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Measure extension for a piecewise invariant map

by

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Abstract. Let P be a piecewise isometry with $n \geq 2$ pieces between two subsets of $\mathbb{R}^3$, and let μ be Lebesgue measure on $\mathbb{R}^3$. If the subsets are disjoint, then μ has a total finitely additive extension $\tilde{\mu}$ such that $\tilde{\mu}(Y) \leq \frac{n}{2}\tilde{\mu}(X)$ whenever $P(X) = Y$. If the subsets are separated by measurable sets, then μ has a total finitely additive extension $\tilde{\mu}$ such that $\frac{2}{n}\tilde{\mu}(X) \leq \tilde{\mu}(Y) \leq \frac{n}{2}\tilde{\mu}(X)$ whenever $P(X) = Y$.

More generally, let P be a piecewise μ-invariant map between two relativized algebras B_a and B_b in a Boolean algebra B, and let μ be a measure on a subring R. If a and b are disjoint then μ has an extension to a measure $\tilde{\mu}$ on B such that $\tilde{\mu}(Y) \leq \frac{n}{2}\tilde{\mu}(X)$ whenever $P(x) = y$. If a and b are separated by elements of R, then μ has an extension to a measure $\tilde{\mu}$ on B such that $\frac{2}{n}\tilde{\mu}(x) \leq \tilde{\mu}(y) \leq \frac{n}{2}\tilde{\mu}(x)$ whenever $P(x) = y$.

Let P be an injective piecewise ε-contraction with $n \geq 2$ pieces between two disjoint subsets of $\mathbb{R}$, and let μ be Lebesgue measure on $\mathbb{R}$. There is a total finitely additive extension $\tilde{\mu}$ of μ such that $\tilde{\mu}(Y) \leq \frac{n\varepsilon}{2}\tilde{\mu}(X)$ whenever $P(X) = Y$.

Introduction. Let A and B be sets in $\mathbb{R}^3$ and let $P : A \to B$ be a piecewise isometry with n pieces. This means that $P : A \to B$ is a bijection and there is a partition $\{A_1, \ldots, A_n\}$ of A such that each restriction $P{\restriction}A_i$ is an isometry. Let $n \geq 2$. Laczkovich has shown that $\mu_*(B) \leq \frac{n}{2}\mu^*(A)$ where μ_* and μ^* are inner and outer Lebesgue measure respectively [L, Theorem 4]. See also [S]. For any subset X of A, the restriction $P{\restriction}X$ is again a piecewise isometry with n pieces. Therefore,

$$\mu_*(Y) \leq \frac{n}{2}\mu^*(X) \quad \text{whenever} \quad P(X) = Y\,. \tag{1}$$

Let μ be Lebesgue measure on $\mathbb{R}^3$. Horn and Tarski have shown that μ has a finitely additive extension $\widetilde{\mu}$ defined on all subsets of $\mathbb{R}^3$ [HT, Theorem 1.22]. Given a piecewise isometry $P : A \to B$ with $n \geq 2$ pieces, we ask whether $\widetilde{\mu}$ may be chosen so as to support Inequality (1) in the sense that

$$\widetilde{\mu}(Y) \leq \frac{n}{2}\widetilde{\mu}(X) \quad \text{whenever} \quad P(X) = Y\,. \tag{2}$$

It follows from a theorem of Robinson that this will not always be possible [R, p. 252]. For example, let $n = 3$. According to Robinson's theorem, the deleted ball $S = \{\mathbf{v} \in \mathbb{R}^3 : 0 < \|\mathbf{v}\| < 1\}$ has a partition $\{S_1, S_2, S_3, S_4\}$ such that the congruences

$$S_2 \cong S_2 \cup S_3 \cup S_4 \quad \text{and} \quad S_3 \cong S_1 \cup S_2 \cup S_3$$

hold via rotations. Let f be a translation such that S and $f(S)$ are disjoint. Write $T = f(S)$, and let $\{T_1, T_2, T_3, T_4\}$ be the partition of T defined by $f(S_i) = T_i$ for each i in $\{1, 2, 3, 4\}$. Let $A = S_2 \cup S_3 \cup S_4$, let $B = S_2 \cup S_3 \cup S_4 \cup T$, and let $P : A \to B$ be defined so as to witness the three congruences

$$S_2 \cong S_2 \cup S_3 \cup S_4, \qquad S_3 \cong T_1 \cup T_2 \cup T_3, \qquad S_4 \cong T_4\,.$$

If there is a measure $\widetilde{\mu}$ which satisfies (2), then

$$\begin{aligned} 2\mu(S) = \mu(S \cup T) &= \widetilde{\mu}(S_1) + \widetilde{\mu}(S_2 \cup S_3 \cup S_4) + \widetilde{\mu}(T_1 \cup T_2 \cup T_3) + \widetilde{\mu}(T_4) \\ &\le 1\widetilde{\mu}(S_1) + \tfrac{3}{2}\widetilde{\mu}(S_2) + \tfrac{3}{2}\widetilde{\mu}(S_3) + \tfrac{3}{2}\widetilde{\mu}(S_4) \le \tfrac{3}{2}\widetilde{\mu}(S)\,. \end{aligned}$$

This counterexample shows that in order to guarantee the existence of a measure $\widetilde{\mu}$ which satisfies (2), some additional condition must be placed on P. A sufficient condition is that A and B are disjoint (Example 1).

