FUNDAMENTA
MATHEMATICAE
142 (1993)

A triple intersection theorem for the varieties $SO(n)/P_d$

by

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Abstract. We study the Schubert calculus on the space of d-dimensional linear subspaces of a smooth n-dimensional quadric lying in the projective space. Following Hodge and Pedoe we develop the intersection theory of this space in a purely combinatorial manner. We prove in particular that if a triple intersection of Schubert cells on this space is nonempty then a certain combinatorial relation holds among the Schubert symbols involved, similar to the classical one. We also show when these necessary conditions are also sufficient to obtain a nontrivial intersection. Several examples are calculated to illustrate the main results.

INTRODUCTION

The aim of the present paper is to establish Schubert calculus on a certain class of homogeneous spaces. To be more precise, let Q_n be a nonsingular quadric hypersurface in $\mathbb{P}^{n+1}$ and let $G(d, Q_n)$ be the set of d-dimensional linear subspaces which lie on Q_n. The orthogonal group $O(n+2)$ acts transitively on $G(d, Q_n)$ in a natural way so that $G(d, Q_n) \simeq O(n+2)/P_{d+1}$, where P_{d+1} is the stabilizer of an arbitrary element in $G(d, Q_n)$. If $d < [n/2]$, then $SO(n+2)$, the special orthogonal group, operates on $G(d, Q_n)$ transitively, and hence $G(d, Q_n) \simeq SO(n+2)/SP_{d+1}$, where $SP_{d+1} = SO(n+2) \cap P_{d+1}$. These spaces $G(d, Q_n)$ are the objects we study in this paper.

These spaces, which are also described as $A_s^{(m)}$, the space of normalized complex s-substructures of $\mathbb{R}^m$, were studied by Dibağ [3], where they appeared as fibers in certain global obstruction problems. He defined some Schubert cells on them which form bases of the cohomology rings of the space in question, and found that these Schubert cells have beautiful duality properties. This discovery was our motivation to establish Schubert symbolism on $G(d, Q_n)$.

$G(d, Q_n)$ is, by definition, a subvariety of $G(d, \mathbb{P}^{n+1})$, the Grassmann variety of d-dimensional linear subspaces in the complex projective space of dimension $n+1$. Our method here is to follow and generalize the classical treatment of Hodge and Pedoe in [7], where they develop the intersection theory on Grassmannians in a purely combinatorial manner. Thus in this paper we prove that if a triple intersection of Schubert cells on $G(d, Q_n)$ is nonempty, then there follows a combinatorial relation, similar to the classical one [7].

In the classical case, the combinatorial relation mentioned above implies the nonempty triple intersection, which amounts to the Pieri formula and the Giambelli formula. In our case, this does not hold in general because of the strange behaviour of the linear subspaces of a quadric. Conditions for this to hold are also discussed here.

Geometrically speaking, we are going to study the Schubert calculus on the space of d-dimensional linear subspaces of a smooth quadric Q_n lying in the projective space $\mathbb{P}^{n+1}$. This variety is denoted by $G(d, Q_n)$. It is a $\frac{1}{2}(d+1)(2n-3d)$-dimensional subspace of $G(d, \mathbb{P}^{n+1})$, the Grassmann space of d-dimensional linear subspaces of the projective space $\mathbb{P}^{n+1}$. The correspondence between the spaces mentioned so far is as follows:

$$SO(n+2)/P_{d+1} = G(d, Q_n) = A_{d+1}^{(n+2)}\,.$$

Throughout the article we let $n = 2m$ or $n = 2m+1$ and d is always a positive integer less than or equal to m. In Section I we define certain points of $\mathbb{P}^{n+1}$ as the skeleton points of Q_n. We define a flag using these skeleton points and interpret the definition of Schubert cells of $G(d, Q_n)$ with respect to this flag. In Section II we quote the classical intersection theorem of Hodge and Pedoe for comparison reasons. Section III gives the proof of our intersection theorem for $G(d, Q_n)$. Since the geometry of smooth quadrics varies depending on the parity of their dimension, our arguments inevitably treat these two cases separately. In Section IV we give explicit examples and discuss the converse of our triple intersection theorem.

Note that Hiller and Boe in [6] treated the case $n = 2m+1$ and $d = m$ and gave a Pieri type formula. A Giambelli type formula in this case was given by Pragacz in [9]. A simple and transparent proof of the main results of [6] can be found in [11]. Finally, we refer the reader to the survey article [10] for recent developments.

The *special Schubert cycle* σ_h, $0 < h \le n-d$, is the set of $[d]$-planes intersecting a given $[n-d-h]$-dimensional space lying on the quadric Q_n. The codimension of σ_h is h. For other definitions needed in the statement of our main result see Section 3.

Main Theorem. *For any two Schubert cycles $\Omega_{a_0\ldots a_d}$ and $\Omega_{b_0\ldots b_d}$ of $A_{d+1}^{(n+2)}$ there exist integers $\lambda_0, \ldots, \lambda_{d+3}$ depending only on $a_0, \ldots, a_d$, $b_0, \ldots, b_d$ and the parity of n such that for any special Schubert cycle σ_h,*

$0 < h \leq n-d$, if

(1) $$\dim_{\mathbb{C}} \Omega_{a_0 \ldots a_d} + \dim_{\mathbb{C}} \Omega_{b_0 \ldots b_d} + \dim_{\mathbb{C}} \sigma_h = 2 \dim_{\mathbb{C}} A^{(n+2)}_{d+1}$$

and

(2) $$\Omega_{a_0 \ldots a_d} \Omega_{b_0 \ldots b_d} \sigma_h \neq 0$$

then

(3) $$(n-d) - \frac{1}{2} d(d+1) + e \leq h + \sum_{i=0}^{d+3} \lambda_i \leq n-d\,,$$

where $e(\Omega_{a_0 \ldots a_d})$ is defined as the cardinality of the set $\{(a_i, a_j) \mid i < j$ and $a_i + a_j < n\}$, and e is $e(\Omega_{a_0 \ldots a_d}) + e(\Omega_{b_0 \ldots b_d})$.

The λ_i's for the $n = 2m$ case are given in Lemmas 6.1, 6.2 and in Section 6.3. The λ_i's for the $n = 2m+1$ case are given in Lemma 7.1. A partial converse to this theorem is given in the last section (see Theorem 13).

We refer to conditions (2) and (3) as MT(2) and MT(3) respectively in the forthcoming discussions.

Acknowledgements. I thank Prof. İ. Dibağ for suggesting the problem and supplying material, and also for his encouragement at several stages. I also thank Prof. P. Pragacz for his numerous comments and generous help.

I. FLAGS A AND B IN Q_n AND SCHUBERT CELLS

1. Flags A and B in the $n = 2m$ case. We first fix $2m+2$ skeleton points on Q_{2m} in Section 1.1 and examine in Sections 1.2 and 1.3 the dimensions of certain spaces constructed from skeleton points. Flags A and B are then constructed in Section 1.4. Schubert cells will be constructed in Section 3. They define homology cycles independent of the flags used, and hence are independent of the skeleton points chosen; this follows from [3] and [7].

1.1. We choose and fix $2m+2$ points $p_0, \ldots, p_{2m+1}$ in Q_{2m}, called the *skeleton points* of Q_{2m}, as follows:

(i) Choose p_0 in Q_{2m} arbitrarily.

(ii) Once $p_0, \ldots, p_{k-1}$ in Q_{2m} are chosen with $k \leq m$, choose p_k as any point in Q_{2m} which is not in the join of $p_0, \ldots, p_{k-1}$ but in the f-orthogonal of the join (f-orthogonal means orthogonal with respect to the form $Q^c(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2$, see [3, pp. 501–502] for further details). In the notation from [3] we have

$$p_k \in \{(p_0 \vee \ldots \vee p_{k-1})^{\perp_{\mathrm{f}}} - (p_0 \vee \ldots \vee p_{k-1})\} \cap Q_{2m},$$

where we have used the notation $\perp_{\mathrm{f}}$ to denote orthogonality with respect to the above form (f-orthogonality).

