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Some variations on the partition property for normal ultrafilters on $\mathbf{P}_\kappa\lambda$

by

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Abstract. Suppose κ is a supercompact cardinal and $\lambda \geq \kappa$. In [3], we studied the relationship between the weak partition property and the partition property for normal ultrafilters on $\mathbf{P}_\kappa\lambda$. In this paper we study a hierarchy of properties intermediate between the weak partition property and the partition property. Given appropriate large cardinal assumptions, we show that these properties are not all equivalent.

1. Introduction. Our set-theoretic notation is quite standard. V denotes the universe of all sets. By "inner model" we mean a transitive class which is a model of ZFC. If M is an inner model and λ is a cardinal, we say that M is *closed under λ-sequences* if and only if for any $x \subseteq M$, if $|x| \leq \lambda$ then $x \in M$.

We assume the reader is familiar with the basic notation, definitions, and techniques involving supercompactness (see Solovay, Reinhardt, and Kanamori [8]). For cardinals $\kappa \leq \lambda$, $\mathbf{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. κ is λ-*supercompact* if and only if there is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$. Equivalently, κ is λ-supercompact if and only if there is an inner model M which is closed under λ-sequences, and an elementary embedding $i : V \to M$ such that κ is the first cardinal moved by i and $i(\kappa) > \lambda$. κ is supercompact if and only if κ is λ-supercompact for every $\lambda \geq \kappa$.

Part of the definition of "U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$" is that U is *fine*: For every $x \in \mathbf{P}_\kappa\lambda$, $\{y \in \mathbf{P}_\kappa\lambda : x \subseteq y\} \in U$.

We assume for the remainder of this paper that κ is a fixed supercompact cardinal.

Suppose U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$ and M is the transitive collapse of the ultrapower $\prod V/U$. Then M is closed under λ-sequences. Also, there is a natural elementary embedding $i : V \to M$.

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Suppose that $\kappa \leq \gamma < \lambda$, U_γ is a normal ultrafilter on $\mathbf{P}_\kappa\gamma$, and U_λ is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$. We say that U_γ is the *restriction* of U_λ to γ, and we write $U_\gamma = U_\lambda\restriction\gamma$, if and only if for every $A \subseteq \mathbf{P}_\kappa\gamma$, $A \in U_\gamma$ if and only if $\{x \in \mathbf{P}_\kappa\lambda : x \cap \gamma \in A\} \in U_\lambda$.

Suppose that U_γ and U_λ are as above, and $i_\gamma : V \to M_\gamma$ and $i_\lambda : V \to M_\lambda$ are the corresponding elementary embeddings and inner models. Then there is a natural elementary embedding $k : M_\gamma \to M_\lambda$ satisfying $k \circ i_\gamma = i_\lambda$. Also, k fixes all sets hereditarily of cardinal less than or equal to γ. For the definition of k, see [8].

In [6], Menas introduced the partition property for normal ultrafilters on $\mathbf{P}_\kappa\lambda$. For any set A, let $[A]^2 = \{\{x, y\} : x, y \in A \text{ and } x \neq y\}$. We say that a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ has the *partition property* if and only if given any $f : [\mathbf{P}_\kappa\lambda]^2 \to 2$, there exists $A \in U$ and $i < 2$ such that $f[\{\{x, y\} \in [A]^2 : x \subseteq y \text{ or } y \subseteq x\}] = \{i\}$. For U a normal ultrafilter on $\mathbf{P}_\kappa\lambda$, let "part(U)" denote the fact that U has the partition property.

The following characterization of part(U) is central to this paper:

Theorem 1.1 (Menas [6]). *Suppose $\lambda \geq \kappa$ and U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$. Then* part(U) *holds if and only if there exists $A \in U$ such that for any $x, y \in A$, if $x \subsetneq y$, then $|x| < |y \cap \kappa|$.*

We shall be concerned with the effect of changing the final "κ" in Theorem 1.1 to some other cardinal γ.

