More generally, we have:

THEOREM. *Let $\alpha$ and $\gamma$ be infinite cardinals such that* $\mathrm{m}(\alpha) \leq \gamma \leq 2^\alpha$. *Then*

(1) $\log\gamma \leq \alpha$, *and every cardinal $\beta$ such that* $\log\gamma \leq \beta \leq \alpha$ *satisfies* $\mathrm{m}(\beta) \leq \gamma \leq 2^\beta$; *and*

(2) $\log\gamma$ *is the least cardinal $\beta$ such that* $\mathrm{m}(\beta) \leq \gamma \leq 2^\beta$.

Proof. (1) follows from 1.6(e), and (2) is obvious since if $\beta < \log\gamma$ then $\mathrm{m}(\beta) \leq \gamma \leq 2^\beta$ fails. ∎

(d) Our methods do not answer the following question, which is suggested by Theorem 3.5. If a group $G$ (say with $|G| = \gamma$) admits a pseudocompact group topology $\mathcal{T}$ such that $\mathrm{w}(G, \mathcal{T}) = \alpha$, so in particular $\mathrm{m}(\alpha) \leq \gamma \leq 2^\alpha$, and if $\beta = \log\gamma$, then must $G$ itself admit a pseudocompact group topology $\mathcal{U}$ such that $\mathrm{w}(G, \mathcal{U}) = \beta$? Let us note that in any case such $\mathcal{U}$ with $\mathcal{U} \subseteq \mathcal{T}$ need not exist. Let $\omega < \alpha \leq 2^{\mathbf{c}}$, so $\mathrm{m}(\alpha) = \mathrm{m}(\omega) = \mathbf{c}$ by 1.6(b), and let $G = \langle G, \mathcal{T} \rangle$ be a $G_\delta$-dense subgroup of $\mathbb{T}^\alpha$ with $|G| = \gamma = \mathbf{c}$. No pseudocompact group topology $\mathcal{U}$ exists on $G$ with $\mathrm{w}(G, \mathcal{U}) = \log\gamma = \omega$ and $\mathcal{U} \subseteq \mathcal{T}$, since (as is easily shown) a one-to-one continuous function from a pseudocompact space onto a metric space is a homeomorphism.

(e) Frequently in what follows we will impose pseudocompact group topologies on groups of the form $\bigoplus_\gamma \mathbb{Q}$ and other divisible Abelian groups. It is helpful to bear in mind the following simple fact.

THEOREM ([CvM](7.4)). *Every pseudocompact group topology on a divisible Abelian group is connected.* ∎

We show now that the groups $\bigoplus_\gamma \mathbb{Q}$ serve as test spaces for the existence of pseudocompact group topologies: If the class $\mathcal{G}(\alpha)$ contains any group of cardinality $\gamma$, then $\bigoplus_\gamma \mathbb{Q} \in \mathcal{G}(\alpha)$.

3.3. THEOREM. *Let $\alpha$ and $\gamma$ be infinite cardinals such that* $\mathrm{m}(\alpha) \le \gamma \le 2^\alpha$, *and let $G = \bigoplus_\gamma \mathbb{Q}$. Then $G$ admits a* (*necessarily connected*) *pseudocompact group topology of weight $\alpha$ with respect to which $\overline{G} = \Sigma_{\mathbf{a}}^\alpha$.*

Proof. Let $K = \Sigma_{\mathbf{a}}^\alpha$. According to Theorem 1.6(c) there is a $G_\delta$-dense subgroup $H$ of $K$ such that $|H| = \gamma$. Since $H$ is torsion-free, from $|H| > \omega$ follows $r_0(H) = |H| = \gamma$. Since $K$ is divisible, there is (cf. [Fu](24.4 and p. 107) a divisible hull $D$ of $H$ in $K$ such that $H \subseteq D \subseteq K$ and $r_0(D) = r_0(H) = \gamma$. Clearly $D \approx G = \bigoplus_\gamma \mathbb{Q}$, and $D$ is $G_\delta$-dense in $K$; it follows from 1.3 and 1.2 that (with the topology inherited from $K$) $D$ is a pseudocompact group of weight $\alpha$. ∎

3.4. Remark. From 1.6(b) and 3.3 it follows that the group $\bigoplus_{\mathbf{c}} \mathbb{Q}$, which is algebraically isomorphic to $\mathbb{R}$, admits a pseudocompact group topology of weight $\alpha$ if (and only if) $\omega \le \alpha \le 2^{\mathbf{c}}$. For $\alpha = \omega$ this result is not new: See in this connection [Hal], [Haw], and [HR](25.26(c)).

With suitable additional hypotheses, Theorem 3.3 furnishes an intrinsic characterization of those cardinals which arise as the cardinality of a pseudocompact group.

3.5. THEOREM. *Let $\gamma$ be an infinite cardinal. The following three conditions are equivalent.*

(a) $\bigoplus_\gamma \mathbb{Q}$ *admits a pseudocompact group topology of weight* $\log \gamma$.
(b) $\bigoplus_\gamma \mathbb{Q}$ *admits a pseudocompact group topology.*
(c) *Some group of cardinality $\gamma$ admits a pseudocompact group topology.*

*If in addition* (M) *is assumed*, *the following condition is also equivalent to those above*:

(d) $(\log\log\gamma)^\omega \le \gamma$.

*If in addition* (†) *is assumed*, *the following condition is also equivalent to those above*:

(e) $(\log\gamma)^\omega \le \gamma$.

Proof. That (a)⇒(b)⇒(c) is obvious, and (c)⇒(a) by 3.2(c)(1) and 3.3. Clearly (e)⇒(d), and (d) (with 1.5(a)) gives

$$\mathrm{m}(\log\gamma) \le (\log\log\gamma)^\omega \le \gamma \le 2^{\log\gamma}$$

and hence (c). (So far we have used neither (M) nor (†).)