(iii) Once $p_0, \ldots, p_m \in Q_{2m}$ are chosen, the remaining points are their complex conjugates, ordered as follows:

$$p_{2m+1-i} = \mathrm{c}(p_i), \qquad i = 0, \ldots, m\,,$$

where $\mathrm{c}(\cdot)$ is the complex conjugate.

1.2. Let I be a subset of $I_m = \{0, 1, \ldots, m\}$. Define S_I as the intersection of Q_{2m} with the join of all skeleton points p_i with i in I, i.e. $S_I = (\bigvee_{i\in I} p_i) \cap Q_{2m}$. Let $\bar{I}$ denote the set of all integers of the form $2m+1-i$ with i in I. Then we have the following lemma.

Lemma 1.2. *If I and J are two nonempty, disjoint subsets of I_m then*:

(i) *S_I is a linear subspace of Q_{2m} and $\dim_{\mathbb{C}} S_I = \#I - 1$, where $\#I$ is the cardinality of I.*

(ii) *$S_{J\cup\bar{J}}$ is a smooth subquadric of Q_{2m} and $\dim_{\mathbb{C}} S_{J\cup\bar{J}} = 2\#J - 2$.*

(iii) *$S_{I\cup J\cup\bar{J}}$ is the join of S_I and $S_{J\cup\bar{J}}$ in Q_{2m}, and $\dim_{\mathbb{C}} S_{I\cup J\cup\bar{J}} = 2\#J + \#I - 2$.* ∎

1.3. For any nonempty subset L of I_{2m+1} define S_L as in 1.2. To find the dimension of S_L we construct two disjoint subsets $I(L)$ and $J(L)$ of I_m as follows:

$$I(L) = \{i \in I_m \mid \text{either } i \in L \text{ or } 2m+1-i \in L, \text{ but not both}\},$$
$$J(L) = \{i \in I_m \mid i \in L \text{ and } 2m+1-i \in L\}\,.$$

The following lemma on the dimension of S_L can now be proved using 1.2.

Lemma 1.3 ($n = 2m$).

$$\dim_{\mathbb{C}} S_L = \begin{cases} \#L - 2 & \text{if } J(L) \neq \emptyset, \\ \#L - 1 & \text{if } J(L) = \emptyset. \end{cases} \ ∎$$

1.4. Flag A consists of a nested sequence of subvarieties

$$A_0 \subset A_1 \subset \ldots \subset A_{m_0}, A_{m_1} \subset A_{m+1} \subset \ldots \subset A_{2m} = Q_{2m}$$

of Q_{2m} such that $A_i - A_{i-1}$ is an open cell of dimension i [3, p. 503]. Using the skeleton points introduced above we define a flag A where each A_i is defined as follows:

(i) $A_i = S_{\{0,1,\ldots,i\}}$ for $i = 0, \ldots, m-1$.

(ii) $A_{m_0} = S_{\{0,1,\ldots,m\}}$ and $A_{m_1} = S_{\{0,1,\ldots,m-1,m+1\}}$.

(iii) $A_{m+i} = S_{\{0,1,\ldots,m+1+i\}}$ for $i = 1, \ldots, m$.

Denote by V_0 and V_1 the two disjoint families of projective $[m]$-planes in Q_{2m}. We have arbitrarily labeled $S_{\{0,1,\ldots,m\}}$ as an element of V_0. Consequently, $S_{\{0,1,\ldots,m-1,m+1\}}$ must belong to V_1 regardless of m being odd or even. Together with a flag A we will consider its "dual" flag B:

$$B_0 \subset B_1 \subset \ldots \subset B_{m_0}, B_{m_1} \subset \ldots \subset B_{2m} = Q_{2m}\,.$$

For a discussion of dual flags on quadrics see [3, p. 512]. Assuming m is even we define B_i as follows:

(i) $B_i = S_{\{2m+1,2m,\dots,2m+1-i\}}$ for $i = 0,\dots,m-1$.
(ii) $B_{m_0} = S_{\{2m+1,2m,\dots,m+2,m\}}$ and $B_{m_1} = S_{\{2m+1,2m,\dots,m+1\}}$.
(iii) $B_{m+i} = S_{\{2m+1,2m,\dots,m-i\}}$ for $i = 1,\dots,m$.

If, however, m is odd, then we redefine B_{m_0} and B_{m_1} as

$$B_{m_0} = S_{\{2m+1,\dots,m+1\}} \quad \text{and} \quad B_{m_1} = S_{\{2m+1,\dots,m+2,m\}}\,.$$

2. Flags A and B in the $n = 2m+1$ case

2.1. The smooth quadric Q_{2m+1} in $\mathbb{P}^{2m+2}$ can be realized as the intersection in $\mathbb{P}^{2m+3}$ of Q_{2m+2} with a hyperplane H. With this in mind the geometric meaning of the skeleton points of Q_{2m+1} as defined below can be visualized as follows: construct a set of skeleton points $p_0,\dots,p_{2m+3}$ of Q_{2m+2} in $\mathbb{P}^{2m+3}$ as explained in 1.1. The hyperplane H is then defined by identifying the coefficients of p_{m+1} with p_{m+2} in the join $p_0\vee\dots\vee p_{2m+3}$. The skeleton points of Q_{2m+1} are then obtained by renumbering the remaining points.

The skeleton points $p_0,\dots,p_{2m+2}$ of Q_{2m+1} are chosen in the following manner:

(i) Choose $p_0 \in Q_{2m+1}$ arbitrarily.

(ii) For $0 < k < m$, p_k is any point in Q_{2m+1} which is in the f-orthogonal of the join $p_0 \vee \dots \vee p_{k-1}$ but not in the join.

(iii) The complex conjugates of $p_0,\dots,p_m$ are also skeleton points with indices set as follows:

$$p_{2m+2-i} = \mathrm{c}(p_i), \qquad i = 0,\dots,m\,.$$

(iv) Choose p_{m+1} as any point in $\mathbb{P}^{2m+2}$ which is f-orthogonal to $p_0 \vee \dots \vee p_m \vee p_{m+2} \vee \dots \vee p_{2m+2}$.

It is easy to see that p_{m+1} is not a point of the quadric and that the points $p_0,\dots,p_{2m+2}$ span the whole space $\mathbb{P}^{2m+2}$.

2.2. Let L be a subset of $I_{2m+2} = \{0,\dots,2m+2\}$. Define the subsets $I(L)$ and $J(L)$ of I_m as

$$I(L) = \{i \in I_m \mid \text{either } i \in L \text{ or } 2m+2-i \in L, \text{ but not both}\}\,,$$
$$J(L) = \{i \in I_m \mid i \in L \text{ and } 2m+2-i \in L\}\,.$$

Notice that neither of these sets can include $m+1$.

We further define a constant that depends on L:

$$\varepsilon = \begin{cases} 0 & \text{if } m+1 \notin L, \\ 1 & \text{if } m+1 \in L. \end{cases}$$

We use this constant to determine the dimension of S_L:

LEMMA 2.2 ($n = 2m+1$).

$$\dim_{\mathbb{C}} S_L = \begin{cases} \#L - 2 & \text{if } J(L) \neq \emptyset, \\ \#L - 1 - \varepsilon & \text{if } J(L) = \emptyset. \end{cases}$$

Proof. It can be shown that $\dim_{\mathbb{C}} S_L = (\#I(L)-1)+(2\#J(L)-2)+1+\varepsilon$ if $J(L) \neq \emptyset$, and $\dim_{\mathbb{C}} S_L = \#I(L) - 1$ if $J(L) = \emptyset$. Combining these equalities with the fact that $\#L = \#I(L) + 2\#J(L) + \varepsilon$ yields the lemma. ∎

2.3. Flag A consists of a nested sequence

$$A_0 \subset \ldots \subset A_{2m+1} = Q_{2m+1}$$

where

(i) $A_i = S_{\{0,\ldots,i\}}$ for $i = 0, \ldots, m$,
(ii) $A_{m+i} = S_{\{0,\ldots,m+1+i\}}$ for $i = 1, \ldots, m+1$.

