A rigid Boolean algebra that admits the elimination of $Q^2_1$

by

Heike Mildenberger (Bonn)

Abstract. Using ♦, we construct a rigid atomless Boolean algebra that has no uncountable antichain and that admits the elimination of the Malitz quantifier $Q^2_1$.

1. Introduction. Malitz quantifiers are introduced in [Mag-Mal]. Let us recall the semantics of $Q^n_\alpha$, $n \geq 1$, $\alpha \in \text{ORD}$: $A \models Q^n_\alpha \phi(\pi, h)$ iff there is a subset $H$ of $A$ such that $\text{card}(H) \geq \aleph_\alpha$ and $A \models \phi(\pi, h)$ for all pairwise different $h_0, h_1, \ldots, h_{n-1} \in H$. Such a set $H$ is called a homogeneous set for $\phi(\pi, h)$. Baldwin and Kueker [Bal-Ku], Rothmaler and Tuschik [Ro-Tu], Bürger [Bü] and Koepke [Ko] consider the question of elimination of some of these quantifiers in certain theories or structures. [Ro-Tu] shows that any saturated model allows the elimination of all $Q^n_\alpha$, $\alpha \in \text{ORD}$, $n \geq 1$.

Saturated models with two elements of the same type are not rigid. On the other hand, there are $L_{\omega\omega}(Q^2_1)$-sentences $\phi$ that have only rigid models and that are satisfiable under CH (see [Ot], [Mil]). We consider $\phi := \text{“the structure is a Boolean algebra with } 0 \neq 1\text{”}$

$\land \forall x(x \neq 0 \rightarrow Q_1yx \subseteq x) \land \neg Q^2_1xy x \subseteq y$.

[Ba-Ko, Theorem 5(a)] shows that all models of $\phi$ are rigid. The search for a model of $\phi$ that contains two different elements of the same $L_{\omega\omega}(Q^2_1)$-type leads, under ♦, to a model of $\phi$ that admits the elimination of $Q^2_1$ and in which therefore any two elements $\neq 0, 1$ have the same $L_{\omega\omega}(Q^2_1)$-type.

In ZFC + ♦ and even in ZFC + CH there are various constructions of uncountable Boolean algebras with no uncountable antichains and with some other algebraic properties (see [Ba-Ko], [Sh], [Ru], but also [Ba]). In the course of showing that additional tasks may be fulfilled along the way given in [Ba-Ko], we get a partition of all formulas $\phi(\bar{x}, x, y) \in L_{\omega\omega}(Q^2_1)$, $r \in \omega$, into two classes $\Phi_1$ and $\Phi_2$ such that...
1. The methods of [Ba-Ko] are applicable to any $\phi(z, x, y) \in \Phi_1$. They will allow us to show that the homogeneous sets for any $\phi(z, x, y) \in \Phi_1$ will grow only during countably many steps in the chain which we build in the next section.

2. For any Boolean algebra $A$ with $A \models \forall x \neq 0 Q_1 y y \subseteq x$ and any $\phi(z, x, y) \in \Phi_2$: $A \models \exists z Q_1^+ x y \phi(z, x, y)$.

“$\phi(z, x, y) \in \Phi_1$” will be shown to be equivalent under the first order theory of atomless Boolean algebras to a first order formula with its free variables among $z_0, z_1, \ldots, z_{r-1}$. The consideration of the possible quantifierfree types of the $z$ leads to a procedure for eliminating $Q_1^+$.

2. The construction

Notation. We will use $A$, $B$, $B_n$ to denote Boolean algebras. Boolean algebras are considered as $\tau_{BA}$-structures with $\tau_{BA} = \{\cap, \cup, -, 0, 1\}$. $x \subseteq y$ is written for $x \cap y = x$, $\subset$ means strict inclusion, $x \setminus y$ is used for $x \cap \neg y$.

$P(\omega)$ denotes the powerset algebra of $\omega$. For $A \subseteq P(\omega)$ we often write $A$ for $A$. The interpretations of the $\tau_{BA}$-symbols in $P(\omega)$ are denoted by the symbols themselves.

$a, b \in A$ are comparable (in $A$) if $a \subseteq A b$ or $b \subseteq A a$. $C \subseteq A$ is a chain (an antichain) if any two distinct elements of $C$ are comparable (not comparable). For $a \subseteq A b \in A$ let $(a, b)_A := \{c \in A | a \subseteq A c \subseteq A b\}$.