If (M) is assumed then from (a) follows

$$\gamma \geq \mathrm{m}(\log\gamma) = (\log\log\gamma)^{\omega},$$

which is (d); thus to complete the proof it is enough to derive (e) from (d) and (†). If $\gamma = \log\gamma$ then (e) is clear from (d), and if $\log\gamma < \gamma$ then since $\mathrm{cf}((\log\gamma)^{+}) = (\log\gamma)^{+} > \omega$ we have

$$(\log\gamma)^{\omega} \leq ((\log\gamma)^{+})^{\omega} = (\log\gamma)^{+} \leq \gamma$$

from (†), as required. ∎

For cardinals of the form $\gamma^{\omega}$, Theorems 3.3 and 3.5 assume this form.

3.6. Corollary. *Let $\gamma = \gamma^{\omega} > \omega$ and let $\alpha \geq \omega$. The following conditions are equivalent.*

(a) $\log\gamma \leq \alpha \leq 2^{\gamma}$;
(b) $\log\alpha \leq \gamma \leq 2^{\alpha}$;
(c) $\bigoplus_{\gamma}\mathbb{Q} \in \mathcal{G}(\alpha)$. ∎

**4. Pseudocompact Abelian groups with large torsion-free rank.** In this section we determine some conditions sufficient to ensure that certain free Abelian groups, and more generally Abelian groups $G$ with large torsion-free rank, admit pseudocompact group topologies.

4.1. Lemma. *Let $\{K_i : i \in I\}$ be a set of (Tikhonov) spaces with each $|K_i| > 1$ and with $|I| = \alpha > \omega$, and let $K = \prod_{i\in I} K_i$. Let $X = \{x(\eta) : \eta < \kappa\} \subseteq K$ and $Y = \{y(\eta) : \eta < \kappa\} \subseteq K$, and for $\eta < \kappa$ let*

$$d(\eta) = \{i \in I : x(\eta)_i \neq y(\eta)_i\}.$$

*If $X$ is $G_\delta$-dense in $K$ and*

$$d(\eta) \cap d(\eta') = \emptyset \quad \text{for } \eta < \eta' < \kappa,$$

*then $Y$ is $G_\delta$-dense in $K$.*

Proof. Every non-empty $G_\delta$-subset of $K$ contains a set of the form

$$U = \Big(\prod_{i\in C} U_i\Big) \times \Big(\prod_{i\in I\setminus C} K_i\Big)$$

with $C \in [I]^{\omega}$ and with $U_i$ a $G_\delta$-subset of $K_i$ for $i \in C$. We claim for each such set $U$ that $|U \cap X| \geq \omega^{+}$. From $|I\setminus C| = \alpha > \omega$ follows $|U| \geq 2^{\alpha} \geq \omega^{+}$, so if $|U \cap X| \leq \omega$ then (since points of $K$ are closed) the set $U\setminus X$ is a non-empty $G_\delta$-subset of $K$ which misses $X$. This contradiction establishes the claim.

Since $d(\eta) \cap d(\eta') = \emptyset$ for $\eta < \eta' < \kappa$ and $|C| \leq \omega$, there is $\eta < \kappa$ such that $d(\eta) \cap C = \emptyset$. Since $y(\eta)_i = x(\eta)_i \in U_i$ for all $i \in C$ we have $y(\eta) \in U \cap Y$, as required. ∎

4.2. Lemma. *Let $K$ be an infinite Abelian group with $r_0(K) = |K|$, and let $H$ be a subgroup of $K$ such that $|H| < |K|$. Then there is $y \in K \backslash \mathrm{tor}(K)$ such that $\langle H \cup \{y\}\rangle = H \oplus \langle y\rangle$.*

Proof. Let $B$ be a maximal independent subset of $H$, and let $C$ be a maximal independent subset of $K$ such that $C \supseteq B$. From

$$r_0(K) = |K| > |H| \geq r(H)$$

there is $y \in C \backslash B$ such that $y \notin \mathrm{tor}(K)$. Then $y$ is as required, since if some integer $n$ satisfies $0 \neq ny \in H$ then $B \cup \{ny\}$ is independent in $H$. ∎

A routine iteration furnishes the following consequence of Lemma 4.2.

4.3. Corollary. *Let $K$ be an infinite Abelian group with $r_0(K) = |K|$ and let $H$ be a free Abelian subgroup of $K$. Then for every cardinal $\gamma$ such that $|H| \leq \gamma \leq |K|$ there is a free Abelian group $G$ such that $H \subseteq G \subseteq K$ and $|G| = \gamma$.* ∎

We would like to be able to prove for infinite cardinals $\alpha > \omega$ and $\gamma$ that if some group $G$ with $|G| = \gamma$ satisfies $G \in \mathcal{G}(\alpha)$, then $FA(\gamma) \in \mathcal{G}(\alpha)$. At present our methods suffice to give this result only under the additional assumption that either $\mathrm{m}(\alpha) \leq \alpha$ or $\gamma \geq \alpha^\omega$. The proof of the first of these results uses Lemma 4.1 while the proof of the second is more direct and does not. In either case the witnessing pseudocompact topology of weight $\alpha$ on $FA(\gamma)$ may be chosen either to be connected, or to be 0-dimensional and linear.

4.4. Lemma. *Let $\alpha$ be an infinite cardinal such that $\mathrm{m}(\alpha) \leq \alpha$, and let $K$ be a compact Abelian group such that $\mathrm{w}K \leq \alpha$ and $K \neq \mathrm{tor}(K)$. Then $K^\alpha$ contains a $G_\delta$-dense copy of $FA(\mathrm{m}(\alpha))$.*

Proof. Algebraically we have $K \supseteq \mathbb{Z}$, so $K^\alpha \supseteq \mathbb{Z}^\alpha$ and hence

$$2^\alpha = |K^\alpha| \geq r_0(K^\alpha) \geq r_0(\mathbb{Z}^\alpha) = 2^\alpha\,.$$

Since $K^\alpha$ is compact and $\mathrm{w}(K^\alpha) = \alpha$ there is a $G_\delta$-dense subset $X = \{x(\eta) : \eta < \mathrm{m}(\alpha)\}$ of $K^\alpha$. Now, using the assumption $\mathrm{m}(\alpha) \leq \alpha$, let $\{A(\eta) : \eta < \mathrm{m}(\alpha)\}$ be a (faithfully indexed) partition of $\alpha$ into pairwise disjoint subsets of cardinality $\alpha$.