Observe that if $P : A \to B$ is a piecewise isometry with $n \ge 2$ pieces, then so is $P^{-1} : B \to A$. Therefore, in addition to (1), we have

$$\mu_*(X) \le \frac{n}{2}\mu^*(Y) \quad \text{whenever} \quad P(X) = Y\,. \tag{3}$$

Given a piecewise isometry with $n \ge 2$ pieces, we now ask whether $\widetilde{\mu}$ may be chosen so as to support both (1) and (3) in the sense that

$$\frac{2}{n}\widetilde{\mu}(X) \le \widetilde{\mu}(Y) \le \frac{n}{2}\widetilde{\mu}(X) \quad \text{whenever} \quad P(X) = Y\,. \tag{4}$$

The condition that A and B are disjoint is not strong enough to guarantee the existence of a measure $\widetilde{\mu}$ which satisfies (4). Robinson's theorem again provides a counterexample. Let S , T and f be as above. Let $U = f(T)$ and let $\{U_1, U_2, U_3, U_4\}$ be the partition of U defined by $f(T_i) = U_i$ for each i in $\{1, 2, 3, 4\}$. Let $A = S \cup T_3 \cup T_4$, let $B = T_1 \cup T_2 \cup U$, and let $P : A \to B$ be defined so as to witness the three congruences

$$S_1 \cup T_4 \cong T_1 \cup U_4, \qquad S_2 \cup S_3 \cup S_4 \cong T_2, \qquad T_3 \cong U_1 \cup U_2 \cup U_3\,.$$

(P agrees with f on $S_1 \cup T_4$.) If there is a measure $\widetilde{\mu}$ which satisfies (4), then

$$\begin{aligned} 2\mu(T) = \mu(S \cup U) &= \widetilde{\mu}(S_1) + \widetilde{\mu}(S_2 \cup S_3 \cup S_4) + \widetilde{\mu}(U_1 \cup U_2 \cup U_3) + \widetilde{\mu}(U_4) \\ &\le \tfrac{3}{2}\widetilde{\mu}(T_1) + \tfrac{3}{2}\widetilde{\mu}(T_2) + \tfrac{3}{2}\widetilde{\mu}(T_3) + \tfrac{3}{2}\widetilde{\mu}(T_4) = \tfrac{3}{2}\mu(T)\,. \end{aligned}$$

For the existence of a measure $\widetilde{\mu}$ which satisfies (4), a sufficient condition on P is that A and B are separated by measurable sets (Example 1). This means that there are disjoint measurable sets M and N such that $A \subset M$ and $B \subset N$.

The main results of the present work are based on a proof of Horn and Tarski's measure extension theorem, which is presented by Wagon [W, Theorem 10.7] and outlined earlier by Mycielski [M, Theorem 4.1]. The extension theorem is proven in the context of Boolean algebras; a measure on a subring R of a Boolean algebra B is extended to a measure on the whole of B. Theorems 1–3 below are also presented in this general context. Lemmas 1 and 2 in the next section are based on a proof of the max-flow min-cut theorem of Ford and Fulkerson [FF] (see Wilson's second proof [Wi, §29]).

Two problems in linear programming. Let $I = \{1,\ldots,m\}$, let $J = \{1,\ldots,n\}$ and let A be a given subset of $I \times J$. For each subset S of I define $N(S) = \{j \in J : \exists i \in S,\ (i,j) \in A\}$, and for each subset T of J define $N(T) = \{i \in I : \exists j \in T,\ (i,j) \in A\}$. For each pair (i,j) in A, let x_{ij} and y_{ij} be variables, and introduce for convenience variables w_i and z_j as follows:

$$w_i = \sum_{j\in N(\{i\})} x_{ij} \quad (i \in I), \qquad z_j = \sum_{i\in N(\{j\})} y_{ij} \quad (j \in J)\,.$$

Let α and β be elements of $(0,\infty)$ with $\alpha\beta \ge 1$, let $k_1,\ldots,k_m$, $l_1,\ldots,l_n$ be elements of $[0,\infty]$, let D be a subset of I, and let E be a subset of J. Define $X = \{i \in I : k_i < \infty\}$ and $Y = \{j \in J : l_j < \infty\}$. Consider the following problems in linear programming:

Problem 1.

maximize $\displaystyle\sum_{j\in E\cap Y} z_j$

subject to $w_i \le k_i$ $(i \in I)$

$z_j \le l_j$ $(j \in J)$

$y_{ij} \le \alpha x_{ij},\ x_{ij} \ge 0,\ y_{ij} \ge 0$ $((i,j) \in A)$

Problem 2.

maximize $\displaystyle\sum_{i\in D\cap X} w_i + \sum_{j\in E\cap Y} z_j$

subject to $w_i \le k_i$ $(i \in I)$

$z_j \le l_j$ $(j \in J)$

$y_{ij} \le \alpha x_{ij},\ x_{ij} \le \beta y_{ij},\ x_{ij} \ge 0,\ y_{ij} \ge 0$ $((i,j) \in A)$

These problems may be considered to be problems in only the real (finite) variables $\{x_{ij} : i \in X, (i,j) \in A\} \cup \{y_{ij} : j \in Y, (i,j) \in A\}$ since only these variables occur in the objective functions, and any feasible (respecting the given inequalities) assignment of real numbers to these variables may be extended to a feasible assignment of all the variables by the following rule:

RULE 1. Let (i, j) be an element of A.
If $i \in X$, assume x_{ij} has been assigned a value.
If $j \in Y$, assume y_{ij} has been assigned a value.
If $i \in X$ and $j \notin Y$, let $y_{ij} = \alpha x_{ij}$.
If $i \notin X$ and $j \in Y$, let $x_{ij} = \beta y_{ij}$.
If $i \notin X$ and $j \notin Y$, let $x_{ij} = \infty$, and let $y_{ij} = \infty$.

The problems may be interpreted as problems of supply and demand. There are m sources with *supplies* $k_1, \ldots, k_m$ of some product and n destinations with *demands* $l_1, \ldots, l_n$. When $(i, j) \in A$, the ith source serves the jth destination, x_{ij} is the amount supplied and y_{ij} is the amount received. If the product were say electric back-scratchers, it would be practical to insist on $x_{ij} = y_{ij}$, but in the present situation, the product is less rigid. The product may be expanded or compressed at the discretion of the problem solver but within the limits prescribed by α and β.

LEMMA 1. *If* $\sum_{j \in T} l_j \le \alpha \sum_{i \in N(T)} k_i$ *for each subset* T *of* E, *then Problem* 1 *has a feasible solution in which* $z_j = l_j$ *for all* j *in* E, *and* $w_i = k_i$ *for all* i *in* $N(J)$.