In this case flag B is defined as

$$B_0 \subset \ldots \subset B_{2m+1} = Q_{2m+1}$$

where

(i) $B_i = S_{\{2m+2,\ldots,2m+2-i\}}$ for $i = 0, \ldots, m$,
(ii) $B_{m+i} = S_{\{2m+2,\ldots,m+1-i\}}$ for $i = 1, \ldots, m+1$.

3. Schubert cells on $A_{d+1}^{(n+2)}$. A reference for the spaces $A_s^{(n)}$ and the Schubert cycles on them is [3]. Here we recall the basic definitions and results. First note that for $d < [n/2]$ we can realize $A_{d+1}^{(n+2)}$ as a $(d+1) \times (n - \frac{3}{2}d)$-dimensional subvariety of $G(d, \mathbb{P}^{n+1})$, the Grassmann variety of $[d]$-planes in $\mathbb{P}^{n+1}$. Any $q \in A_{d+1}^{(2m+2)}$ can hence be considered as a $[d]$-plane, and using this interpretation we can define a sequence of subspaces in Q_{2m},

$$q_0 \subset \ldots \subset q_{m-1} \subset q_{m_0}, q_{m_1} \subset q_{m+1} \subset \ldots \subset q_{2m}$$

where $q_i = q \cap A_i$ if $i = 0, 1, \ldots, \widehat{m}, \ldots, 2m$ and $q_{m_j} = q \cap A_{m_j}$ for $j = 0$ or 1. The (closed) *Schubert cell* corresponding to the integers $0 \leq a_0 < \ldots < a_d \leq n$, with $a_i + a_j \neq n$ for $i < j$, is defined as

$$\Omega_{a_0 \ldots a_d} = \{q \in A_{d+1}^{(2m+2)} \mid \dim_{\mathbb{C}} q_{a_i} \geq i\}\,.$$

We do not lose any generality by using only those $\Omega_{a_0 \ldots a_d}$'s for which $a_i + a_j \neq n$. This only avoids duplication (see [3, p. 506]).

The homology cycle represented by this cell, denoted by the same notation, is independent of the skeleton points used in its definition. The dimension of the cycle depends only on the Schubert symbol used:

$$\dim_{\mathbb{C}} \Omega_{a_0 \ldots a_d} = a_0 + \ldots + a_d - d(d+1) + e$$

where

$$e = \#\{(a_i, a_j) \mid i < j \text{ and } a_i + a_j < n\}\,.$$

In the above notation the special Schubert cycle σ_h appearing in the main theorem (see Introduction) can be expressed as

$$\begin{array}{ll} \Omega_{n-d-h\ n-d+1\ldots n} & \text{for } 0 < h \leq n-2d\,,\ \text{and} \\ \Omega_{n-d-h\ n-d\ldots\widehat{d+h}\ldots n} & \text{for } n-2d < h \leq n-d\,, \end{array}$$

where $\widehat{d+h}$ means that $d+h$ is to be omitted.

If $n-d-h=m$, then we necessarily need to distinguish between m_0 and m_1, but in the triple intersection arguments we do not need this distinction for the special Schubert cycles.

The Schubert cycles for the odd-dimensional case, $A_{d+1}^{(2m+1)}$, are defined similarly using the corresponding flag defined earlier.

II. DEFINITIONS AND RESULTS FROM STANDARD INTERSECTION THEORY

The results of this section are classical (see for example [4], [7], [8]). We include this section with the sole purpose of comparing the main theorem of this paper with the classical triple intersection theorem on Grassmannian manifolds.

4. Summary. Let $0 = V_0 \subset V_1 \subset \ldots \subset V_{n+1} = \mathbb{C}^{n+1}$ be a nested sequence of vector subspaces of $\mathbb{C}^{n+1}$ where $\dim_{\mathbb{C}} V_i = i$ for $i = 0, \ldots, n+1$. If we define $A_i = \mathbb{P}(V_{i+1})$, the projectivization of V_{i+1}, for $i = 0, \ldots, n$, then

$$A_0 \subset A_1 \subset \ldots \subset A_n = \mathbb{P}^n$$

is a cellular decomposition of $\mathbb{P}^n$. The variety of projective $[d]$-planes in $\mathbb{P}^n$ is denoted by $G(d, \mathbb{P}^n)$. The *Schubert variety* corresponding to the integers $0 \leq a_0 < \ldots < a_d \leq n$ is defined as

$$\Omega^c_{a_0\ldots a_d} = \{q \in G(d, \mathbb{P}^n) \mid \dim_{\mathbb{C}}(q \cap A_{a_i}) \geq i, i = 0, \ldots, d\}\,.$$

Recall that the homology cycle represented by $\Omega^c_{a_0\ldots a_d}$ is independent of the flag chosen and

$$\dim_{\mathbb{C}} \Omega^c_{a_0\ldots a_d} = a_0 + \ldots + a_d - \tfrac{1}{2}d(d+1)\,.$$

The *special Schubert cycle* σ_h^c is defined to be the cycle $\Omega^c_{n-d-h\ n-d+1\ldots n}$ and its codimension is h. Schubert cycles give a $\mathbb{Z}$-basis of the cohomology ring of $G(d, \mathbb{P}^n)$. As for the cohomology ring structure, we have equalities of the form

$$\Omega^c_{a_0\ldots a_d}\Omega^c_{b_0\ldots b_d} = \sum \alpha(a,b,c)\Omega^c_{c_0\ldots c_d}$$

where $\alpha(a,b,c)$ is an integer and the summation is over all $\Omega^c_{c_0\ldots c_d}$ such that

$$\dim_{\mathbb{C}} \Omega^c_{c_0\ldots c_d} = \dim_{\mathbb{C}} \Omega^c_{a_0\ldots a_d} + \dim_{\mathbb{C}} \Omega^c_{b_0\ldots b_d} - \dim_{\mathbb{C}} G(d, \mathbb{P}^n)\,.$$

One has

$$\alpha(a,b,c) = \Omega^c_{a_0\dots a_d}\Omega^c_{b_0\dots b_d}\Omega^c_{n+1-c_d\dots n+1-c_0}\,.$$

The triple intersection theorem for $G(d,\mathbb{P}^n)$ decides on the value of $\alpha(a,b,c)$ when c is the Schubert symbol for the dual of a special Schubert cycle. To be precise, the theorem [7, Thm. III, p. 333] states that given $\Omega^c_{a_0\dots a_d}$ and $\Omega^c_{b_0\dots b_d}$ there exist integers $\lambda^c_0,\dots,\lambda^c_{d+1}$ such that for any special Schubert cycle σ^c_h, if

$$\dim_{\mathbb{C}}\Omega^c_{a_0\dots a_d} + \dim_{\mathbb{C}}\Omega^c_{b_0\dots b_d} + \dim_{\mathbb{C}}\sigma^c_h = 2\dim_{\mathbb{C}} G(d,\mathbb{P}^n) \tag{1}$$

and

$$\Omega^c_{a_0\dots a_d}\Omega^c_{b_0\dots b_d}\sigma^c_h = 1 \tag{2}$$

then

$$h + \sum_{i=1}^{d}\lambda^c_i = n - d\,. \tag{3}$$

Conversely, if (1) and (3) hold, then (2) holds. Here the λ^c_i's are defined as

$$\begin{aligned}\lambda^c_i &= \max\{0, n-a_{d-i}-b_{i-1}-1\}, \qquad i=1,\dots,d\,,\\ \lambda^c_0 &= n-a_d\,,\\ \lambda^c_{d+1} &= n-b_0\,.\end{aligned}$$

III. TRIPLE INTERSECTION THEOREM FOR $A^{(n+2)}_{d+1}$

In Section 5 we give a general argument which explains the role λ_i's play in deriving the main theorem (MT). The values of λ_i's for the case $n=2m$ are determined in Section 6. The corresponding statements for the $n=2m+1$ case are listed without proof in Section 7. Finally, in Section 8 we put all this together to establish the necessary conditions for having nonzero triple intersections.