Choose $y(0) \in K^\alpha \backslash \mathrm{tor}(K^\alpha)$ such that

$$y(0)_i = x(0)_i \qquad \text{for } i \in \alpha \backslash A(0)$$

and recursively, if $\zeta < \mathrm{m}(\alpha)$ and $y(\eta)$ has been defined for all $\eta < \zeta$, set

$$H(\zeta) = \langle\{y_\eta : \eta < \zeta\}\rangle$$

and use Lemma 4.2 (with $\pi_\zeta[H(\zeta)]$ and $K^{A(\zeta)}$ replacing $H$ and $K$, respectively) to find $y'(\zeta) \in K^{A(\zeta)} \backslash \mathrm{tor}(K^{A(\zeta)})$ such that

$$\langle \pi_\zeta[H_\zeta] \cup \{y'(\zeta)\}\rangle = \pi_\zeta[H_\zeta] \oplus \langle y'(\zeta)\rangle \tag{$*$}$$

(here $\pi_\zeta$ denotes the projection from $K^\alpha$ onto $K^{A(\zeta)}$); and define $y(\zeta) \in K^\alpha$ by the rule

$$y(\zeta)_i = \begin{cases} y'(\zeta)_i & \text{if } i \in A(\zeta), \\ x(\zeta)_i & \text{if } i \in \alpha \backslash A(\zeta). \end{cases}$$

This defines $y(\eta)$ for all $\eta < \mathrm{m}(\alpha)$. We set

$$H = \langle \{y(\eta) : \eta < \mathrm{m}(\alpha)\} \rangle .$$

It is clear from condition $(*)$ that $H = FA(\{y(\eta) : \eta < \mathrm{m}(\alpha)\})$. Since

$$\{i < \alpha : y(\eta)_i \neq x(\eta)_i\} \subseteq A(\eta)$$

and the sets $A(\eta)$ are pairwise disjoint, the group $H$ is $G_\delta$-dense in $K^\alpha$ by Lemma 4.1. ∎

4.5. Lemma. *Let $\alpha > \omega$ and let $K$ be a compact Abelian group such that* $\mathrm{w}K \leq \alpha$ *and $K \neq \mathrm{tor}(K)$. Then $K^\alpha$ contains a $G_\delta$-dense copy of $FA(\alpha^\omega)$.*

Proof. From $\mathrm{w}(K^\alpha) = \alpha$ we have $\mathrm{w}(\mathrm{P}(K^\alpha)) \leq \alpha^\omega$, so there is a base $\{U(\eta) : \eta < \alpha^\omega\}$ for $\mathrm{P}(K^\alpha)$ with each $U(\eta)$ of the form

$$U(\eta) = V(\eta) \times K^{\alpha \backslash C(\eta)} ;$$

here $C(\eta) \in [\alpha]^\omega$ and $V(\eta) = \prod_{i \in C(\eta)} V(\eta, i)$ with each $V(\eta, i)$ a $G_\delta$-subset of $K_i$. For $\eta < \alpha^\omega$ we choose $t(\eta) \in V(\eta)$.

Since $\alpha \backslash C(0) \neq \emptyset$ there is $y(0) \in K^\alpha \backslash \mathrm{tor}(K^\alpha)$ such that $y(0)_i = t(0)_i$ for $i \in C(0)$. Now recursively, if $\zeta < \alpha^\omega$ and $y(\eta)$ has been defined for all $\eta < \zeta$, set $H(\zeta) = \langle \{y(\eta) : \eta < \zeta\} \rangle$, note that

$$|\pi_{\alpha \backslash C(\zeta)}[H(\zeta)]| \leq |H(\zeta)| < \alpha^\omega \leq 2^\alpha = |K^{\alpha \backslash C(\zeta)}| ,$$

and use Lemma 4.2 to choose $y'(\zeta) \in K^{\alpha \backslash C(\zeta)} \backslash \pi_{\alpha \backslash C(\zeta)}[H(\zeta)]$ such that

$$(*) \qquad \langle \pi_{\alpha \backslash C(\zeta)}[H(\zeta)] \cup \{y'(\zeta)\} \rangle = \pi_{\alpha \backslash C(\zeta)}[H(\zeta)] \oplus \langle y'(\zeta) \rangle;$$

then define $y(\zeta) \in K^\alpha$ by the rule

$$y(\zeta) = \begin{cases} y'(\zeta) & \text{if } i \in \alpha \backslash C(\zeta), \\ t(\zeta)_i & \text{if } i \in C(\zeta). \end{cases}$$

This defines $y(\eta)$ for all $\eta < \alpha^\omega$. We set $H = \langle \{y(\zeta) : \zeta < \alpha^\omega\} \rangle$. It is clear from condition $(*)$ that $H = FA(\{y(\eta) : \eta < \alpha^\omega\})$. That $H$ is $G_\delta$-dense in $K^\alpha$ follows from the relation $y(\eta) \in U(\eta) \cap H$. ∎

4.6. Theorem. *Let $\alpha$ and $\gamma$ be infinite cardinals such that either*

(i) $\mathrm{m}(\alpha) \leq \alpha$ *and* $\mathrm{m}(\alpha) \leq \gamma \leq 2^\alpha$, *or*

(ii) $\alpha > \omega$ *and* $\alpha^\omega \leq \gamma \leq 2^\alpha$.

*Then $FA(\gamma) \in \mathcal{G}(\alpha)$. Further, the witnessing pseudocompact group topology on $FA(\gamma)$ may be chosen either connected, or* 0*-dimensional and linear.*

P r o o f. For the "connected" conclusion take $K = \mathbb{T}$ or $K = \Sigma_{\mathbf{a}}$ in what follows; for the "0-dimensional and linear" conclusion take $K = \Delta_p$.

Using 4.4 and $\delta = \mathrm{m}(\alpha)$ under hypothesis (i), and using 4.5 and $\delta = \alpha^\omega$ under hypothesis (ii), there is a $G_\delta$-dense subgroup $H$ of $K^\alpha$ such that $H \approx FA(\delta)$. Since $r_0(K^\alpha) = 2^\alpha$, by Corollary 4.3 there is $G \approx FA(\gamma)$ such that $H \subseteq G \subseteq K^\alpha$. Clearly $G$ itself is $G_\delta$-dense in $K^\alpha$, hence pseudocompact by 1.3; and $\mathrm{w}G = \alpha$ by 1.2.