LEMMA 2. *If* $\sum_{i \in S} k_i \le \beta \sum_{j \in N(S)} l_j$ *for each subset* S *of* D, *and* $\sum_{j \in T} l_j \le \alpha \sum_{i \in N(T)} k_i$ *for each subset* T *of* E, *then Problem* 2 *has a feasible solution in which* $w_i = k_i$ *for all* i *in* D, *and* $z_j = l_j$ *for all* j *in* E.

In order to prove Lemmas 1 and 2, it suffices to prove the following proposition:

PROPOSITION 1. *Let* $\{x_{ij}, y_{ij}\}$ *be an optimal solution to Problem* 2, *obtained by first finding an optimal solution in the real variables and then extending to the other variables by Rule* 1. *If* $\sum_{j \in T} l_j \le \alpha \sum_{i \in N(T)} k_i$ *for each subset* T *of* E, *then* $z_j = l_j$ *for all* j *in* E.

Proof of Lemma 1. Start by letting $\{x_{ij}, y_{ij}\}$ be as in Proposition 1, so $z_j = l_j$ for all j in E, and $w_i \le k_i$ for all i in I. Then for each i in $N(J)$, choose j such that (i, j) is in A and increase x_{ij} until $w_i = k_i$. This process does not violate the constraint $y_{ij} \le \alpha x_{ij}$. ∎

Proof of Lemma 2. Let $\{x_{ij}, y_{ij}\}$ be as in Proposition 1, so $z_j = l_j$ for all j in E. By the symmetry of Problem 2 and Rule 1, conclude that $w_i = k_i$ for all i in D, from Proposition 1 and the assumption that $\sum_{i \in S} k_i \le \beta \sum_{j \in N(S)} l_j$ for each subset S of D. ∎

Proof of Proposition 1. Suppose $z_j < l_j$ for some j in E. Without loss of generality, assume $1 \in E$ and $z_1 < l_1$. Construct T as follows:

Case 1: $1 \notin Y$ ($l_1 = \infty$). Let $T = \{1\}$. Suppose $i \in N(\{1\})$. Since z_1 is finite, so is y_{i1} by the definition of z_1. Therefore $i \in X$, by Rule 1, which means that k_i is finite. This is true for all i in $N(\{1\})$. Therefore

$$\sum_{j \in T} l_j = l_1 = \infty > \alpha \sum_{i \in N(T)} k_i \, .$$

Case 2: $1 \in Y$ ($l_1 < \infty$). Construct T recursively as follows: Put 1 in T. If $t \in T$, $(s,t) \in A$, $(s,u) \in A$, and $x_{su} > 0$, then put u in T. We establish some properties of T. Suppose $t \in T$. Then

(i) $t \in E$.
(ii) $y_{st} = \alpha x_{st}$ for all s in $N(\{t\})$.
(iii) $w_s = k_s < \infty$ for all s in $N(\{t\})$.

We argue that if one of these properties does not hold, then there is a perturbation of the solution $\{x_{ij}, y_{ij}\}$, which is still feasible and for which the value of the objective function in Problem 2 is larger. This then contradicts maximality.

First suppose $t = 1$. Then (i) holds by assumption. Suppose $s \in N(\{1\})$. If $y_{s1} < \alpha x_{s1}$, increase the value of the objective function by ε as follows: Let $\varepsilon = \min\{l_1 - z_1, \alpha x_{s1} - y_{s1}\}$. Increase y_{s1} by ε. (All other variables x_{ij} and y_{ij} are unchanged.) This proves (ii). Now assume $y_{s1} = \alpha x_{s1}$. If $w_s < k_s$ or if $k_s = \infty$, increase the value of the objective function as follows: Let $\varepsilon = \min\{l_1 - z_1, \alpha(k_s - w_s)\}$. Increase x_{s1} by ε/α and increase y_{s1} by ε. This proves (iii).

Now let t be an element of T other than 1. Without loss of generality, assume that t is in T because $(1,1)$, $(1,2)$, $(2,2)$, $(2,3), \ldots, (t-1,t-1)$, $(t-1,t)$ are all elements of A, and $x_{12}, x_{23}, \ldots, x_{t-1,t}$ are all positive. Assume inductively that $y_{ij} = \alpha x_{ij}$ for all i in $N(\{j\})$ when j is in $\{1, \ldots, t-1\}$.

Suppose $s \in N(\{t\})$. Consider first the case when $s = t-1$. If $y_{t-1,t} < \alpha x_{t-1,t}$, increase the value of the objective function as follows:

Let $\varepsilon = \min\{l_1 - z_1; y_{12}, y_{23}, \ldots, y_{t-2,t-1}; \alpha x_{t-1,t} - y_{t-1,t}\}$.
Increase $y_{11}, y_{22}, \ldots, y_{t-2,t-2}, y_{t-1,t-1}$ (each) by ε.
Increase $x_{11}, x_{22}, \ldots, x_{t-2,t-2}, x_{t-1,t-1}$ by ε/α.
Decrease $x_{12}, x_{23}, \ldots, x_{t-2,t-1}, x_{t-1,t}$ by ε/α.
Decrease $y_{12}, y_{23}, \ldots, y_{t-2,t-1}$ by ε.
This proves (ii) for the case $s = t-1$.

Now assume $y_{t-1,t} = \alpha x_{t-1,t}$. If $t \notin E$, increase the value of the objective function as follows:

Let $\varepsilon = \min\{l_1 - z_1, y_{12}, y_{23}, \ldots, y_{t-2,t-1}, y_{t-1,t}\}$. Make all the adjustments to x_{ij} and y_{ij} described as yet, and in addition, decrease $y_{t-1,t}$ by ε. (This last adjustment does not decrease the value of the objective function.) This proves (i).

Suppose $s \in N(\{t\})$ and $s \neq t-1$. If $y_{st} < \alpha x_{st}$, increase the value of the objective function as follows:

Let $\varepsilon = \min\{l_1 - z_1, y_{12}, y_{23}, \ldots, y_{t-1,t}, \alpha x_{st} - y_{st}\}$. Make all the adjustments to x_{ij} and y_{ij} described as yet and in addition, increase y_{st} by ε. This completes the proof of (ii).