5. General arguments for the $n=2m$ case. We start with two cycles $\Omega_{a_0\dots a_d}$ and $\Omega_{b_0\dots b_d}$ and we assume that the Schubert condition for the former is expressed with respect to a flag A and that of the latter is expressed with respect to the corresponding dual flag B. Our arguments are independent of the choice of skeleton points used in the construction of the flags.

The two Schubert cycles $\Omega_{a_0\dots a_d}$ and $\Omega_{b_0\dots b_d}$ are disjoint unless $a_{d-i}+b_i \ge n$ for all $i=0,\dots,d$, hence we assume this throughout. Any point of the intersection $\Omega_{a_0\dots a_d}\cap\Omega_{b_0\dots b_d}$ represents a $[d]$-plane lying inside $A_{a_{d-i}}\vee B_{b_{i-1}}$ for all $i=1,\dots,d$. Clearly this plane also lies in A_{a_d} and B_{b_d}, hence in the intersection

$$\Lambda = A_{a_d}\cap(A_{a_{d-1}}\vee B_{b_0})\cap\dots\cap(A_{a_0}\vee B_{b_{d-1}})\cap B_{b_d}\subset Q_n\,.$$

Recall that $p_0, \ldots, p_{n+1} \in Q_n$ denote the skeleton points described in Section 1.1. Using them we define auxiliary subsets of $I_{n+1} = \{0, 1, \ldots, n+1\}$:

$$\begin{aligned} L(0) &= \{r \in I_{n+1} \mid p_r \in A_{a_d}\}\,, \\ L(i) &= \{r \in I_{n+1} \mid p_r \in A_{a_{d-i}} \vee B_{b_{i-1}}\}\,, \qquad i = 1, \ldots, d\,, \\ L(d+1) &= \{r \in I_{n+1} \mid p_r \in A_{b_d}\}\,. \end{aligned}$$

This is one of the key steps where we translate geometry into arithmetic. Observe in particular that $A_{a_d} = S_{L(0)}$, $A_{a_{d-i}} \vee B_{b_{i-1}} = S_{L(i)}$ for $i = 1, \ldots, d$, and $B_{b_d} = S_{L(d+1)}$. We can thus rewrite Λ as

$$\Lambda = S_{L(0)} \cap S_{L(1)} \cap \ldots \cap S_{L(d+1)}\,.$$

Furthermore, if we let

$$L = L(0) \cap L(1) \cap \ldots \cap L(d+1)$$

then clearly

$$\Lambda = S_L\,.$$

It is the dimension of S_L that we wish to calculate. For this we proceed as follows: we first calculate the cardinality of $L(0)$; then with the intersection of each $L(i)$ certain points of $L(0)$ are left out, leaving us finally with only the points of L. Thus we define λ_i's as

$$\lambda_i = \#(I_{n+1} - L(i)) = n + 2 - \#L(i), \qquad i = 0, \ldots, d+1\,.$$

Note that each λ_i, $i = 1, \ldots, d$, counts the number of skeleton points which do not belong to the set $A_{a_{d-i}} \vee B_{b_{i-1}}$. Moreover,

$$\lambda_0 = \begin{cases} n - a_d & \text{if } a_d > m, \\ n - a_d + 1 & \text{if } a_d \le m, \end{cases}$$

$$\lambda_{d+1} = \begin{cases} n - b_d & \text{if } b_d > m, \\ n - b_d + 1 & \text{if } b_d \le m. \end{cases}$$

Normally the sum of these λ_i's should correctly count the number of points left out while forming the intersection $L(0) \cap \ldots \cap L(d+1)$, but due to the geometric anomalies that occur in the middle dimension of smooth quadrics, the point p_m in the even-dimensional case can be counted twice. To correct this oversight of $\lambda_0, \ldots, \lambda_{d+1}$ we introduce λ_{d+2}, which is -1 when a certain combination of the Schubert conditions is present and 0 otherwise. We will need one more correction factor λ_{d+3} which will decide when a jump in dimension occurs as observed in Lemmas 1.3 and 2.2.

6. Calculation of λ_i's for the $n = 2m$ case. We now give a lemma with a table to calculate the λ_i's using the a_i's and b_i's.

LEMMA 6.1. *When $n = 2m$ the λ_i's, $i = 1, \ldots, d$, are as in the table below*:

$a_{d-i} < m$	$b_{i-1} \le m$		$\lambda_i = n - a_{d-i} - b_{i-1}$
	$b_{i-1} > m$	$a_{d-i} + b_{i-1} \ge n$	$\lambda_i = 0$
		$a_{d-i} + b_{i-1} < n$	$\lambda_i = n - a_{d-i} - b_{i-1} - 1$
$a_{d-i} = m_t$	$b_{i-1} = m_t$	m even	$\lambda_i = 1$
		m odd	$\lambda_i = 0$
	$b_{i-1} = m_s$	m even	$\lambda_i = 0$
		m odd	$\lambda_i = 1$
$a_{d-i} > m$	$b_{i-1} \ge m$		$\lambda_i = 0$

Here $s, t \in \{0, 1\}$ and $s \ne t$. To find the λ_i corresponding to the case when $a_{d-i} > m$ and $b_{i-1} \le m$ we must observe that λ_i is a symmetric function of a_{d-i} and b_{i-1}. (Note that λ_0 and λ_{d+1} were calculated in Section 5.)

Proof. Case 1: $a_{d-i} < m$, $b_{i-1} \le m$. We have $L(i) = \{0, 1, \ldots, a_{d-i}, n+1, n, \ldots, n+1-b_{i-1}\} \in I_{n+1}$. Assume for the time being that $a_{d-i} < b_{i-1} < m$. Then the skeleton points missing from $S_{L(i)}$ have indices $a_{d-i}+1$, $a_{d-i}+2, \ldots, n-b_{i-1}$, and there are $(n-b_{i-1})-(a_{d-i}+1)+1 = \lambda_i$ of them. Hence $\lambda_i = n - a_{d-i} - b_{i-1}$ as claimed. If $b_{i-1} = m$, then depending on whether $B_{b_{i-1}}$ is in V_0 or in V_1, the element $m+1$ of $L(i)$ will be replaced by m, or vice versa depending on the parity of m. This changes $L(i)$ but not $\#L(i)$ and hence λ_i still has the same value. Finally, the argument is symmetric in a_{d-i} and b_{i-1}, and the assumption that one is less than the other is redundant.

Case 2: $a_{d-i} < m$, $b_{i-1} > m$. If $a_{d-i} + b_{i-1} \ge n$, then $L(i) = I_{n+1}$ and $\lambda_i = 0$. If, however, $a_{d-i} + b_{i-1} < n$, then $L(i) = \{0, 1, \ldots, a_{d-i}, n+1, n, \ldots, n-b_{i-1}\}$ and consequently $\lambda_i = n - a_{d-i} - b_{i-1} - 1$.

Case 3: $a_{d-i} = m_0$, $b_{i-1} = m_0$. If m is even, then $L(i) = I_{n+1} - \{m+1\}$, and if m is odd then $L(i) = I_{n+1}$. Hence λ_i is 1 or 0 accordingly.