When $K = \mathbb{T}$ or $K = \Sigma_{\mathbf{a}}$ the group $K^\alpha$ is connected. When $K = \Delta_p$ the group $K^\alpha$ is 0-dimensional and linear. Since $K^\alpha = \beta G$ by 1.3, the corresponding properties are inherited from $K^\alpha$ by $G$. ∎

In our abstract [CRe2], the hypothesis $\alpha > \omega$ was inadvertently omitted ([CR2] Theorem 3(ii)). Concerning the necessity of this hypothesis see 4.9 below.

4.7. R e m a r k. A topological space $X$ is said to be *locally connected* if $X$ has a base of connected, open sets. It is known [Ban], [HI], [Wu] that a Tikhonov space $X$ is locally connected and pseudocompact if and only if $\beta X$ is locally connected. Since $\mathbb{T}^\alpha$ is locally connected and $\Sigma_{\mathbf{a}}^\alpha$ is not, it follows that some of the connected pseudocompact group topologies imposed on the groups $FA(\gamma)$ in Theorem 4.6 are locally connected and others are not. At the other extreme it is easy to see that every (Tikhonov) space $X$ such that $\beta X$ is 0-dimensional is itself *strongly* 0-*dimensional* in the sense that disjoint zero-sets of $X$ are separated by a partition (cf. [GJ](16.17))—that is, the Čech–Lebesgue dimension function dim satisfies $\dim X = 0$. Thus of necessity the witnessing 0-dimensional topologies afforded by Theorem 4.6 on the groups $FA(\gamma)$ are strongly 0-dimensional. In this connection, the referee has contributed the following remark. "There is even more necessity for the strong zero-dimensionality of the topologies under consideration: if $G$ is a zero-dimensional pseudocompact group then $e$ has a local base of clopen sets and hence so does every point of $\beta G$; it follows that $\beta G$ is zero-dimensional and that $G$ is strongly zero-dimensional."

4.8. Corollary. *Let $\alpha$ and $\gamma$ be infinite cardinals.*

(a) *If* $\alpha^\omega \le \gamma \le 2^{2^{(\alpha^\omega)}}$, *then* $FA(\gamma) \in \mathcal{G}(2^{(\alpha^\omega)})$.

(b) *If* $2^\alpha \le \gamma \le 2^{2^{2^\alpha}}$, *then* $FA(\gamma) \in \mathcal{G}(2^{2^\alpha})$.

P r o o f. From 1.5(a) and 4.6. ∎

4.9. Discussion. We showed in Theorem 3.3 that the condition $\mathrm{m}(\alpha) \le \gamma \le 2^\alpha$ is equivalent to the condition $\bigoplus_\gamma \mathbb{Q} \in \mathcal{G}(\alpha)$. One's initial speculation that the condition $FA(\gamma) \in \mathcal{G}(\alpha)$ may also be equivalent to these is thwarted in the particular case $\alpha = \omega$, $\gamma = \mathbf{c}$ by the concatenation of these two facts: (1) Every pseudocompact metrizable space is compact (cf. [GJ](3D.2)); and

(2) every locally compact group topology on a group of the form $FA(\gamma)$ is discrete (see [D]). Thus $FA(\mathbf{c}) \notin \mathcal{G}(\omega)$, while $\bigoplus_{\mathbf{c}} \mathbb{Q} \in \mathcal{G}(\omega)$. (This example shows that the implication (iii)⇒(iv) of Theorem 6.2 of [DS] fails when $\tau = \mathbf{c}, \sigma = \omega$.) We have been unable to find other pairs $\langle \alpha, \gamma \rangle$ of cardinals for which the implication $\mathrm{m}(\alpha) \le \gamma \le 2^\alpha \Rightarrow FA(\gamma) \in \mathcal{G}(\alpha)$ fails, so the following question is open in ZFC.

4.10. Question. If some group $G$ with $|G| = \gamma$ admits a pseudocompact group topology $\mathcal{T}$ with $\mathrm{w}(G, \mathcal{T}) = \alpha > \omega$, must $FA(\gamma)$ admit such a topology? ■

In case (M) is assumed, we give a positive answer to Question 4.10 in 4.13 below.

4.11. Lemma. *Let $B$ be a maximal independent subset of an infinite Abelian group $G$, and let $D$ be a divisible hull for $G$. Then $D$ is a divisible hull for the subgroup $\langle B \rangle$ generated by $B$ in $G$.*

Proof. If $E$ is divisible with $\langle B \rangle \subseteq E \subseteq D$ and $E \neq D$, then since every divisible group is a direct summand there is a non-degenerate subgroup $F$ of $G$ such that $D = E \oplus F$. But every non-degenerate subgroup of $D$ has non-trivial intersection with $G$ (cf. [Fu](24.3)), and with $0 \neq x \in F \cap G$ it is clear that $B \cup \{x\}$ is independent in $G$. ■

4.12. Theorem. *Let $\alpha$ and $\gamma$ be infinite cardinals and let $G$ be an Abelian group such that $|G| = \gamma \le 2^\alpha$. Suppose either*

(i) $\mathrm{m}(\alpha) \le \alpha$ *and* $\mathrm{m}(\alpha) \le r_0(G)$, *or*
(ii) $\alpha > \omega$ *and* $\alpha^\omega \le r_0(G)$.