Now assume $y_{st} = \alpha x_{st}$. If $w_s < k_s$, or if $k_s = \infty$, increase the value of the objective function as follows:

Let $\varepsilon = \min\{l_1 - z_1, y_{12}, y_{23}, \ldots, y_{t-2,t-1}, y_{t-1,t}, \alpha(k_s - w_s)\}$. Make all the adjustments to x_{ij} and y_{ij} described as yet and in addition, increase x_{st} by ε/α. This proves (iii).

We have established (i)–(iii) for all t in T.

Subcase 1: $\sum_{j\in T} l_j = \infty$. By (iii), $k_i < \infty$ for each i in $N(T)$. Therefore $\sum_{i\in T} l_j = \infty > \alpha \sum_{i\in N(T)} k_i$.

Subcase 2: $\sum_{j\in T} l_j < \infty$. Then

$$\begin{aligned}
\sum_{j\in T} l_j > \sum_{j\in T} z_j & \qquad \text{(since } z_1 < l_1\text{)}\\
= \sum_{j\in T} \sum_{i\in N(\{j\})} y_{ij} & \qquad \text{(by the definition of } z_j\text{)}\\
= \alpha \sum_{j\in T} \sum_{i\in N(\{j\})} x_{ij} & \qquad \text{(by (ii))}\\
= \alpha \sum_{i\in N(T)} \sum_{j\in N(\{i\})} x_{ij} & \qquad \text{(by the definition of } T\text{)}\\
= \alpha \sum_{i\in N(T)} w_i & \qquad \text{(by the definition of } w_i\text{)}\\
= \alpha \sum_{i\in N(T)} k_i & \qquad \text{(by (iii))}\,.
\end{aligned}$$

This completes the proof of Proposition 1. ■

Laczkovich's inequality in the context of a Boolean algebra. We develop a version of Inequality (1) in the context of a Boolean algebra (Lemma 3).

Let R be a subring of a Boolean algebra B. Let μ be a measure on R. This means that $\mu : R \to [0,\infty]$ is a function such that $\mu(\mathbf{0}) = 0$, and $\mu(r_1 \vee r_2) = \mu(r_1) + \mu(r_2)$ whenever r_1 and r_2 are disjoint ($r_1 \wedge r_2 = \mathbf{0}$). For each element b of B define

$$\begin{aligned}
&\mu^*(b) = \inf\{\mu(r) : b \le r \text{ and } r \in R\} \quad \text{and}\\
&\mu_*(b) = \sup\{\mu(r) : r \le b \text{ and } r \in R\}\,.
\end{aligned}$$

PROPOSITION 2. *Let* $y_1, \ldots, y_n$ *be elements of* B. *If* $y_1 \vee \ldots \vee y_n = s \in R$, *then*

$$2\mu(s) \le \sum_{i=1}^{n} (\mu^*(y_i) + \mu_*(y_i))\,.$$

Proof. If $\mu(s) = \infty$, there is some i such that $\mu^*(y_i) = \infty$, so assume $\mu(s) < \infty$. Let $\varepsilon > 0$ be given. For each i in $\{1, \ldots, n\}$ choose an element q_i of R such that $y_i \le q_i \le s$ and $\mu(q_i) \le \mu^*(y_i) + \varepsilon$. If q is an element of R, write q^1 and q^0 for the element q and its complement q' respectively. For each i in $\{1, \ldots, n\}$ let

$$p_i = q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}$$

where $k_i = 1$ and $k_j = 0$ for $j \neq i$. Note that $p_i \le y_i$, so $\mu(p_i) \le \mu_*(y_i)$. Consider a general n-tuple $(k_1, \ldots, k_n)$ of zeros and ones. If $(k_1, \ldots, k_n)$ has a single one in the ith place, we may write

$$2\mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}) = \mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}) + \mu(p_i)$$

by adding the same quantity to both sides in the definition of p_i. If at least two of the k_i's are ones, then

$$2\mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}) \le (k_1 + \ldots + k_n)\mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n})\,.$$

Summing over all n-tuples except $(0, \ldots, 0)$ we obtain

$$\begin{aligned} 2\mu(s) = 2\mu(q_1 \vee \ldots \vee q_n) &= \sum 2\mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}) \\ &\le \sum (k_1 + \ldots + k_n)\mu(q_1^{k_1} \wedge \ldots \wedge q_n^{k_n}) + \sum_{i=1}^{n} \mu(p_i) \\ &= \sum_{i=1}^{n} \mu(q_i) + \sum_{i=1}^{n} \mu(p_i) \le \sum_{i=1}^{n} (\mu^*(y_i) + \varepsilon) + \sum_{i=1}^{n} \mu_*(y_i)\,. \end{aligned}$$

Since this is true for all ε, the result follows. ∎

PROPOSITION 3. *Let* $x_1, \ldots, x_n$ *be pairwise disjoint elements of* R. *If* $x_1 \vee \ldots \vee x_n \le r \in R$, *then*

$$\mu^*(x_1) + \sum_{i=2}^{n} \mu_*(x_i) \le \mu(r)\,.$$

Proof. If $\mu(r) = \infty$, there is nothing to prove, so assume $\mu(r) < \infty$. Let $\varepsilon > 0$ be given. For each i in $\{2, \ldots, n\}$, choose an element p_i of R such that $p_i \le x_i$ and $\mu_*(x_i) \le \mu(p_i) + \varepsilon$. Let $q = r - (p_2 \vee \ldots \vee p_n)$. Note that $x_1 \le q$, so $\mu^*(x_1) \le \mu(q)$. Therefore

$$\mu^*(x_1) + \sum_{i=2}^{n} \mu_*(x_i) \le \mu(q) + \sum_{i=2}^{n} (\mu(p_i) + \varepsilon) = \mu(r) + (n-1)\varepsilon\,.$$

Since this is true for all ε, the result follows. ∎

A bijection g from one subset of B to another is said to be an *isomorphism* if it preserves Boolean operations. An isomorphism $g : B \to B$ is *μ-invariant* if for all x in B, either x and $g(x)$ both lie in R and have the same measure, or neither x nor $g(x)$ lie in R. Note that if g is μ-invariant and $g(x) = y$, then $\mu^*(y) = \mu^*(x)$ and $\mu_*(y) = \mu_*(x)$.