Case 4: $a_{d-i} = m_0$, $b_{i-1} = m_1$. Similar to case 3.

Case 5: $a_{d-i} > m$, $b_{i-1} \ge m$. In this case $a_{d-i}+b_{i-1} > n$ so $L(i) = I_{n+1}$ and λ_i is 0. ∎

Lemma 6.2 (Calculation of λ_{d+2} when n is even). *Assume that there exist two numbers a_i, b_j with $i + j > d - 1$, such that $a_i = m_t$, $b_j = m_s$ where $t, s \in \{0, 1\}$. Then, for even m,*

$$\lambda_{d+2} = \begin{cases} -1 & \text{if } s = t, \\ 0 & \text{if } s \ne t, \end{cases}$$

and for odd m,

$$\lambda_{d+2} = \begin{cases} 0 & \text{if } s = t, \\ -1 & \text{if } s \ne t. \end{cases}$$

Proof. For general indices x and z let $a_{d-x} = m_t$ and $b_{z-1} = m_s$ where $t, s \in \{0, 1\}$. If $x = z$ then the middle dimension complications are already incorporated into the considerations leading to the calculation of λ_x. If, however, $x \neq z$ then a complication will arise in the intersection $L(x) \cap L(z)$, and we intend to correct this with λ_{d+2}.

First assume $x > z$; then $b_{x-1} > b_{z-1} = m$ and λ_x will be zero since $a_{d-x} + b_{x-1} \geq n$. Similarly $a_{d-z} > a_{d-x} = m$ and λ_z is also zero. In this case $L(x) \cap L(z) = I_{n+1}$, and $\lambda_x + \lambda_z$ correctly counts the number of missing skeleton points.

Next assume that $x < z$; then $a_{d-z} < a_{d-x} = m$ and $b_{x-1} < b_{z-1} = m$, which in turn gives $\lambda_z = m - a_{d-z}$ and $\lambda_x = m - b_{x-1}$ according to the previous lemma. Assume now that m is even. When $s \neq t$ the spaces $A_{a_{d-x}}$ and $B_{b_{z-1}}$ do not have a point in common and again $\lambda_x + \lambda_z$ correctly counts the number of missing skeleton points from the intersection $L(x) \cap L(z)$. However, if $t = s$, then the spaces $A_{a_{d-x}}$ and $B_{b_{z-1}}$ share a point. Without loss of generality assume that t is such that $A_{a_{d-x}} \cap B_{b_{z-1}} = p_{m+1}$. This shows that the sets of skeleton points that are left out by $L(x)$ and $L(z)$ both contain the point p_{m+1}, i.e. λ_x and λ_z both count p_{m+1}. Hence the number of skeleton points left out by $L(x) \cap L(z)$ is $\lambda_x + \lambda_z - 1$. This correction factor is λ_{d+2}. If m is odd we argue similarly. Thus when $x < z$ we let $i = d - x$ and $j = z - 1$ to obtain the statement of the lemma. ∎

6.3. We are now in a position to calculate $\dim_{\mathbb{C}} S_L$ in terms of λ_i's. This is where we need the correction factor λ_{d+3} which registers the shift in dimension due to Lemma 1.3. First we observe that

$$\begin{aligned} \#L &= \#L(0) - (\lambda_1 + \ldots + \lambda_{d+2}) \\ &= (n + 2 - \lambda_0) - (\lambda_1 + \ldots + \lambda_{d+2}) = n - (\lambda_0 + \ldots + \lambda_{d+2}) + 2\,. \end{aligned}$$

On the other hand,

$$\dim_{\mathbb{C}} S_L = \begin{cases} \#L - 2 & \text{if } J(L) \neq \emptyset, \\ \#L - 1 & \text{if } J(L) = \emptyset. \end{cases}$$

Therefore define λ_{d+3} as

$$\lambda_{d+3} = \begin{cases} 0 & \text{if } J(L) \neq \emptyset, \\ -1 & \text{if } J(L) = \emptyset. \end{cases}$$

Then we finally have

$$\dim_{\mathbb{C}} S_L = n - (\lambda_0 + \ldots + \lambda_{d+3})\,.$$

To calculate λ_{d+3} we must observe that $J(L)$ will be empty if either $\{0, 1, \ldots, m\}$ or $\{n+1, n, \ldots, m+1\}$ is disjoint from L, i.e. if either of these sets is ignored by the intersection $L(0) \cap L(1) \cap \ldots \cap L(d+1)$. We therefore define an algorithm which checks if this is the case.

Algorithm. Define the following subintervals of I_m:

$$I(0) = \begin{cases} I_m & \text{if } a_d \leq m, \\ \{j \in I_m \mid j < n - a_d\} & \text{if } a_d > m, \end{cases}$$

$$I(d+1) = \begin{cases} I_m & \text{if } b_d \leq m, \\ \{j \in I_m \mid j < n - b_d\} & \text{if } b_d > m. \end{cases}$$

For $i = 1, \ldots, d$ define $I(i)$ as

$$I(i) = \begin{cases} \{j \in I_m \mid j > \min\{a_{d-i}, b_{i-1}\}\} & \text{if } a_{d-i}, b_{i-1} \leq m, \\ \{j \in I_m \mid a_{d-i} < j < n - b_{i-1}\} & \text{if } a_{d-i} < m < b_{i-1}, \\ \{j \in I_m \mid b_{i-1} < j < n - a_{d-i}\} & \text{if } a_{d-i} > m > b_{i-1}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Conclusion of the algorithm ($n = 2m$).

$$\lambda_{d+3} = \begin{cases} -1 & \text{if } \bigcup_{i=0}^{d+1} I(i) = I_m, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the calculation of the λ_i's in the $n = 2m$ case.

7. The λ_i's for the $n = 2m+1$ case. In this section we give without proof the corresponding statements for the case $n = 2m+1$. We also remind that λ_0 and λ_{d+1} were calculated in Section 5 (regardless of the parity of n).

Lemma 7.1. *When $n = 2m+1$ the λ_i's, $i = 1, \ldots, d$, are as in the table below*:

$a_{d-i} < m$	$b_{i-1} \leq m$		$\lambda_i = n - a_{d-i} - b_{i-1}$
	$b_{i-1} > m$	$a_{d-i} + b_{i-1} \geq n$	$\lambda_i = 0$
		$a_{d-i} + b_{i-1} < n$	$\lambda_i = n - a_{d-i} - b_{i-1} - 1$
$a_{d-i} = m$	$b_{i-1} = m$		$\lambda_i = 1$
$a_{d-i} > m$	$b_{i-1} > m$		$\lambda_i = 0$

Once again we remind that λ_i is a symmetric function of a_{d-i} and b_{i-1}.

We have $\lambda_{d+2} = 0$ when n is odd: Recall that we need this correction factor when A_m and B_m share a point which the other λ_i's fail to count. But when n is odd, then A_m is always disjoint from B_m, hence the other λ_i's do their job correctly.

λ_{d+3} when n is odd is calculated using the same algorithm as before except that we need the following modification.

Conclusion of the algorithm ($n = 2m+1$).

$$\lambda_{d+3} = \begin{cases} -1 & \text{if } \bigcup_{i=0}^{d+1} I(i) = I_m, \text{ and } m+1 \notin L, \\ 0 & \text{otherwise.} \end{cases}$$

8. Completion of the proof of the main theorem. We will describe the inequalities of the main theorem for the case $n = 2m$. The arguments for the $n = 2m+1$ case follow very closely the proof given here using this time the λ_i's defined for the odd-dimensional case, and we leave it to the reader.