*Then $G \in \mathcal{G}(\alpha)$. Indeed, $G$ admits a connected pseudocompact group topology of weight $\alpha$ with respect to which $\overline{G} = \mathbb{T}^\alpha$.*

Proof. Let $B$ be a maximal independent subset of $G$ and let $T = \mathrm{tor}(\langle B \rangle)$. We claim that $\langle B \rangle$ embeds algebraically as a $G_\delta$-dense subgroup of $\mathbb{T}^\alpha$. Note first that the divisible hull $D$ of $T$ satisfies $D \approx \bigoplus_{p \in \mathbb{P}} \bigoplus_{r_p(G)} \mathbb{Z}(p^\infty)$ with $r_p(G) \le |G| = \gamma \le 2^\alpha$, so algebraically we have

$$T \subseteq D \subseteq \bigoplus_{2^\alpha} \mathbb{Q} \oplus \bigoplus_{p \in \mathbb{P}} \bigoplus_{2^\alpha} \mathbb{Z}(p^\infty) \approx \mathbb{T}^\alpha .$$

Now with $T$ considered as a subgroup of $\mathbb{T}^\alpha$ we have from (the proof of) 4.6 that there is $F \subseteq \mathbb{T}^\alpha$ with $F \approx FA(r_0(G))$ and $F$ $G_\delta$-dense. From $F \cap T = \{1\}$ we have $F \oplus T \approx \langle B \rangle$, so $F \subseteq F \times T \approx \langle B \rangle \subseteq \mathbb{T}^\alpha$ and the claim is proved.

Now let $E$ be a divisible hull of $\langle B \rangle$ in $\mathbb{T}^\alpha$, and let $D$ be a divisible hull of $G$. Then $D$ is a divisible hull for $\langle B \rangle$ by Lemma 4.11, and according to [Fu](24.4) there is an isomorphism $\eta$ from $D$ onto $E$ leaving $\langle B\rangle$ fixed pointwise. We have

$$\langle B\rangle \subseteq \eta[G] \subseteq \eta[D] = E \subseteq \mathbb{T}^\alpha$$

with $\langle B\rangle$ $G_\delta$-dense in $\mathbb{T}^\alpha$, so $\eta[G]$ is (an isomorph of) $G$ which is $G_\delta$-dense in $\mathbb{T}^\alpha$, as required.

The statements $\mathrm{w}G = \alpha$, $G$ is connected, $\overline{G} = \mathbb{T}^\alpha$ follow as before from 1.2(c), 1.3, and the uniqueness of the Weil completion. ∎

For cardinal numbers $\alpha$ of the form $\alpha = 2^\sigma$, Theorem 4.12 is closely related to Theorem 6.4 of [DS]. Indeed,

$$\mathrm{m}(\alpha) \le (\log\alpha)^\omega \le \sigma^\omega \le 2^\sigma = \alpha$$

for such cardinals and the hypothesis $r_0(G) \le \alpha$ of [DS] is inessential.

We see next that if (M) is assumed then every Abelian group of full torsion-free rank can serve as a test space for the question whether $\mathcal{G}(\alpha)$ contains a group of cardinality $\gamma$.

4.13. Theorem. *Assume* (M), *let $\alpha$ and $\gamma$ be infinite cardinals with $\alpha > \omega$, and let $G$ be an Abelian group such that $|G| = r_0(G) = \gamma$. Then the following are equivalent.*

(i) $\mathrm{m}(\alpha) \le \gamma \le 2^\alpha$;
(ii) *$G$ admits a connected*, *pseudocompact group topology of weight* $\alpha$;
(iii) $G \in \mathcal{G}(\alpha)$.

Proof. That (ii)⇒(iii)⇒(i) is clear (without the hypothesis (M)). Now assume (i). If $\mathrm{m}(\alpha) \le \alpha$ then (ii) follows from 4.12(i). If $\omega < \alpha < \mathrm{m}(\alpha)$ then (M) gives $\alpha^\omega \le (\mathrm{m}(\alpha))^\omega = \mathrm{m}(\alpha) \le \gamma \le 2^\alpha$, so 4.12(ii) gives (ii). ∎

We show in 4.15 that the cardinality restriction on $r_0(G)$ in Theorems 4.12 and 4.13 cannot be omitted, and in 4.16(b) we offer an algebraic characterization of those Abelian groups $G$ with $r_0(G) = (r_0(G))^\omega$ which admit a pseudocompact, connected group topology of pre-assigned weight.

We use the following simple lemma from [CvM](2.16).

4.14. Lemma. *Every connected*, *totally bounded Abelian group $G$ with $|G| > 1$ satisfies $r_0(G) \ge$* $\mathbf{c}$. ∎

4.15. Theorem. *Let $G$ be a connected*, *pseudocompact Abelian group with* $\mathrm{w}G = \alpha \ge \omega$. *Then $r_0(G) \ge \mathrm{m}(\alpha)$.*

Proof. It is shown in [CvM](6.1), taking a subgroup $A$ of $G$ which is maximal with respect to the property $A \cap \mathrm{tor}(G) = \{0\}$, that there is $F \subseteq G$ such that $|F| \le \mathbf{c}$ and $\langle A \cup F\rangle$ is $G_\delta$-dense in $G$. (This argument does not depend on the hypothesis $|\mathrm{tor}(G)| > \mathbf{c}$ of [CvM](6.1).) The maximality of

$A$ and 4.14 give $r_0(A) = r_0(G) \geq \mathbf{c}$ so from $|F| \leq \mathbf{c}$ and 1.2(c) follows

$$r_0(G) = r_0(A) = |A| = |A \cup F| \geq \mathrm{d}(\mathrm{P}(G)) \geq \mathrm{d}(\mathrm{P}(\overline{G})) = \mathrm{m}(\alpha)\,. \ \blacksquare$$

4.16. Corollary. *Let $\alpha$ and $\gamma$ be infinite cardinals with $\alpha > \omega$, and let $G$ be an infinite Abelian group such that $|G| = \gamma$.*

(a) *Assume* (M). *Then $G$ admits a connected pseudocompact group topology with* $\mathrm{w}G = \alpha$ *if and only if* $r_0(G) \geq (\log \alpha)^\omega$ *and* $\alpha \geq \log \gamma$.

(b) *If* $r_0(G) = (r_0(G))^\omega$, *then $G$ admits a connected pseudocompact group topology with* $\mathrm{w}G = \alpha$ *if and only if* $r_0(G) \geq \log \alpha$ *and* $\alpha \geq \log \gamma$.

Proof. (a) ($\Rightarrow$) From 4.15 and (M) follows $r_0(G) \geq \mathrm{m}(\alpha) = (\log \alpha)^\omega$. ($\Leftarrow$) If $\mathrm{m}(\alpha) \leq \alpha$, use 4.12(i); if $\omega < \alpha \leq \mathrm{m}(\alpha)$ use

$$r_0(G) \geq (\log \alpha)^\omega = \mathrm{m}(\alpha) = (\mathrm{m}(\alpha))^\omega \geq \alpha^\omega$$

and then 4.12(ii).