Kunen [4] (see also Kunen–Pelletier [5]) introduced the weak partition property for normal ultrafilters on $\mathbf{P}_\kappa\lambda$. We say that a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ has the *weak partition property* if and only if there exists $A \in U$ such that for all $x, y \in A$, x is not a proper initial segment of y. For U a normal ultrafilter on $\mathbf{P}_\kappa\lambda$, let "wpart(U)" denote the fact that U has the weak partition property.

It is easy to see that part(U) implies wpart(U): Suppose that U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$ and part(U) holds. Let A be as in Theorem 1.1. Let $B = \{x \in A : \kappa \in x\}$. Then $B \in U$ and B witnesses wpart(U).

In [3], we studied the large cardinal strength of the assertion that there is a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ such that wpart(U) holds and part(U) fails.

It is known that many normal ultrafilters have the partition property.

Theorem 1.2 (Menas [6]). *Suppose $\lambda \geq \kappa$. Then $|\{U : U$ is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$ such that* part(U) *holds$\}| = 2^{2^{\lambda^{\underline{\kappa}}}}$.*

Since a normal ultrafilter on $\mathbf{P}_\kappa\lambda$ is an element of $\mathbb{P}\mathbb{P}\mathbf{P}_\kappa\lambda$, this is the maximal possible number of such normal ultrafilters.

It is also known that the assertion that there exists a normal ultrafilter which does not have the partition property is a large cardinal assertion. See [4] or [5].

We shall be considering properties which result from making certain changes in the characterization of the partition property given in Theorem 1.1. In particular, for fixed $\lambda \geq \kappa$, we define the binary relation $\mathrm{sep}(-,-)$ as follows:

$\mathrm{sep}(U,\gamma)$ holds if and only if U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$, γ is a cardinal, and there exists $A \in U$ such that for any $x, y \in A$, if $x \subsetneq y$, then $|x| < |y \cap \gamma|$.

"sep" is meant to denote "separation", since we think of $\mathrm{sep}(U,\gamma)$ as asserting that there exists $A \in U$ such that the elements of A are "sufficiently separated".

2. Some beginning facts on sep

THEOREM 2.1. *Suppose $\lambda \geq \kappa$ and U is a normal ultrafilter on $\mathbf{P}_\kappa\lambda$.*

(a) *If $\delta \leq \gamma$ and $\mathrm{sep}(U,\delta)$ holds, then $\mathrm{sep}(U,\gamma)$ holds.*
(b) *$\mathrm{sep}(U,\kappa)$ holds if and only if $\mathrm{part}(U)$.*
(c) *If $\mathrm{sep}(U,\gamma)$ holds for some $\gamma < \lambda$, then $\mathrm{wpart}(U)$ holds.*
(d) *If $\gamma < \kappa$ then $\mathrm{sep}(U,\gamma)$ fails.*
(e) *If $\gamma \geq \lambda$ then $\mathrm{sep}(U,\gamma)$ holds.*

Proof. Parts (a) and (b) follow immediately from the relevant definitions.

The proof of (c) is similar to the proof, given in the previous section, that $\mathrm{part}(U)$ implies $\mathrm{wpart}(U)$.

For (d), fix $\gamma < \kappa$, and suppose, by way of contradiction, that $\mathrm{sep}(U,\gamma)$ holds. Then there exists $A \in U$ such that for any $x, y \in A$, if $x \subsetneq y$, then $|x| < |y \cap \gamma|$. Let $B = \{x \in A : \gamma \subseteq x\}$. Then, by the fineness of U, it follows that $B \in U$. Pick any $x, y \in B$ with $x \subsetneq y$. Then, since $\gamma \subseteq x$ and $\gamma \subseteq y$, we have $|x \cap \gamma| = \gamma = |y \cap \gamma|$. Clearly $|x| \geq |x \cap \gamma|$. Hence, $|x| \geq |y \cap \gamma|$. This contradicts the fact that $x, y \in A$ and $x \subsetneq y$.