8.1. We have shown that all the $[d]$-spaces that are represented by points of $\Omega_{a_0\ldots a_d}\cap\Omega_{b_0\ldots b_d}$ lie in the $n-(\lambda_0+\ldots+\lambda_{d+3})$-dimensional subvariety S_L of Q_{2m}. These $[d]$-spaces also belong to σ_h if they intersect a certain $[n-d-h]$-dimensional space in Q_{2m} which belongs to a flag used in the description of σ_h. Generically this intersection is empty if $(n-d-h)+(n-\sum\lambda_i)<n$, i.e. for nonempty intersection we must have $h+(\lambda_0+\ldots+\lambda_{d+3})\le n-d$. This proves the second inequality of the main theorem.

8.2. We rewrite the dimension condition (1) of the main theorem and rearrange it to obtain

$$(*)\qquad a_d+b_d+\sum_{i=1}^{d}(a_{d-i}+b_{i-1}+1)-d-(d+1)(n+\tfrac{1}{2}d)+e=h$$

where e is as given in the statement of the theorem. Recall that $a_d=n-\lambda_0$ and $b_d=n-\lambda_{d+1}$. For $a_{d-i}+b_{i-1}+1, i=1,\ldots,d$, we have four cases to consider. We list these cases first and then examine them:

Case 1: $a_{d-i}+b_{i-1}+1=n-\lambda_i$ if either "$a_{d-i}<m$, $b_{i-1}>m$ and $a_{d-i}+b_{i-1}<n$" or "$a_{d-i}>m$, $b_{i-1}<m$ and $a_{d-i}+b_{i-1}<n$".

Case 2: $a_{d-i}+b_{i-1}+1=n-\lambda_i+1$ if either "$a_{d-i}<m$, $b_{i-1}\le m$" or "$a_{d-i}\le m$, $b_{i-1}<m$".

Case 3: $a_{d-i}+b_{i-1}+1=n-\lambda_i+2$ if $a_{d-i}=b_{i-1}=m_t$, $t=0$ or 1, when m is even. When m is odd the same expression for λ_i holds if $a_{d-i}=m_t$, $b_{i-1}=m_s$, $t,s\in\{0,1\}$ and $s\ne t$.

Case 4: $a_{d-i}+b_{i-1}+1\ge n-\lambda_i+1$ if $a_{d-i}+b_{i-1}\ge n$.

We now examine these cases. If case 1 holds for all $i=1,\ldots,d$, then no a_i or b_j is m so $\lambda_{d+2}=0$. Since either a_{d-i} or b_{i-1} is greater than m, the interval $I(i)$ does not contain the integer m, for $i=1,\ldots,d$. Hence $\lambda_{d+3}=0$, and

$$(**)\qquad a_d+b_d+\sum_{i=1}^{d}(a_{d-i}+b_{i-1}+1)$$
$$\ge n-\lambda_0+n-\lambda_{d+1}+\sum_{i=1}^{d}(n-\lambda_i)-\lambda_{d+2}-\lambda_{d+3}\,.$$

If case 2 holds only once, and the rest is case 1, then there is a single occurrence of m among $a_0,\ldots,a_d,b_0,\ldots,b_d$, and hence $\lambda_{d+2}=0$. Assume either a_{d-k} or b_{k-1} is $\le m$. Then

$$a_{d-k}+b_{k-1}+1=n-\lambda_k-\lambda_{d+3},$$

hence $(**)$ holds.

If case 3 holds, say when $i = k$, then

$$a_{d-k} + b_{k-1} + 1 = n - \lambda_k + 2 \geq n - \lambda_k - \lambda_{d+2} - \lambda_{d+3},$$

hence $(**)$ holds.

If case 4 holds at least once and the rest is case 1, we can have at most one occurrence of m, so $\lambda_{d+2} = 0$. If case 4 holds for $i = k$,

$$a_{d-k} + b_{k-1} + 1 \geq n - \lambda_k + 1 \geq n - \lambda_k - \lambda_{d+3}$$

and $(**)$ holds. In any other combination of cases from 1 to 4 the inequality $(**)$ is easily seen to hold. Substituting $(**)$ into $(*)$ we obtain

$$(n-d) - \tfrac{1}{2}d(d+1) + e \leq h + \sum_{i=0}^{d+3} \lambda_i\,,$$

which completes the proof. ∎

IV. EXAMPLES

In this section we use the notation $G(d, Q_n)$ to denote the subvariety of the Grassmannian manifold consisting of the $[d]$-planes in the smooth quadric Q_n. Due to the representation theorem of Dibağ [3, p. 501] we have $A_d^{(n)} \simeq G(d-1, Q_{n-2})$. The notation for Schubert varieties is explained in Section 3.

Note. In the following intersection-product tables Schubert cycles appearing in the intersection are given without multiplicities, e.g. in Table 1, $\Omega_{14} \cdot \Omega_{2_04}$ is given as $\Omega_{12_1}, \Omega_{03}$ and $\Omega_{14} \cdot \Omega_{2_03}$ is given as Ω_{02_1}, meaning that $\Omega_{14} \cdot \Omega_{2_04} = c_1\Omega_{12_1} + c_2\Omega_{03}$ and $\Omega_{14} \cdot \Omega_{2_03} = c_3\Omega_{02_1}$, where c_1, c_2 and c_3 are nonzero integers which we omit. For example, in the products involving special Schubert varieties, the multiplicities in the examples below are 1, 2 or 4 as Pragacz (private communication) points out.

9. Cohomology ring structure of $A_2^{(6)} \simeq G(1, Q_4)$. We give the homology intersection structure. The 0-dimensional cycle Ω_{01} and the 5-dimensional cycle Ω_{34} are dual, $\Omega_{01}\Omega_{34} = 1$; and we omit them in Table 1. The numbers in the rightmost column are homological dimension.

10. Cohomology ring structure of $A_3^{(6)} \simeq G(2, Q_4)$. $A_3^{(6)}$ consists of two isomorphic connected components V_0, V_1, say. The dimension of each component is 3 and planes from different components do not generically intersect (see [5, p. 735]). For example, $\Omega_{12_04}\Omega_{02_03} = 1$ but $\Omega_{12_04}\Omega_{02_13} = 0$. In general $\Omega_{a_0a_1a_2}\Omega_{b_0b_1b_2} = 0$ if both 2_0 and 2_1 appear in the set of indices $\{a_0, \ldots, b_2\}$. For this reason we give in Table 2 the homology intersection table for one of the components only. The table for the other component

T a b l e 1. Intersection products for $A_2^{(6)}$

	Ω_{2_03}	Ω_{2_13}	Ω_{14}	Ω_{2_04}	Ω_{2_14}	dim
Ω_{02_0}	0	0	0	1	0	1
Ω_{02_1}	0	0	0	0	1	1
Ω_{12_0}	1	0	0	Ω_{02_0}	0	2
Ω_{12_1}	0	1	0	0	Ω_{02_1}	2
Ω_{03}	0	0	1	Ω_{02_1}	Ω_{02_0}	2
Ω_{2_03}	0	Ω_{02_0},Ω_{02_1}	Ω_{02_1}	Ω_{12_1}	Ω_{12_1},Ω_{03}	3
Ω_{2_13}	Ω_{02_0},Ω_{02_1}	0	Ω_{02_0}	Ω_{12_0},Ω_{03}	Ω_{12_0}	3
Ω_{14}	Ω_{02_1}	Ω_{02_0}	Ω_{02_0},Ω_{02_1}	Ω_{12_1},Ω_{03}	Ω_{12_0},Ω_{03}	3
Ω_{2_04}	Ω_{12_1}	Ω_{12_0},Ω_{03}	Ω_{12_1},Ω_{03}	Ω_{2_03}	Ω_{14}	4
Ω_{2_14}	Ω_{12_1},Ω_{03}	Ω_{12_0}	Ω_{12_0},Ω_{03}	Ω_{14}	Ω_{2_13}	4

T a b l e 2. Intersection products for $A_3^{(6)}$

	Ω_{012}	Ω_{023}	Ω_{124}	Ω_{234}	dim
Ω_{012}	0	0	0	1	0
Ω_{023}	0	0	1	Ω_{023}	1
Ω_{124}	0	1	Ω_{023}	Ω_{124}	2
Ω_{234}	1	Ω_{023}	Ω_{124}	Ω_{234}	3

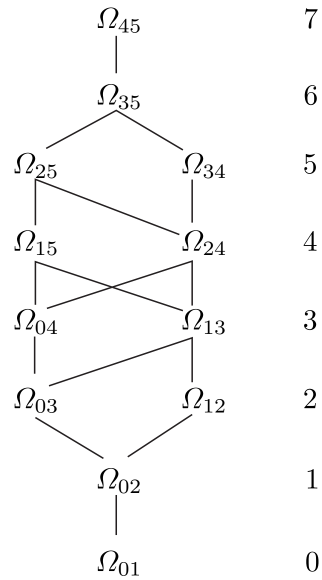
is identical. All the 2's appearing in the table are either all 2_0, for the component V_0, or all 2_1, for the component V_1, hence we omit this labeling.