The proof of (b) is similar to that of (a). $\blacksquare$

4.17. Remarks. (a) It is not difficult to see, using [HR](25.33), that when $\alpha, \gamma$ and $G$ are given as in Theorem 4.12, the conclusion of 4.12 remains valid when $\mathbb{T}$ is replaced by any infinite, compact, connected Abelian group $K$ with $\mathrm{w}K \leq \alpha$ for which $r_p(K) > 0$ whenever $r_p(G) > 0$.

(b) Any non-divisible Abelian group of cardinality $\mathbf{c}$ constitutes a counterexample to the statements "if" of 4.16(a), (b) for the case $\alpha = \omega$: Every pseudocompact, metrizable group is compact, hence (if connected) is divisible.

(c) From 4.15 and 4.16 one derives the following characterization without difficulty.

Theorem. *Assume* (M). *An infinite Abelian group $G$ admits a connected, pseudocompact group topology if and only if* $r_0(G) \geq (\log \log |G|)^\omega$.

(d) After this paper was completed we learned by a letter received from D. Dikranjan that he and D. Shakhmatov will answer Question 4.10 positively in a forthcoming paper. That same anticipated paper will contain also (in ZFC, with no additional set-theoretic assumptions) a characterization of those Abelian groups which admit a pseudocompact, connected group topology of pre-assigned weight.

**5. Pseudocompact Abelian groups with small torsion-free rank.** We shall see in §6 below that it is not unusual that an Abelian group of cardinality $\gamma \geq \mathbf{c}$ will admit the maximal number of pseudocompact topological group topologies—that is, $2^{2^\gamma}$-many. It is amusing that for certain divisible groups the estimate provided by Theorem 4.15 imposes a smaller upper bound on the number of pseudocompact group topologies. We show this in Theorem 5.2 below.

In this section and the next, given an Abelian group $G$ with $|G| = \gamma$ we reserve the symbol $\mathbb{H}$ to denote the group $\mathbb{H} = \mathrm{Hom}(G, \mathbb{T})$. For an Abelian group $N$ we write

$$\mathcal{S}(N) = \{A : A \text{ is a subgroup of } N\}$$

and for $N = \mathbb{H} = \mathrm{Hom}(G, \mathbb{T})$ we write

$$\mathcal{S}^*(\mathbb{H}) = \{A \in \mathcal{S}(\mathbb{H}) : A \text{ separates points of } G\}\,.$$

For $A \in \mathcal{S}^*(\mathbb{H})$ we denote by $\mathcal{T}_A$ the (Hausdorff) topology induced on $G$ by $A$.

The following facts will be useful. See [Fu](47.5) or [HR](24.47) for (a), [BCR](4.3) for (b), and [CRos1](1.3) for (c) and (d).

5.1. Theorem. *Let $G$ be an Abelian group with $|G| = \gamma \geq \omega$. Then*

(a) $|\mathbb{H}| = 2^\gamma$;

(b) *if* $A \in \mathcal{S}^*(\mathbb{H})$ *then* $\mathrm{w}(G, \mathcal{T}_A) = |A|$;

(c) *every totally bounded topological group topology* $\mathcal{T}$ *on* $G$ *has the form* $\mathcal{T} = \mathcal{T}_A$ *for* (*suitable*) $A \in \mathcal{S}^*(\mathbb{H})$; *and*

(d) *if* $A \in \mathcal{S}^*(\mathbb{H})$ *and* $B \in \mathcal{S}^*(\mathbb{H})$ *with* $A \neq B$, *then* $\mathcal{T}_A \neq \mathcal{T}_B$. ∎

5.2. Theorem. *Let $G$ be an infinite divisible Abelian group with $|G| = \gamma$. Then*

(a) *the number of pseudocompact group topologies on $G$ does not exceed* $(2^\gamma)^{2^{r_0(G)}}$; *and*

(b) *if* $2^{r_0(G)} \leq \gamma$ *then $G$ admits at most* $2^\gamma$*-many pseudocompact group topologies.*

Proof. (a) Let $\mathcal{P}$ be the set of pseudocompact group topologies for $G$, and set

$$X = \{A \in \mathcal{S}(\mathbb{H}) : |A| \leq 2^{r_0(G)}\}\,.$$

According to 5.1 for every $\mathcal{T} \in \mathcal{P}$ there is (a unique) $A \in \mathcal{S}^*(\mathbb{H})$ such that $\mathcal{T} = \mathcal{T}_A$; further, from 3.2(e) and Theorem 4.15 we have $A \in X$. Thus the map $\mathcal{T} = \mathcal{T}_A \to A$ is one-to-one from $\mathcal{P}$ into $X$, and we have $|\mathcal{P}| \leq |X| \leq (2^\gamma)^{2^{r_0(G)}}$, as required.

(b) is immediate from (a) and the relation $(2^\gamma)^\gamma = 2^\gamma$. ∎

5.3. Discussion. By way of illustration of the content of Theorem 5.2, let $G$ be a divisible Abelian group with $|G| = \gamma \geq \mathbf{c}$ and set $\gamma_0 = r_0(G), \gamma_1 = \sum_{p\in\mathbb{P}} r_p(G)$; then $\gamma = \gamma_0 + \gamma_1$.

(1) If $\gamma_0 < \mathbf{c}$ then (from 4.14) the group $G$ supports no pseudocompact group topology.

(2) If $\mathbf{c} \le \gamma_0$ and $2^{2^{\gamma_0}} < \gamma_1$ then $G$ supports no pseudocompact group topology (for the weight $\alpha$ of such a topology must satisfy $\alpha \le 2^{\gamma_0}$ and $\gamma_1 \le \gamma_0 + \gamma_1 \le 2^\alpha \le 2^{2^{\gamma_0}}$).