For (e), it suffices (by (a)) to show that $\mathrm{sep}(U,\lambda)$ holds. Obviously, for any $y \in \mathbf{P}_\kappa\lambda$, $|y \cap \lambda| = |y|$. Hence, we must show that there exists $A \in U$ such that for any $x, y \in A$, if $x \subsetneq y$, then $|x| < |y|$. This follows from a result of Solovay involving ω-Jonsson functions (see [6] or [7]). ∎

Parts (a), (d), and (e) of Theorem 2.1 suggest that we consider the following function from normal ultrafilters to cardinals: For $\lambda \geq \kappa$ and U a normal ultrafilter on $\mathbf{P}_\kappa\lambda$, we define $S(U)$ to be the least γ such that $\mathrm{sep}(U,\gamma)$ holds. By part (e) of Theorem 2.1, for any such normal ultrafilter U, $S(U)$ is defined. In particular, parts (d) and (e) of Theorem 2.1 tell us that $\kappa \leq S(U) \leq \lambda$.

In a previous paper (see [2]), we showed that if λ is a subtle cardinal, then, for almost every (with respect to the subtle ideal on λ) $\gamma < \lambda$, part$(U\restriction\gamma)$ fails. A closer examination of our proof in [2] reveals that for almost every (with respect to the subtle ideal on λ) $\gamma<\lambda$, $S(U\restriction\gamma)=\gamma$. This observation leads to the following natural question: For U a normal ultrafilter on $\mathbf{P}_\kappa\lambda$, what values of $S(U)$ are possible? In particular, can we have $\kappa<S(U)<\lambda$, or are $S(U)=\kappa$ and $S(U)=\lambda$ the only possibilities? Our main result is that, given certain large cardinal assumptions, there do exist normal ultrafilters U on $\mathbf{P}_\kappa\lambda$ such that $\kappa<S(U)<\lambda$.

3. Intermediate values of $S(U)$. Our main result is the following:

THEOREM 3.1. *Suppose $\kappa\le\gamma\le\lambda$, and γ is $2^{\lambda^{\underset{\smile}{\gamma}}}$-supercompact. Then there exists a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ such that $S(U)=\gamma$.*

In order to prove the theorem, we first describe a construction due to Solovay (see [7]), and then establish four lemmas that are central to the proof. The first two lemmas give us information about Solovay's construction, and the last two lemmas generalize work of Menas (see [6]).

Suppose $\alpha<\gamma\le\beta$, α is β-supercompact, and γ is β-supercompact. Let W_0 be any normal ultrafilter on $\mathbf{P}_\alpha\beta$, let W_1 be any normal ultrafilter on $\mathbf{P}_\gamma\beta$, and let $i:V\to M$ be the elementary embedding and inner model corresponding to W_0. Define a normal ultrafilter U on $\mathbf{P}_\alpha\beta$ as follows: For $A\subseteq\mathbf{P}_\alpha\beta$, $A\in U$ if and only if $\{x\in\mathbf{P}_\gamma\beta : i[x]\in i(A)\}\in W_1$. We refer the reader to [6] for a proof that U is indeed a normal ultrafilter on $\mathbf{P}_\kappa\lambda$.

We define the relation $\Psi(-,-,-,-)$ as follows:

$\Psi(\alpha,\gamma,\beta,U)$ holds if and only if $\alpha<\gamma\le\beta$, α is β-supercompact, γ is β-supercompact, and U is a normal ultrafilter on $\mathbf{P}_\alpha\beta$ obtained from normal ultrafilters on $\mathbf{P}_\alpha\beta$ and $\mathbf{P}_\gamma\beta$ as above.

Solovay developed the construction above to obtain normal ultrafilters which fail to satisfy the partition property. We observe that his construction actually provides a stronger result.