DEFINITION 1. Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be elements of B, and let $B_a = \{x \in B : x \le a\}$ and $B_b = \{x \in B : x \le b\}$ be the corresponding relativized algebras. We say that an isomorphism $P : B_a \to B_b$ is *piecewise μ-invariant* with n pieces if P is defined piecewise by μ-invariant isomorphisms $g_1, \ldots, g_n$. This means that there are elements $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ of B such that $x_1, \ldots, x_n$ are pairwise disjoint, $y_1, \ldots, y_n$ are pairwise disjoint, $x_1 \vee \ldots \vee x_n = a$, $y_1 \vee \ldots \vee y_n = b$, and for each i in $\{1, \ldots, n\}$, $g_i(x_i) = y_i$ and $P(z) = g_i(z)$ for all $z \le x_i$.

LEMMA 3. *Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be elements of B. Let $P : B_a \to B_b$ be a piecewise μ-invariant isomorphism with $n \ge 2$ pieces. Then $\mu(s) \le \frac{n}{2}\mu(r)$ whenever $P^{-1}(s) \le r$, $r \in R$, and $s \in R$.*

It is understood here that $s \le b$, but it is not necessarily true that $r \le a$.

Proof of Lemma 3. Let P be as above and suppose $P^{-1}(s) \le r$, $r \in R$, and $s \in R$. Then there are elements $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ of B and μ-invariant isomorphisms $g_i : B \to B$ such that $x_1, \ldots, x_n$ are pairwise disjoint, $x_1 \vee \ldots \vee x_n \le r$, $y_1 \vee \ldots \vee y_n = s$, and $g_i(x_i) = y_i$ for each i in $\{1, \ldots, n\}$. For each i in $\{1, \ldots, n\}$ we have $\mu^*(y_i) = \mu^*(x_i)$, and $\mu_*(y_i) = \mu_*(x_i)$. If $\mu(r) = \infty$, there is nothing to prove, so assume $\mu(r) < \infty$. Each difference $\mu^*(x_i) - \mu_*(x_i)$ is then defined. Without loss of generality, assume $\mu^*(x_1) - \mu_*(x_1) \ge \mu^*(x_i) - \mu_*(x_i)$ for all i in $\{2, \ldots, n\}$. Then

$$\begin{aligned}
2\mu(s) &\le \sum_{i=1}^{n} (\mu^*(y_i) + \mu_*(y_i)) \qquad \text{(Proposition 2)} \\
&= \sum_{i=1}^{n} (\mu^*(x_i) + \mu_*(x_i)) = \sum_{i=1}^{n} (\mu^*(x_i) - \mu_*(x_i)) + 2\sum_{i=1}^{n} \mu_*(x_i) \\
&\le (\mu^*(x_1) - \mu_*(x_1)) + n\sum_{i=1}^{n} \mu_*(x_i) = n\mu^*(x_1) + n\sum_{i=2}^{n} \mu_*(x_i) \\
&\le n\mu(r) \qquad \text{(Proposition 3)}\,.
\end{aligned}$$

This completes the proof of Lemma 3. ■

Inequality (1) is in fact a special case of a more general inequality proven by Laczkovich in order to deal with piecewise Lipschitz functions (see [L, Theorem 4]). A version of the more general inequality can also be formulated in the context of a Boolean algebra, but the Lipschitz conditions are replaced by measure-theoretic conditions:

Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be elements of B. For each i in $\{1,\dots,n\}$, let ε_i be an element of $(0,\infty)$ and let $g_i : B \to B$ be an isomorphism with the property that for all z in B, either z and $g_i(z)$ both lie in R and satisfy $\mu(g_i(z)) \le \varepsilon_i\mu(z)$, or neither z nor $g_i(z)$ lie in R. Let $P : B_a \to B_b$ be defined piecewise by $g_1,\dots,g_n$. Then $\mu(s) \le M\mu(r)$ whenever $P^{-1}(s) \le r$, $r \in R$, and $s \in R$, where

$$M = \max\left\{\frac{1}{2}\sum_{i=1}^{n}\varepsilon_i, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\right\}.$$

Proof. For each i in $\{1,\dots,n\}$ we have $\mu^*(y_i) \le \varepsilon_i\mu^*(x_i)$ and $\mu_*(y_i) \le \varepsilon_i\mu_*(x_i)$. Assume $\mu^*(x_1) - \mu_*(x_1) \ge \mu^*(x_i) - \mu_*(x_i)$ for all i in $\{2,\dots,n\}$. Then

$$\begin{aligned}
\sum_{i=1}^{n}(\mu^*(y_i) + \mu_*(y_i)) &\le \sum_{i=1}^{n}\varepsilon_i(\mu^*(x_i) - \mu_*(x_i)) + 2\sum_{i=1}^{n}\varepsilon_i\mu_*(x_i)\\
&\le \Big(\sum_{i=1}^{n}\varepsilon_i\Big)(\mu^*(x_1) - \mu_*(x_1)) + 2\varepsilon_1\mu_*(x_1) + 2\varepsilon_2\mu_*(x_2) + 2\varepsilon_n\mu_*(x_n)\\
&\le 2M\Big(\mu^*(x_1) + \sum_{i=2}^{n}\mu_*(x_i)\Big),
\end{aligned}$$

and Propositions 2 and 3 are applied as in the proof of Lemma 3. ∎

Main results

THEOREM 1. *Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be disjoint elements of B. Let $P : B_a \to B_b$ be an isomorphism. Let α be an element of $(0,\infty)$. The following are equivalent*:

(1) *μ has an extension to a measure $\widetilde{\mu}$ on B such that $\widetilde{\mu}(y) \le \alpha\widetilde{\mu}(x)$ whenever $P(x) = y$.*
(2) *$\mu_*(y) \le \alpha\mu^*(x)$ whenever $P(x) = y$.*
(3) *$\mu(s) \le \alpha\mu(r)$ whenever $P^{-1}(s) \le r$, $r \in R$ and $s \in R$.*

The point of this theorem is the implication (3)⇒(1). For the other implications, the assumption that a and b are disjoint is not necessary.