(3) If $\mathbf{c} \le \gamma_0$ and $2^{\gamma_0} \le \gamma_1 \le 2^{2^{\gamma_0}}$, then $G$ admits a pseudocompact group topology of weight $\alpha = 2^{\gamma_0}$ (for $\mathrm{m}(\alpha) \le (\gamma_0)^\omega \le 2^{\gamma_0} = \alpha \le \gamma_1 = \gamma \le 2^\alpha$ and 4.12(i) applies), and $G$ admits at most $2^{\gamma_1} = 2^\gamma$-many pseudocompact group topologies by 5.2(b).

(4) If $\mathbf{c} \le \gamma_0$ and $\gamma_1 < 2^{\gamma_0}$, no general statement is available. Taking $\gamma_1 = 0$, for example, so that $\gamma = \gamma_0$, we see from 3.5 that the condition $G \in \mathcal{G}(\alpha)$ for some $\alpha$ is equivalent to the condition $\mathrm{m}(\log \gamma) \le \gamma$. This holds for many $\gamma$, as we have seen, but according to Theorem 3.1(a) it fails for $\gamma \in \mathrm{SLC}$ with $\mathrm{cf}(\gamma) = \omega$.

**6. Chains and anti-chains.** Of course, a set of infinite cardinality $\gamma$ admits at most $2^{2^\gamma}$ topologies. We show that for many of the Abelian groups $G$ considered in this paper (with $|G| = \gamma$) this upper bound is achieved by a family of pseudocompact group topologies—and even, in suitable cases, by pseudocompact group topologies of pre-assigned weight.

In the interest of simplicity we restrict our attention here to groups of the form $\bigoplus_\gamma \mathbb{Q}$ and $FA(\gamma)$. The interested reader will experience no difficulty using the results of §4 to achieve statements parallel to Theorem 6.2 for groups $G$ with $r_0(G)$ suitably constrained.

We say as usual that a collection $\mathcal{C}$ of sets is a *chain* if $x, y \in \mathcal{C}$ implies $x \subseteq y$ or $y \subseteq x$; and $\mathcal{A}$ is an *anti-chain* if $x, y \in \mathcal{A}$ with $x \ne y$ implies that $x$ and $y$ are not $\subseteq$-comparable.

The following lemma is well-known. For proofs of (b) and (c), and remarks on the impossibility (for some $\kappa$, in some models of ZFC) of replacing $\kappa^+$ by $2^\kappa$, even when $\kappa$ has the form $\kappa = 2^\lambda$, see [Bau] or [CRe1](§1). For (a) it is enough to consider $\{\mathrm{graph}(f) : f \in \kappa^\kappa\}$, an anti-chain of subsets of $\kappa \times \kappa$.

6.1. Lemma. *Let $\kappa$ be an infinite cardinal. Then*

(a) *there is an anti-chain $\mathcal{A}$ of subsets of $\kappa$ such that $|\mathcal{A}| = 2^\kappa$ and each $C \in \mathcal{A}$ satisfies $|C| = \kappa$;*

(b) *there is a chain $\mathcal{B}$ of subsets of $\kappa$ such that $|\mathcal{B}| = \kappa^+$ and each $C \in \mathcal{B}$ satisfies $|C| = \kappa$;*

(c) *if $\kappa = 2^\lambda$ then there is a chain $\mathcal{C}$ of subsets of $\kappa$ such that $|\mathcal{C}| = 2^{(\lambda^+)}$ and each $C \in \mathcal{C}$ satisfies $|C| = \kappa$.* ■

In what follows we denote by $\mathcal{P}(G, \kappa)$ the set of pseudocompact group topologies of weight $\kappa$ on the group $G$.

6.2. Theorem. *Let $\alpha$ and $\gamma$ be cardinals such that either*

(i) $\alpha^\omega \le \gamma < \kappa \le 2^{(\alpha^\omega)}$, *or*

(ii) $2^\alpha \le \gamma < \kappa \le 2^{2^\alpha}$,

*and let $G = \bigoplus_\gamma \mathbb{Q}$ or $G = FA(\gamma)$. Then*

(a) *there is an anti-chain $\mathcal{A} \subseteq \mathcal{P}(G,\kappa)$ such that $|\mathcal{A}| = 2^\kappa$;*

(b) *there is a chain $\mathcal{B} \subseteq \mathcal{P}(G,\kappa)$ such that $|\mathcal{B}| = \kappa^+$;*

(c) *if $\kappa = 2^\lambda$ (for example, with $\lambda = \alpha^\omega$ in* (i) *and $\lambda = 2^\alpha$ in* (ii)), *then there is a chain $\mathcal{C} \subseteq \mathcal{P}(G,\kappa)$ such that $|\mathcal{C}| = 2^{(\lambda^+)}$; and*

(d) $|\mathcal{P}(G,\kappa)| = 2^\kappa$.

Proof. We begin with ($\le$) in (d). According to 5.1 we have $|\mathbb{H}| = 2^\gamma$, and every $\mathcal{T} \in \mathcal{P}(G,\kappa)$ satisfies $\mathcal{T} = \mathcal{T}_A$ for some $A \in \mathcal{S}^*(\mathbb{H})$ with $|A| = \kappa$. Now

$$|\{A \subseteq \mathbb{H} : |A| = \kappa\}| = (2^\gamma)^\kappa = 2^\kappa\,,$$

so the inequality ($\le$) of (d) is proved.

To prove ($\ge$) in (d), and (a) and (b) and (c), note first that

$$\mathrm{m}(\kappa) \le (\log\kappa)^\omega \le (\alpha^\omega)^\omega = \alpha^\omega \le \gamma \le 2^\kappa$$

in (i), and

$$\mathrm{m}(\kappa) \le (\log\kappa)^\omega \le (2^\alpha)^\omega = 2^\alpha \le \gamma \le 2^\kappa$$

in (ii), so by Theorem 3.3 in case $G = \bigoplus_\gamma \mathbb{Q}$ and by Theorem 4.6(i) in case $G = FA(\gamma)$ there is $\mathcal{U} \in \mathcal{P}(G,\kappa)$. Let $\mathcal{U} = \mathcal{T}_X$ with $X \in \mathcal{S}^*(\mathbb{H})$, choose $S \in \mathcal{S}^*(\mathbb{H})$ such that $S \subseteq X$ and $|S| \le \gamma$, and let $\phi : X \to X/S = F$ be the natural homomorphism. From $\gamma < \kappa = |X|$ it follows that $|F| = \kappa$, so according to [Fu](16.1) the group $F$ contains a subgroup of the form $\bigoplus_{i<\kappa} F_i$ with each $|F_i| > 1$. For $Y \subseteq \kappa$ let

$$H(Y) = \phi^{-1}\Big(\bigoplus_{i\in Y} F_i\Big) = \Big\langle \Big(\bigcup_{i\in Y} \phi^{-1}(F_i)\Big)\cdot S\Big\rangle \in \mathcal{S}^*(\mathbb{H})\,.$$