LEMMA 3.2. *If $\Psi(\alpha,\gamma,\beta,U)$ holds, then $S(U)\ge\gamma$.*

Proof. Assume $\Psi(\alpha,\gamma,\beta,U)$ holds, and let W_0 and W_1 be the normal ultrafilters on $\mathbf{P}_\alpha\beta$ and $\mathbf{P}_\gamma\beta$ respectively that were used in the construction of U, as in the definition of $\Psi(\alpha,\gamma,\beta,U)$. We must show that for every $\delta<\gamma$, sep(U,δ) fails. Fix some $\delta<\gamma$. Suppose, by way of contradiction, that there exists $A\in U$ such that for any $x,y\in A$, if $x\subsetneq y$, then $|x|<|y\cap\delta|$. Let $i:V\to M$ be the elementary embedding and inner model corresponding to W_0. Then, by elementarity, $M\models$ "For any $x,y\in i(A)$, if $x\subsetneq y$, then $|x|<|y\cap i(\delta)|$".

Since $A\in U$, it follows from the definition of U that $B\in W_1$, where $B=\{x\in\mathbf{P}_\gamma\beta : i[x]\in i(A)\}$. Let $C=\{x\in B : \delta\subseteq x\}$. Then, by the fineness of W_1, $C\in W_1$. Pick $x,y\in C$ with $x\subsetneq y$. Then we have $\delta\subseteq x\subsetneq y$

and hence $i[\delta] \subseteq i[x] \subsetneq i[y]$. Also, since $x, y \in B$, we have $i[x] \in i(A)$ and $i[y] \in i(A)$. This implies that $|i[x]| < |i[y] \cap i(\delta)|$. But this is a contradiction, since $|i[x]| \geq |i[x] \cap i(\delta)| = |i[y] \cap i(\delta)|$. ∎

LEMMA 3.3. *Suppose $\alpha < \gamma \leq \beta < \delta$, W is a normal ultrafilter on $\mathbf{P}_\alpha\delta$ such that $\Psi(\alpha, \gamma, \delta, W)$ holds, and $U = W \restriction \beta$. Then $\Psi(\alpha, \gamma, \beta, U)$ holds.*

Proof. Suppose that W_0 and W_1 are the normal ultrafilters on $\mathbf{P}_\alpha\delta$ and $\mathbf{P}_\gamma\delta$ respectively that are used in the construction of W, as in the definition of $\Psi(\alpha, \gamma, \delta, W)$. Let $i_\delta : V \to M_\delta$ and $i_\beta : V \to M_\beta$ be the elementary embeddings and inner models corresponding to W_0 and $W_0 \restriction \beta$ respectively. Also, let $k : M_\beta \to M_\delta$ be the usual elementary embedding. Recall that $k \circ i_\beta = i_\delta$.

We first claim that for any $x \in \mathbf{P}_\gamma\beta$, and any $A \subseteq \mathbf{P}_\alpha\beta$, $i_\delta[x] \in i_\delta(A)$ if and only if $i_\beta[x] \in i_\beta(A)$. We establish this as follows:

$$\begin{aligned}
&i_\delta[x] \in i_\delta(A) \quad \text{iff}\\
&(k \circ i_\beta)[x] \in (k \circ i_\beta)(A) \quad \text{iff}\\
&k[i_\beta[x]] \in k(i_\beta(A)) \quad \text{iff}\\
&k(i_\beta[x]) \in k(i_\beta(A)) \quad \text{iff}\\
&i_\beta[x] \in i_\beta(A)\,.
\end{aligned}$$

All of the above equivalences are straightforward, except possibly the third. This equivalence follows from the fact that, since k fixes all ordinals less than or equal to β, and $|i_\beta[x]| = |x| < \gamma \leq \beta$, it follows that $k[i_\beta[x]] = k(i_\beta[x])$.