Proof of Theorem 1. (1)⇒(2). $\mu_*(y) \le \widetilde{\mu}(y) \le \alpha\widetilde{\mu}(x) \le \alpha\mu^*(x)$.
(2)⇒(3). $\mu(s) = \mu_*(s) \le \alpha\mu^*(P^{-1}(s)) \le \alpha\mu^*(r) = \alpha\mu(r)$.

(3)⇒(2). Fix x and y such that $P(x) = y \le b$, and consider all elements r and s of R such that $s \le y$ and $x \le r$. Then $\sup\{\mu(s)\} \le \inf\{\alpha\mu(r)\}$.

(3)⇒(1). Let $\mathcal{F} = \{C \subset B : C$ is a finite subalgebra of B, $a \in C$, $b \in C$, and $x \in C \Leftrightarrow y \in C$ whenever $P(x) = y\}$. Since a and b are disjoint and P is an isomorphism, every finite subset of B is a subset of some set C in $\mathcal{F}$.

For each $C \in \mathcal{F}$, define $\mathcal{M}(C)$ to be the set of all functions $\nu : B \to [0, \infty]$ satisfying the following conditions:

(i) ν is finitely additive on C.
(ii) ν agrees with μ on $R \cap C$.
(iii) $\nu(y) \le \alpha\nu(x)$ whenever $P(x) = y$, $x \in C$ and $y \in C$.

Step 1. *If $C_1, \ldots, C_k$ are elements of $\mathcal{F}$, then $\bigcap_{i=1}^k \mathcal{M}(C_i)$ is not empty.*

Proof. Since $\bigcup_{i=1}^k C_i$ is a finite subset of B, it is a subset of some C in $\mathcal{F}$. Since $\mathcal{M}(C) \subset \bigcap_{i=1}^k \mathcal{M}(C_i)$, it suffices to show that $\mathcal{M}(C)$ is not empty.

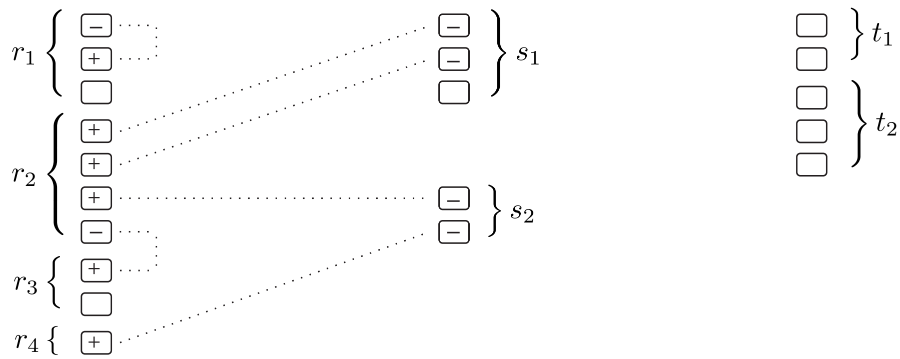

Fig. 1

Let $r_1, \ldots, r_m$, $s_1, \ldots, s_n$, $t_1, \ldots, t_p$ be all the atoms of $R \cap C$ named so as to satisfy

$$\begin{aligned} &r_i \wedge a \neq \mathbf{0} && (i \in I = \{1, \ldots, m\}), \\ &s_j \wedge a = \mathbf{0}, \quad s_j \wedge b \neq \mathbf{0} && (j \in J = \{1, \ldots, n\}), \\ &t_k \wedge a = \mathbf{0}, \quad t_k \wedge b = \mathbf{0} && (k \in K = \{1, \ldots, p\}). \end{aligned}$$

Each atom of $R \cap C$ is a join of atoms of C. The structure of an element C of $\mathcal{F}$ can be represented by a figure such as Figure 1. The small discs are the atoms of C. The element a is the join of the atoms marked "+" and the element b is the join of the atoms marked "−". Atoms of C are grouped together and labelled accordingly when their join is an atom of $R \cap C$. The connecting arcs indicate the action of P.

Let $A = \{(i, j) \subset I \times J : c \le r_i,\ d \le s_j$ and $P(c) = d$, for some atoms c, d of $C\}$. Let $E = \{j \in J : s_j \le b\}$. In Figure 1, for example, $A = \{(2, 1), (2, 2), (4, 2)\}$ and $E = \{2\}$.

For each i in I, let $k_i = \mu(r_i)$, and for each j in J, let $l_j = \mu(s_j)$. Suppose $T \subset E$. Then $P^{-1}(\bigvee_{j\in T} s_j) \le \bigvee_{i\in N(T)} r_i$. Therefore

$$\sum_{j\in T} l_j = \mu\Big(\bigvee_{j\in T} s_j\Big) \le \alpha\mu\Big(\bigvee_{i\in N(T)} r_i\Big) = \alpha \sum_{i\in N(t)} k_i\,.$$

Therefore, by Lemma 1, Problem 1 has a feasible solution $\{x_{ij}, y_{ij}\}$ with $w_i = k_i$ for all i in $N(J)$ and $z_j = l_j$ for all j in E.

For each pair (i,j) in A, choose atoms c and d of C such that $c \le r_i$, $d \le s_j$ and $P(c) = d$, and define $\nu(c) = x_{ij}$ and $\nu(d) = y_{ij}$. For each i in $I \setminus N(J)$, choose an atom c of C such that $c \le r_i \wedge a$, and define $\nu(c) = \mu(r_i)$. For each j in $J \setminus E$, choose an atom d of C such that $d \le s_j \wedge b'$, and define $\nu(d) = l_j - z_j$. (If $l_j = \infty$, define $\nu(d) = \infty$.) For each k in K, choose an atom c of C such that $c \le t_k$, and define $\nu(c) = \mu(t_k)$. For each atom c of C on which ν is not yet defined, define $\nu(c) = 0$. Define ν on the nonatomic elements of C by additivity. Define ν arbitrarily on $B \setminus C$.