11. Cohomology ring structure of $A_2^{(7)} \simeq G(1, Q_5)$. The Hasse diagram for the Schubert cycles of $A_2^{(7)}$ is given in Table 4 with the dimensions given in the right hand column. Intersection products are given in Table 3.

T a b l e 3. Intersection products for $A_2^{(7)}$

	Ω_{15}	Ω_{24}	Ω_{25}	Ω_{34}	Ω_{35}	dim
Ω_{02}	0	0	0	0	1	1
Ω_{03}	0	0	1	0	Ω_{02}	2
Ω_{12}	0	0	0	1	Ω_{02}	2
Ω_{04}	1	0	Ω_{02}	0	Ω_{03}	3
Ω_{13}	0	1	Ω_{02}	Ω_{02}	Ω_{03},Ω_{12}	3
Ω_{15}	Ω_{02}	Ω_{02}	Ω_{03},Ω_{12}	Ω_{03}	Ω_{04},Ω_{15}	4
Ω_{24}	Ω_{02}	Ω_{02}	Ω_{03},Ω_{12}	Ω_{03},Ω_{12}	Ω_{04},Ω_{13}	4
Ω_{25}	Ω_{03},Ω_{12}	Ω_{03},Ω_{12}	Ω_{04},Ω_{13}	Ω_{04},Ω_{13}	Ω_{15},Ω_{24}	5
Ω_{34}	Ω_{03}	Ω_{03},Ω_{12}	Ω_{04},Ω_{13}	Ω_{13}	Ω_{24}	5
Ω_{35}	Ω_{04},Ω_{13}	Ω_{04},Ω_{13}	Ω_{15},Ω_{24}	Ω_{24}	Ω_{25},Ω_{34}	6

Table 4. Hasse diagram for $A_2^{(7)}$



12. Examples of triple intersections. We verify the necessity of the condition (3) in MT by some examples.

1. $\Omega_{14}\Omega_{14}\Omega_{2_04} \neq \emptyset$ in $A_2^{(6)} \simeq G(1, Q_4)$; $e = 0$, $h = 1$, $n = 4$, $d = 1$, $m = 2$. Then

$$\begin{array}{llll}
a_1 = 4 & & \lambda_0 = n - a_1 = 0 & \text{Section 5} \\
a_0 = 1 & b_0 = 1 & \lambda_1 = n - a_0 - b_0 = 2 & \text{Lemma 6.1} \\
 & b_1 = 4 & \lambda_2 = n - b_1 = 0 & \text{Section 5} \\
 & & \lambda_3 = 0 & \text{Lemma 6.2} \\
 & & \lambda_4 = 0 & \text{Algorithm 6.3}
\end{array}$$

In this case MT(3) holds with $2 \leq 3 \leq 3$, showing among other things that the upper bound of $h + \sum_{i=0}^{d+3} \lambda_i$ cannot be improved.

2. $\Omega_{124}\Omega_{023}\Omega_{234} \neq \emptyset$ in one component of $A_3^{(6)} \simeq G(2, Q_4)$; $e = 1+2+0$, $h = 0$, $n = 4$, $d = 2$, $m = 2$. Then

$$\begin{array}{llll}
a_2 = 4 & & \lambda_0 = n - a_2 = 0 & \text{Section 5} \\
a_1 = 2 & b_0 = 0 & \lambda_1 = n - a_1 - b_0 = 2 & \text{Lemma 6.1} \\
a_0 = 1 & b_1 = 2 & \lambda_2 = n - a_0 - b_1 = 1 & \text{Lemma 6.1} \\
 & b_2 = 3 & \lambda_3 = n - b_2 = 1 & \text{Section 5} \\
 & & \lambda_4 = -1 & \text{Lemma 6.2} \\
 & & \lambda_5 = -1 & \text{Algorithm 6.3}
\end{array}$$

Hence MT(3) is satisfied as $2 \leq 2 \leq 2$, showing also that the lower bound cannot be improved either. Note also that if Ω_{124} and Ω_{023} are taken in different components of $A_3^{(6)}$ then $\lambda_4 = 0$ and (3) of MT is not satisfied, implying that the above intersection is zero, which also follows from the fact that [2]-planes of different families in Q_4 do not generically intersect. Hence A_{2_s} and B_{2_t} cannot have a line in common for a generic choice of flags.

3. $\Omega_{34}\Omega_{34}\Omega_{15} = 0$ in $A_2^{(7)} \simeq G(1, Q_5)$; $e = 0$, $h = 3$, $n = 5$, $d = 1$, $m = 2$. Then

$a_1 = 4$		$\lambda_0 = n - a_1 = 1$	Section 5
$a_0 = 3$	$b_0 = 3$	$\lambda_1 = 0$	Lemma 7.1
	$b_1 = 4$	$\lambda_2 = n - b_1 = 1$	Section 5
		$\lambda_3 = 0$	Section 7
		$\lambda_4 = 0$	Algorithm 7

Here $h + \sum_{i=0}^{4} \lambda_i = 5 \not\leq n - d$. In this example the algebra predicts that the cycles will not intersect, and indeed we can check from Table 4 that $(\Omega_{34}\Omega_{34})\Omega_{15} = \Omega_{13}\Omega_{15} = 0$.

4. We show that MT(3) is not sufficient: consider $\Omega_{12_1}\Omega_{2_14}\Omega_{2_04} = 0$ in $A_2^{(6)} \simeq G(1, Q_4)$. Then $e = 1 + 0 + 0$, $h = 1$, $n = 4$, $d = 1$, $m = 2$ and

$a_1 = 2_1$		$\lambda_0 = n - a_1 + 1 = 3$	Section 5
$a_0 = 1$	$b_0 = 2_1$	$\lambda_1 = n - a_0 - b_0 = 1$	Lemma 6.1
	$b_1 = 4$	$\lambda_2 = n - b_1 = 0$	Section 5
		$\lambda_3 = -1$	Lemma 6.2
		$\lambda_4 = -1$	Algorithm 6.3

In this case MT(3) is satisfied with equality holding on both sides, $3 \leq 3 \leq 3$, hence MT(3) alone is not sufficient for MT(2).