We must show that $\Psi(\alpha, \gamma, \beta, U)$ holds. We claim that U is constructed, using Solovay's method, from the normal ultrafilters $W_0 \restriction \beta$ and $W_1 \restriction \beta$ on $\mathbf{P}_\alpha\beta$ and $\mathbf{P}_\gamma\beta$ respectively. Fix $A \subseteq \mathbf{P}_\alpha\beta$. Hence, we must show that $A \in U$ if and only if $\{x \in \mathbf{P}_\gamma\beta : i_\beta[x] \in i_\beta(A)\} \in W_1 \restriction \beta$. We establish this as follows:

$$\begin{aligned}
&A \in U \quad \text{iff}\\
&A \in W \restriction \beta \quad \text{iff}\\
&\{z \in \mathbf{P}_\alpha\delta : z \cap \beta \in A\} \in W \quad \text{iff}\\
&\{y \in \mathbf{P}_\gamma\delta : i_\delta[y] \in i_\delta(\{z \in \mathbf{P}_\alpha\delta : z \cap \beta \in A\})\} \in W_1 \quad \text{iff}\\
&\{y \in \mathbf{P}_\gamma\delta : i_\delta[y] \cap i_\delta(\beta) \in i_\delta(A)\} \in W_1 \quad \text{iff}\\
&\{y \in \mathbf{P}_\gamma\delta : i_\delta[y \cap \beta] \in i_\delta(A)\} \in W_1 \quad \text{iff}\\
&\{y \in \mathbf{P}_\gamma\delta : y \cap \beta \in \{x \in \mathbf{P}_\gamma\beta : i_\delta[x] \in i_\delta(A)\}\} \in W_1 \quad \text{iff}\\
&\{x \in \mathbf{P}_\gamma\beta : i_\delta[x] \in i_\delta(A)\} \in W_1 \restriction \beta \quad \text{iff}\\
&\{x \in \mathbf{P}_\gamma\beta : i_\beta[x] \in i_\beta(A)\} \in W_1 \restriction \beta\,.
\end{aligned}$$

Hence, $\Psi(\alpha, \gamma, \beta, U)$ holds. ∎

Define the relation $\chi(-,-,-,-,-)$ as follows:

$\chi(\alpha,\gamma,\beta,U,f)$ holds if and only if $\alpha<\gamma<\beta$, $\Psi(\alpha,\gamma,\beta,U)$ holds, $f:\alpha\to\alpha$, and $M\models$ "$i(f)\gamma=\beta$ and, for any $\delta<\gamma$, if $i(f)\delta<\beta$, then $i(f)\delta<\gamma$", where $i:V\to M$ is the elementary embedding and inner model associated with U.

Lemma 3.4. *Suppose α, γ, β, U, and f, are such that $\chi(\alpha,\gamma,\beta,U,f)$ holds. Then $S(U)\le\gamma$.*

Proof. We assume that $\chi(\alpha,\gamma,\beta,U,f)$ holds. We must show that $\operatorname{sep}(U,\gamma)$ holds.

By Theorem 2.1(e), $\operatorname{sep}(U,\beta)$ holds. Hence, there exists $A\in U$ such that for $x,y\in A$ with $x\subsetneq y$, we have $|x|<|y|$.

Next, we recall that β is represented by the function $x\to|x|$, and γ is represented by the function $x\to|x\cap\gamma|$ in the ultrapower $\prod V/U$. Then it follows that $B\in U$, where $B=\{x\in A: f(|x\cap\gamma|)=|x|$ and, for any $\delta<|x\cap\gamma|$, if $f(\delta)<|x|$, then $f(\delta)<|x\cap\gamma|\}$.

We claim that B witnesses that $\operatorname{sep}(U,\gamma)$ holds. Pick $x,y\in B$ with $x\subsetneq y$. We must show that $|x|<|y\cap\gamma|$. Since $x,y\in A$ we have $|x|<|y|$. Then, since $f(|x\cap\gamma|)=|x|$, $f(|y\cap\gamma|)=|y|$, and obviously $|x\cap\gamma|\le|y\cap\gamma|$, it follows that $|x\cap\gamma|<|y\cap\gamma|$.