The function ν satisfies condition (i) by definition. For condition (ii), it suffices to observe that for each atom r of $R \cap C$, we have $\sum \nu(c) = \mu(r)$, where the sum is taken over all atoms c of C such that $c \le r$. For condition (iii), it suffices to observe that $\nu(d) \le \alpha\nu(c)$ whenever $P(c) = d$ and c and d are atoms of C. Then if $P(x) = y$, for nonatomic elements x and y of C, we have

$$\nu(y) = \sum \nu(P(c)) \le \alpha \sum \nu(c) = \alpha\nu(x)\,,$$

where the sums are taken over all atoms c of C such that $c \le x$.

Step 2. *$\mathcal{M}(C)$ is a closed subset of $[0,\infty]^B$ for each C in $\mathcal{F}$.*

Proof. Each condition on ν defines a closed subset of $[0,\infty]^B$.

Step 3. The set $[0,\infty]^B$ is compact by Tikhonov's theorem. Therefore, by Steps 1 and 2, the intersection $\bigcap\{\mathcal{M}(C) : C \in \mathcal{F}\}$ is not empty. Let $\widetilde{\mu}$ be any element of $\bigcap\{\mathcal{M}(C) : C \in \mathcal{F}\}$. For any C in $\mathcal{F}$, and hence for any finite subset C of B, the function $\widetilde{\mu}$ agrees with some measure ν satisfying conditions (i)–(iii). ■

Definition 2. Let R be a subring of a Boolean algebra B. Two elements a and b of B are said to be *separated by elements of* R if there are elements r and s of R such that $a \le r$, $b \le s$, and $r \wedge s = \mathbf{0}$.

Theorem 2. *Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be elements of B which are separated by elements of R. Let $P : B_a \to B_b$ be an isomorphism. Let α and β be elements of $(0,\infty)$ with $\alpha\beta \ge 1$. The following are equivalent*:

(1) *μ has an extension to a measure $\widetilde{\mu}$ on B such that $\frac{1}{\beta}\widetilde{\mu}(x) \le \widetilde{\mu}(y) \le \alpha\widetilde{\mu}(x)$ whenever $P(x) = y$.*

(2) $\mu_*(y) \le \alpha\mu^*(x)$ *and* $\mu_*(x) \le \beta\mu^*(y)$ *whenever* $P(x) = y$.

(3) $\mu(s) \le \alpha\mu(r)$ *whenever* $P^{-1}(s) \le r$, $r \in R$ *and* $s \in R$, *and* $\mu(r) \le \beta\mu(s)$ *whenever* $P(r) \le s$, $r \in R$ *and* $s \in R$.

Proof. (3)⇒(1). The proof is similar to that of Theorem 1. We indicate only the definition of $\mathcal{M}(C)$ and the construction of an element ν of $\mathcal{M}(C)$. Let $\mathcal{F}$ be as in Theorem 1, but assume also that each C contains the separating elements. For each C in $\mathcal{F}$, define $\mathcal{M}(C)$ to be the set of all functions ν satisfying the following conditions:

(i) ν is finitely additive on C.
(ii) ν agrees with μ on $R \cap C$.
(iii) $\frac{1}{\beta}\nu(x) \le \nu(y) \le \alpha\nu(x)$ whenever $P(x) = y$, $x \in C$ and $y \in C$.

We show that $\mathcal{M}(C)$ is not empty. Let $r_1, \ldots, r_m$, $s_1, \ldots, s_n$, $t_1, \ldots, t_p$ be all the atoms of $R \cap C$ named so as to satisfy

$$\begin{array}{lll} r_i \wedge a \neq \mathbf{0} & & (i \in I = \{1, \ldots, m\})\,, \\ & s_j \wedge b \neq \mathbf{0} & (j \in J = \{1, \ldots, n\})\,, \\ t_k \wedge a = \mathbf{0}, & t_k \wedge b = \mathbf{0} & (k \in K = \{1, \ldots, p\})\,. \end{array}$$

Since C contains the separating elements, there is no ambiguity here, that is, there is no atom r of $R \cap C$ such that both $r \wedge a \neq \mathbf{0}$ and $r \wedge b \neq \mathbf{0}$. As in the proof of Theorem 1, let $A = \{(i, j) \subset I \times J : c \le r_i,\ d \le s_j$ and $P(c) = d$, for some atoms c, d of $C\}$, for each i in I, let $k_i = \mu(r_i)$, and for each j in J, let $l_j = \mu(s_j)$. Let $D = \{i \in I : r_i \le a\}$ and let $E = \{j \in J : s_j \le b\}$.

If $S \subset D$, then $P(\bigvee_{i \in S} r_i) \le \bigvee_{j \in N(S)} s_j$, and therefore $\sum_{i \in S} k_i \le \beta \sum_{j \in N(S)} l_j$. Similarly if $T \subset E$, then $P^{-1}(\bigvee_{j \in T} s_j) \le \bigvee_{i \in N(T)} r_i$, and therefore $\sum_{j \in T} l_j \le \alpha \sum_{i \in N(T)} k_i$. By Lemma 2, Problem 2 has a feasible solution $\{x_{ij}, y_{ij}\}$ with $w_i = k_i$ for all i in D, and $z_j = l_j$ for all j in E.

For each pair (i, j) in A, choose atoms c and d of C such that $c \le r_i$, $d \le s_j$ and $P(c) = d$, and define $\nu(c) = x_{ij}$ and $\nu(d) = y_{ij}$. For each i in $I \setminus D$, choose an atom c of C such that $c \le r_i \wedge a'$, and define $\nu(c) = k_i - w_i$. (If $k_i = \infty$, define $\nu(c) = \infty$.) For each j in $J \setminus E$, choose an atom d of C such that $d \le s_j \wedge b'$, and define $\nu(d) = l_j - z_j$. For each k in K, choose an atom c of C such that $c \le t_k$, and define $\nu(c) = \mu(t_k)$. For each atom c of C on which ν is not yet defined, define $\nu(c) = 0$. Define ν on the nonatomic elements of C by additivity. Define ν arbitrarily on $B \setminus C$. ■

We now obtain the measure extension theorem for a piecewise invariant map.