Let $\mathcal{A}', \mathcal{B}'$ and $\mathcal{C}'$ be families of subsets of $\kappa$ as guaranteed by Lemma 6.1: $|\mathcal{A}'| = 2^\kappa$, $|\mathcal{B}'| = \kappa^+$, and if $\kappa = 2^\lambda$ then $|\mathcal{C}'| = 2^{(\lambda^+)}$—and with $|C| = \kappa$ for each element $C$ of $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$. Finally, define

$$\mathcal{A} = \{\mathcal{T}_{H(C)} : C \in \mathcal{A}'\}, \quad \mathcal{B} = \{\mathcal{T}_{H(C)} : C \in \mathcal{B}'\}, \quad \text{and}$$
$$\mathcal{C} = \{\mathcal{T}_{H(C)} : C \in \mathcal{C}'\}\,.$$

Clearly $\mathcal{A}$ is an anti-chain, and $\mathcal{B}$ and $\mathcal{C}$ are chains, of topological group topologies on $G$, with $|\mathcal{A}| = 2^\kappa$, $|\mathcal{B}| = \kappa^+$, and $|\mathcal{C}| = 2^{(\lambda^+)}$. Each group $H(C)$ satisfies

$$|H(C)| = |C|\cdot|S| = \kappa\cdot\gamma = \kappa\,,$$

so $\mathrm{w}(G, \mathcal{T}_{H(C)}) = \kappa$. For each $C$ we have $H(C) \subseteq X$ and hence $\mathcal{T}_{H(C)} \subseteq \mathcal{T}_X = \mathcal{U}$, so the identity function $i : \langle G, \mathcal{U}\rangle \to \langle G, \mathcal{T}_{H(C)}\rangle$ is continuous and $\mathcal{T}_{H(C)}$ is a pseudocompact group topology for $G$. ∎

Two special cases of 6.2(d) are worth recording.

6.3. COROLLARY. *Let $\alpha$ be an infinite cardinal.*

(a) *If $\alpha^\omega \le \gamma < 2^{(\alpha^\omega)}$, then both $\bigoplus_\gamma \mathbb{Q}$ and $FA(\gamma)$ admit $2^{2^{(\alpha^\omega)}}$-many pseudocompact group topologies of weight $2^{(\alpha^\omega)}$.*

(b) *If $2^\alpha \le \gamma < 2^{2^\alpha}$, then both $\bigoplus_\gamma \mathbb{Q}$ and $FA(\gamma)$ admit $2^{2^{2^\alpha}}$-many pseudocompact group topologies of weight $2^{2^\alpha}$.* ∎

6.4. Remark. It has been announced by M. G. Tkachenko [T1] (Theorem 3) that for every infinite cardinal $\gamma$ with $\gamma = \gamma^\omega$ the group $FA(\gamma)$ admits a chain of pseudocompact group topologies of length $2^{(\gamma^+)}$; see [T2] for a proof. This is a special case of 6.2(c) with $\alpha = \gamma, \kappa = 2^\gamma$.

**7. A remark on countably compact group topologies.** The problem of finding and counting countably compact group topologies appears more delicate than the pseudocompact considerations undertaken here. In one special case, however, the techniques we have developed give some information. In parallel with earlier notation we denote by $\mathcal{CC}(\alpha)$ the class of all groups which admit a countably compact group topology of weight $\alpha$; and, given a group $G$, we denote by $\mathcal{CC}(G,\alpha)$ the set of countably compact group topologies of weight $\alpha$ on $G$.

7.1. THEOREM. *Let $\alpha$ and $\gamma$ be infinite cardinals such that $\gamma = \gamma^\omega$ and $\log\alpha \le \gamma \le 2^\alpha$, and let $G = \bigoplus_\gamma \mathbb{Z}(p)$ with $p \in \mathbb{P}$. Then $G$ admits a countably compact group topology of weight $\alpha$. If in addition $\gamma < \kappa \le \alpha$, then*

(a) *there is an anti-chain $\mathcal{A} \subseteq \mathcal{CC}(G,\kappa)$ such that $|\mathcal{A}| = 2^\kappa$;*

(b) *there is a chain $\mathcal{B} \subseteq \mathcal{CC}(G,\kappa)$ such that $|\mathcal{B}| = \kappa^+$;*

(c) *if $\kappa = 2^\lambda$ then there is a chain $\mathcal{C} \subseteq \mathcal{CC}(G,\kappa)$ such that $|\mathcal{C}| = 2^{(\lambda^+)}$; and*

(d) *$|\mathcal{CC}(G,\kappa)| = 2^\kappa$.*

Proof. We have $\mathrm{d}((\mathbb{Z}(p))^\alpha) = \log\alpha \le \gamma$ by Theorem 1.2(a). It is then easy, as noted in [vD](§3), to find a dense, countably compact subgroup $G$ of $(\mathbb{Z}(p))^\alpha$ such that $|G| = \gamma$. Since $G$ is an elementary $p$-group we have $G \approx \bigoplus_\gamma \mathbb{Z}(p)$. We note from Theorems 5.1 and 1.2(c) that the (totally bounded) topology $\mathcal{T}$ inherited by $G$ from $(\mathbb{Z}(p))^\alpha$ has the form $\mathcal{T} = \mathcal{T}_X$ for suitable $X \in \mathcal{S}^*(\mathbb{H})$ with $|X| = \alpha$.

The proofs of (a)–(d) when $\gamma < \kappa \le \alpha$ is assumed now proceed much as in 6.2 above. ∎

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*Received 11 February 1992;*
*in revised form 2 September 1992*