Next, we note that since $f(|x\cap\gamma|)=|x|<|y|$, it follows from the definition of B that $f(|x\cap\gamma|)<|y\cap\gamma|$. But $f(|x\cap\gamma|)=|x|$, and hence $|x|<|y\cap\gamma|$. ∎

Lemma 3.5. *Suppose $\kappa<\gamma<\lambda$ and γ is $2^{\lambda^{<\gamma}}$-supercompact. Then there exists a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$, and a function $f:\kappa\to\kappa$, such that $\chi(\kappa,\gamma,\lambda,U,f)$ holds.*

Proof. Suppose that κ, γ, and λ are as given, and assume, by way of contradiction, that the conclusion of the lemma is false. Let $\beta>\gamma$ be minimal such that for some $\alpha<\gamma$, we have that α is β-supercompact, and it is not the case that there exists a normal ultrafilter U on $\mathbf{P}_\alpha\beta$, and a function $f:\alpha\to\alpha$, such that $\chi(\alpha,\gamma,\beta,U,f)$ holds. Fix such an α. We note that we must have $\beta\le\lambda$.

Claim. *For any $\delta<\gamma$, if there exists a cardinal η with $\delta<\eta<\beta$ such that for some $\sigma<\delta$, σ is η-supercompact, δ is η-supercompact and it is not the case that there exists a normal ultrafilter F on $\mathbf{P}_\sigma\eta$, and a function $g:\sigma\to\sigma$, such that $\chi(\sigma,\delta,\eta,F,g)$ holds, then the least such η is less than γ.*

Fix some $\delta<\gamma$, and suppose that there exists a cardinal η with $\delta<\eta<\beta$ such that for some $\sigma<\delta$, σ is η-supercompact, δ is η-supercompact and it is not the case that there exists a normal ultrafilter F on $\mathbf{P}_\sigma\eta$, and a function $g:\sigma\to\sigma$, such that $\chi(\sigma,\delta,\eta,F,g)$ holds. Let $j:V\to N$ be any elementary embedding witnessing that γ is $2^{\lambda^{\underset{\smile}{\alpha}}}$-supercompact. Then, using the fact that $\beta \leq \lambda$, that N is closed under $2^{\lambda^{\underset{\smile}{\alpha}}}$-sequences, and that $j(\gamma) > 2^{\lambda^{\underset{\smile}{\alpha}}} > \beta$, we have

$N \models$ "There exists a cardinal η with $\delta < \eta < j(\gamma)$ such that for some $\sigma < \delta$, σ is η-supercompact, δ is η-supercompact and it is not the case that there exists a normal ultrafilter F on $\mathbf{P}_\sigma \eta$, and a function $g : \sigma \to \sigma$, such that $\chi(\sigma, \delta, \eta, F, g)$ holds".

But then, by the elementarity of j, it is true (in V) that there exists a cardinal η with $\delta < \eta < \gamma$ such that for some $\sigma < \delta$, σ is η-supercompact, δ is η-supercompact and it is not the case that there exists a normal ultrafilter F on $\mathbf{P}_\sigma \eta$, and a function $g : \sigma \to \sigma$, such that $\chi(\sigma, \delta, \eta, F, g)$ holds. This establishes the claim.

We return to the proof of Lemma 3.5. Since α is β-supercompact and $\beta > \gamma$, it follows that α is γ-supercompact. This, and the fact that γ is $2^{\lambda^{\underset{\smile}{\alpha}}}$-supercompact, tell us that α is $2^{\lambda^{\underset{\smile}{\alpha}}}$-supercompact. Then we may obtain a normal ultrafilter W on $\mathbf{P}_\alpha(2^{\lambda^{\underset{\smile}{\alpha}}})$ such that $\Psi(\alpha, \gamma, 2^{\lambda^{\underset{\smile}{\alpha}}}, W)$ holds.

Let $U = W \restriction \beta$. By Lemma 3.3, $\Psi(\alpha, \gamma, \beta, U)$ holds.

Let $i_W : V \to M_W$ and $i_U : V \to M_U$ be the elementary embeddings and inner models corresponding to W and U respectively, and let $k : M_U \to M_W$ be the elementary embedding discussed previously.