THEOREM 3. *Let R be a subring of a Boolean algebra B. Let μ be a measure on R. Let a and b be elements of B. Let $P : B_a \to B_b$ be a piecewise*

μ-invariant map with $n \ge 2$ pieces. If a and b are disjoint then μ has an extension to a measure $\widetilde{\mu}$ on B such that $\widetilde{\mu}(y) \le \frac{n}{2}\widetilde{\mu}(x)$ whenever $P(x) = y$. If a and b are separated by elements of R then μ has an extension to a measure $\widetilde{\mu}$ on B such that $\frac{2}{n}\widetilde{\mu}(x) \le \widetilde{\mu}(y) \le \frac{n}{2}\widetilde{\mu}(x)$ whenever $P(x) = y$.

Proof. By Lemma 3 we have $\mu(s) \le \frac{n}{2}\mu(r)$ whenever $P^{-1}(s) \le r$, $r \in R$, and $s \in R$. By Lemma 3 applied to P^{-1} we have $\mu(r) \le \frac{n}{2}\mu(s)$ whenever $P(r) = s$, $r \in R$, and $s \in R$. Let $\alpha = \beta = n/2$ and apply Theorems 1 and 2. ∎

Applications

Example 1. Let A and B be bounded sets in $\mathbb{R}^3$ with nonempty interiors. Banach and Tarski have shown that for some finite n, there exists a piecewise isometry $P : A \to B$ with n pieces [BT, Theorem 24]. More generally, let $P : A \to B$ be a piecewise isometry with n pieces between any subsets A and B of $\mathbb{R}^3$. Assume $n \ge 2$ and let μ be Lebesgue measure on $\mathbb{R}^3$. Let $B = \mathcal{P}(\mathbb{R}^3)$ and let R be the subring of Lebesgue measurable sets. (At this point it is possible to apply Theorem 3, but it is worth noting that we can avoid Lemma 3.) By Laczkovich's inequality in $\mathbb{R}^3$, we have $\mu_*(Y) \le \frac{n}{2}\mu^*(X)$ and $\mu_*(X) \le \frac{n}{2}\mu^*(Y)$ whenever $P(X) = Y$.

If A and B are disjoint then by Theorem 1 with $\alpha = n/2$ there is a total (defined on all subsets) finitely additive extension $\widetilde{\mu}$ of μ such that

$$\widetilde{\mu}(Y) \le \frac{n}{2}\widetilde{\mu}(X) \quad \text{whenever} \quad P(X) = Y\,. \tag{2}$$

If A and B are separated by measurable sets then by Theorem 2 with $\alpha = \beta = n/2$ there is a total finitely additive extension $\widetilde{\mu}$ of μ such that

$$\frac{2}{n}\widetilde{\mu}(X) \le \widetilde{\mu}(Y) \le \frac{n}{2}\widetilde{\mu}(X) \quad \text{whenever} \quad P(X) = Y\,. \tag{4}$$

For any $n \ge 2$ it is possible to choose A, B and P such that $\mu(B) = \frac{n}{2}\mu(A)$ (see [S]). In this case Inequality (2) becomes an equality. For example, in Robinson's duplication of the deleted ball with four pieces [R, p. 254], $\widetilde{\mu}$ may be chosen so that roughly speaking, each piece and each part of each piece is exactly doubled (in $\widetilde{\mu}$ measure), assuming that the original deleted ball and the two copies are pairwise disjoint.

Example 2. Let A and B be bounded sets in $\mathbb{R}^2$ with nonempty interiors. Let G be the group generated by $\mathrm{SL}_2(\mathbb{R}) \cup T$ where T is the group of translations on $\mathbb{R}^2$. Then there is a bijection $P : A \to B$ defined piecewise by n elements of G, for some n (see [Wg, Theorem 7.3]). Assume $n \ge 2$ and let μ be Lebesgue measure on $\mathbb{R}^2$. P is piecewise μ-invariant with n pieces.

Apply Theorem 3 with $B = \mathcal{P}(\mathbb{R}^2)$. If A and B are disjoint then there is a total finitely additive extension $\widetilde{\mu}$ of μ which satisfies Inequality (2). If A and B are separated by measurable sets then there is a total finitely additive extension $\widetilde{\mu}$ of μ which satisfies Inequality (4).

Example 3. Let A and B be bounded sets in $\mathbb{R}$ with nonempty interiors. Let G be the group of bijections $g : \mathbb{R} \to \mathbb{R}$ such that g and g^{-1} are both Lebesgue measurable and preserve Lebesgue measure μ. There is a bijection $P : A \to B$ defined piecewise by n elements of G, for some n (see [Wg, Theorem 7.9]). Assume $n \geq 2$. P is piecewise μ-invariant with n pieces. Apply Theorem 3 with $B = \mathcal{P}(\mathbb{R})$. For example, let $A = [0, 1)$ and let $B = [1, 3)$. By incorporating Robinson's four piece decomposition into Wagon's proof (7.9) we see that there is a bijection $P : A \to B$ defined piecewise by four elements of G. By Theorem 3 there is a total finitely additive extension $\widetilde{\mu}$ of μ such that $\widetilde{\mu}(Y) = 2\widetilde{\mu}(X)$ whenever $P(X) = Y$.

Example 4. Let A and B be bounded sets in $\mathbb{R}$ with nonempty interiors. Let S be the set of functions $g : I_g \to \mathbb{R}$ where I_g is an interval containing A, and g is an ε-contraction with respect to I_g. This means that $|g(x)-g(y)| \leq \varepsilon|x-y|$ for all x and y in I_g. There is a bijection $P : A \to B$ defined piecewise by n elements of S, for some n (see [Wg, Theorem 7.12]). Assume $n \geq 2$ and let μ be Lebesgue measure on $\mathbb{R}$. By Laczkovich's Inequality for Lipschitz functions, $\mu_*(Y) \leq \frac{n\varepsilon}{2}\mu^*(X)$ whenever $P(X) = Y$ [L, Theorem 4]. If A and B are disjoint, then by Theorem 1 with $B = \mathcal{P}(\mathbb{R})$ there is a total finitely additive extension $\widetilde{\mu}$ of μ such that $\widetilde{\mu}(Y) \leq \frac{n\varepsilon}{2}\widetilde{\mu}(X)$ whenever $P(X) = Y$.

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