13. Sufficiency of MT(3). We start this section by analyzing the last example of the previous section. Using the notation of Section 5, all the lines in Q_4 which simultaneously belong to the Schubert cells Ω_{12_1} and Ω_{2_14} lie in the space S_L where $L = \{0, 1, 3\}$. S_L is hence a [2]-plane which belongs to V_1. We want these lines also to belong to the Schubert cell Ω_{2_04}, i.e. we want to know if there is a line in S_L which intersects a $[2_0]$-plane, an element of V_0. Since in Q_4 elements of V_1 do not generically intersect elements of V_0 there is no such line in S_L. This explains why MT(3) alone is not sufficient for MT(2). But in this particular example there is some relief (!): using the commutativity of intersection we can write $\Omega_{12_1}\Omega_{2_14}\Omega_{2_04} = \Omega_{12_1}\Omega_{2_04}\Omega_{2_14}$, and we try our main theorem on this new order of intersection:

$\Omega_{12_1}\Omega_{2_04}\Omega_{2_14} = 0$ in $A_2^{(6)} \simeq G(1, Q_4)$; $e = 1$, $h = 1$, $n = 4$, $d = 1$,

$m = 2$. Then

$$\begin{array}{llll} a_1 = 2_1 & & \lambda_0 = n - a_1 + 1 = 3 & \text{Section 5} \\ a_0 = 1 & b_0 = 2_0 & \lambda_1 = n - a_0 - b_0 = 1 & \text{Lemma 6.1} \\ & b_1 = 4 & \lambda_2 = n - b_1 = 0 & \text{Section 5} \\ & & \lambda_3 = 0 & \text{Lemma 6.2} \\ & & \lambda_4 = -1 & \text{Algorithm 6.3} \end{array}$$

Here $h + \sum_{i=0}^{4} \lambda_i = 4 \not\leq n - d = 3$. Hence the algebra tells us that the intersection is zero.

The key questions for the sufficiency of MT(3) are the following:

(i) Is S_L big enough to intersect a generic $[n - d - h]$-plane? (This is the condition imposed by σ_h.)

(ii) Is S_L big enough to contain a $[d]$-plane at all?

The first of these questions gives rise to the familiar necessary condition for MT(3):

$$(*) \qquad h + \sum_{i=0}^{4} \lambda_i \leq n - d\,.$$

This condition is also sufficient for an affirmative answer to (i) when n is odd, or when $h + \sum_{i=0}^{4} \lambda_i \neq m$ in case $n = 2m$. While S_L is sufficiently large to intersect a generic $[n-d-h]$-plane, it may not be large enough to contain any $[d]$-plane. And even if it does contain some $[d]$-planes we may not conclude that any of these $[d]$-planes also satisfies the given Schubert conditions. However, if $\dim S_L < m$, when $n = 2m$, then S_L is an $[n - \sum_{i=0}^{d+3} \lambda_i]$-plane, and the inequality $(*)$ guarantees that S_L intersects a generic $[n - d - h]$-plane in Q_{2m}. If furthermore S_L is large enough to contain a $[d]$-plane, i.e. if $\dim S_L = n - \sum_{i=0}^{d+3} \lambda_i \geq d$, then we can conclude that $\Omega_{a_0 \ldots a_d} \Omega_{b_0 \ldots b_d} \sigma_h \neq \emptyset$.

We collect these arguments in the following theorem. Assume here that $\Omega_{a_0 \ldots a_d}$, $\Omega_{b_0 \ldots b_d}$ and σ_h are as given in the statement of the main theorem.

Theorem 13. *The condition* MT(3) *is sufficient for having a nontrivial triple intersection*, $\Omega_{a_0 \ldots a_d} \Omega_{b_0 \ldots b_d} \sigma_h \neq 0$, *if one of the following conditions holds*:

(i) $\lambda_{d+3} = -1$ *and* $\sum_{i=0}^{d+3} \lambda_i > m$ *when* $n = 2m$, *or*

(ii) $\lambda_{d+3} = -1$ *when* $n = 2m + 1$. ∎

Note that when $\lambda_{d+3} = -1$ then $J(L) = \emptyset$ and in that case S_L is an $[n - \sum_{i=0}^{d+3} \lambda_i]$-plane. In the even-dimensional case we want to exclude the case when $\sum_{i=0}^{d+3} \lambda_i = m$ since the cases $m = m_0$ or $m = m_1$ are different (see Section 3). If, for example, $\lambda_{d+3} = -1$, $n - \sum_{i=0}^{d+3} \lambda_i = m_s$ and $n - d - h = m_t$, then MT(3) is sufficient for MT(2) when

(i) $s = t$ and m is even, or
(ii) $s \neq t$ and m is odd.

When n is odd on the other hand, we do not have such middle dimension complications and $\lambda_{d+3} = -1$ is enough to assure the sufficiency of MT(3).

Now applying Theorem 13 to Example 4 of Section 12, we find that MT(3) holds, $\lambda_{d+3} = -1$ but $\sum_{i=0}^{d+3} \lambda_i \not> m$, so as Theorem 13 above predicts, MT(2) does not hold.

It is important to observe that $\lambda_{d+3} = -1$ is not a necessary condition for MT(2). Hence if MT(3) holds but $\lambda_{d+3} = 0$, then we can conclude nothing about the triple intersection. Compare the following two examples for this purpose. In Example 1 of Section 12, MT(3) holds, $\lambda_{d+3} = 0$ but MT(2) also holds. In $G(1, Q_6)$ on the other hand, if we consider $\Omega_{04}\Omega_{45}\Omega_{46}$ we see that MT(3) holds, and $\lambda_{d+3} = 0$, but this intersection is zero, i.e. MT(2) does not hold.

These two examples show us that when $\lambda_{d+3} = 0$ the inequalities of MT(3) do not necessarily imply MT(2). However, when $\lambda_{d+3} = -1$ and $\sum_{i=0}^{d+3} \lambda_i > m$ then MT(3) safely implies MT(2), as it does in the following example.

In $A_2^{(8)} \simeq G(1, Q_6)$ consider $\Omega_{13_0}\Omega_{45}\Omega_{46}$. Then $e = 1$, $h = 1$, $n = 6$, $d = 1$, $m = 3$ and

$$
\begin{array}{llll}
a_1 = 3_0 & & \lambda_0 = n - a_1 + 1 = 4 & \text{Section 5} \\
a_0 = 1 & b_0 = 4 & \lambda_1 = n - a_0 - b_0 - 1 = 0 & \text{Lemma 6.1} \\
 & b_1 = 5 & \lambda_2 = n - b_1 = 1 & \text{Section 5} \\
 & & \lambda_3 = 0 & \text{Lemma 6.2} \\
 & & \lambda_4 = -1 & \text{Algorithm 6.3}
\end{array}
$$

Here MT(3) holds with $5 \leq 5 \leq 5$. We also have $\sum_{i=0}^{d+3} \lambda_i = 4 > 3 = m$ and $\lambda_4 = -1$. From these algebraic considerations we conclude that $\Omega_{13_0}\Omega_{45}\Omega_{46} \neq 0$.

References

[1] E. Artin, *Geometric Algebra*, Interscience, New York 1988 (c1957).

[2] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells and the cohomology of G/P spaces*, Russian Math. Surveys 28 (1973), 1–26.

[3] İ. Dibağ, *Topology of the complex varieties $A_s^{(n)}$*, J. Differential Geom. 11 (1976), 499–520.

[4] W. Fulton, *Intersection Theory*, Springer, 1984.

[5] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York 1978.

[6] H. Hiller and B. Boe, *Pieri formulas for SO_{2n+1}/U_n and Sp_n/U_n*, Adv. in Math. 62 (1986), 49–67.

[7] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Vol. II, Cambridge University Press, 1968.
[8] S. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly 79 (1972), 1061–1082.
[9] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*, in: Topics in Invariant Theory, Séminaire d'Algèbre Dubreil–Malliavin 1989–1990, Lecture Notes in Math. 1478, Springer, 1991, 130–191.
[10] —, *Geometric applications of symmetric polynomials*, preprint, Max-Planck Institut für Mathematik, Bonn 1992.
[11] P. Pragacz and J. Ratajski, *Pieri for isotropic Grassmannians*: *the operator approach*, preprint, Max-Planck Institut für Mathematik, Bonn 1992.

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Received 5 November 1991;
in revised form 7 April, 19 May and 28 October 1992