Define a function $f : \alpha \to \alpha$ as follows: For any $\delta < \alpha$,

$f(\delta) =$ the least η with $\delta < \eta < \alpha$ such that for some $\sigma < \delta$, σ is η-supercompact, δ is η-supercompact, and it is not the case that there exists a normal ultrafilter F on $\mathbf{P}_\sigma \eta$, and a function $g : \sigma \to \sigma$, such that $\chi(\sigma, \delta, \eta, F, g)$ holds, if such an η exists,

and $f(\delta) = 0$ otherwise.

We recall that $\beta \leq \lambda$ and M_W is closed under $2^{\lambda^{\underset{\smile}{\alpha}}}$-sequences. Then, by the claim and the elementarity of $i_W : V \to M_W$ applied to the function f, we have $M_W \models$ "$i_W(f)\gamma = \beta$ and, for any $\delta < \gamma$, if $i_W(f)\delta < \beta$, then $i_W(f)\delta < \gamma$".

Next we recall that the elementary embedding $k : M_U \to M_W$ fixes all ordinals less than or equal to β. This tells us that $M_U \models$ "$i_U(f)\gamma = \beta$ and, for any $\delta < \gamma$, if $i_U(f)\delta < \beta$, then $i_U(f)\delta < \gamma$".

We have already observed that $\Psi(\alpha, \gamma, \beta, U)$ holds. But this contradicts our assumption that it is not the case that there exists a normal ultrafilter U on $\mathbf{P}_\alpha \beta$, and a function $f : \alpha \to \alpha$, such that $\chi(\alpha, \gamma, \beta, U, f)$ holds. ∎

Proof of Theorem 3.1. We assume that $\kappa \leq \gamma \leq \lambda$, and γ is $2^{\lambda^{\underset{\smile}{\alpha}}}$-supercompact. We must show that there exists a normal ultrafilter U on $\mathbf{P}_\kappa \lambda$ such that $S(U) = \gamma$.

If $\kappa = \gamma$, the result follows immediately from Theorem 1.2, since $S(U) = \kappa$ if and only if $\mathrm{part}(U)$ holds.

If $\gamma = \lambda$, we can use Solovay's construction to obtain a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ such that $\Psi(\kappa, \gamma, \lambda, U)$. Then, by Lemma 3.2, $S(U) \ge \gamma$. But $\gamma = \lambda$ and we know, by Theorem 2.1(e), that $S(U) \le \lambda$. Hence $S(U) = \gamma$.

Assume then that $\kappa < \gamma < \lambda$. By Lemma 3.5, we know that there exists a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$, and a function $f : \kappa \to \kappa$, such that $\chi(\kappa, \gamma, \lambda, U, f)$ holds. Then $\Psi(\kappa, \gamma, \lambda, U)$ holds. By Lemma 3.2, this implies that $S(U) \ge \gamma$.

On the other hand, by Lemma 3.4, $\chi(\kappa, \gamma, \lambda, U, f)$ tells us that $S(U) \le \gamma$.

We have established that $S(U) = \gamma$. ∎

There is a well-known method (see Solovay, Reinhardt, and Kanamori [8], Menas [6], or Barbanel [1]) for showing that there exist $2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$ many normal ultrafilters on $\mathbf{P}_\kappa\lambda$ having some desired property. We note that this technique can be used to show that, given the assumptions of the theorem, there actually exist $2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$ many normal ultrafilters U on $\mathbf{P}_\kappa\lambda$, each satisfying that $S(U) = \gamma$. We chose to concentrate on the existence, rather than on the actual number of such normal ultrafilters, so as not to obscure the main ideas of the proof.

We close by asking whether some sort of converse to Theorem 3.1 is true. In particular, we make the following conjecture:

Conjecture. Suppose $\kappa < \gamma < \lambda$, and there exists a normal ultrafilter U on $\mathbf{P}_\kappa\lambda$ such that $S(U) = \gamma$. Then γ is λ-supercompact.

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