Using $\phi$, we shall construct a Boolean algebra $B$ such that $B$ is a model of the sentence $\phi$ from the introduction and $B$ admits the elimination of $Q_1^+$. As the construction of our Boolean algebra $B$ follows the pattern of [Ba-Ko], we restrict ourselves to a short description, heavily referring to [Ba-Ko].

Inductively on $\alpha \in \omega_1$, we shall build a chain $(B_\alpha, M_\alpha)_{\alpha \in \omega_1}$, where the $B_\alpha$ are countable atomless subalgebras of $P(\omega)$ and each $M_{\alpha+1}$ is a countable collection of pairs $(M, \phi(\bar{r}, x, y))$, where $M \subseteq B_\alpha$ and $\phi(\bar{r}, x, y)$ is a quantifierfree (qf) $L_{\omega\omega}[\tau_{BA}]$-formula with a property that will be defined later on, and $\bar{r}$ are elements of $B_\alpha$. At limit steps we take unions. $B_{\alpha+1}$ will be the Boolean algebra that is generated by $B_\alpha \cup \{x_\alpha\}$ in $P(\omega)$, where the $x_\alpha$ is chosen by the same forcing $P(B_\alpha)$ as in [Ba-Ko], namely: $P(B_\alpha) = \{\alpha, b_\alpha \mid a \subseteq A b_\alpha \}$.

We shall define $D_A(M, \phi(\bar{r}, x, y), c, f)$ and $M_{\alpha+1}$. Then we take a $\{D_A(M, \phi(\bar{r}, x, y), c, f) \mid c, f \in B_\alpha, (M, \phi(\bar{r}, x, y)) \in M_{\alpha+1}\}$-generic subset $\{(a_\alpha, b_\alpha) \mid n \in \omega\}$ of $P(B_\alpha)$ such that $\{(a_\alpha, b_\alpha) \mid n \in \omega\}$ additionally satisfies the properties described in [Ba-Ko] and set $x_\alpha = \bigcup\{a_\alpha \mid n \in \omega\}$. In [Ba-Ko], $M_{\alpha+1}$ is chosen so that chains and antichains are countable. Our $M_{\alpha+1}$ differs from that of [Ba-Ko], because we also want all homogeneous sets for
any $\phi(z, x, y) \in \Phi_3$ to be countable. The next items are the generalizations of the corresponding points of [Ba-Ko].

**Definition 2.1.** Let $A \subseteq \mathcal{P}(\omega)$ and $\tau, e, f \in A$. Let $\phi(\tau, x, y)$ be qf.

(i) $D_A(M, \phi(\tau, x, y), e, f) := \{(a, b)_A \in \mathcal{P}(A) \mid$ for any $u \in \langle a, b \rangle_{\mathcal{P}(\omega)}$ one of the following points is true:

1. $(u \cap e) \cup (f \setminus u) \in M$.
2. There is some $y \in M$ such that $\mathcal{P}(\omega) \models \neg \phi(\tau, (u \cap e) \cup (f \setminus u), y) \lor \neg \phi(\tau, (u \cap e) \cup (f \setminus u))$.

(ii) $M$ is called maximally homogeneous for $\phi(\tau, x, y)$ in $\mathfrak{A}$ iff $M \subseteq A$ is homogeneous for $\phi(\tau, x, y)$ and for all $a \in A \setminus M$ there is some $b \in M$ such that $\mathfrak{A} \models \neg \phi(\tau, a, b) \lor \neg \phi(\tau, b, a)$.

(iii) $\phi(\tau, x, y)$ is small in $\mathfrak{A}$ iff for any $\emptyset \neq M \subseteq A$ that is maximally homogeneous for $\phi(\tau, x, y)$ in $\mathfrak{A}$, $D_A(M, \phi(\tau, x, y), 1, 0)$ is dense in $\mathcal{P}(A)$.

**Lemma 2.2.** Let $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ be atomless, $\tau \in A^{<\omega}$, $\phi(\tau, x, y)$ qf and small in $\mathfrak{A}$, $e, f \in A$ and $\emptyset \neq M \subseteq A$ is maximally homogeneous for $\phi(\tau, x, y)$ in $\mathfrak{A}$. Then $D_A(M, \phi(\tau, x, y), e, f)$ is dense in $\mathcal{P}(A)$ for any $e, f$ in $A$.

**Proof.** [Ba-Ko, Lemmas 2.3 and 2.4].

Also the proof of the next lemma can be carried out as in [Ba-Ko]: just take a $u$ for $\mathfrak{A}$ and $\bar{M}$ in the same way as they take $x_\alpha$ for $\mathfrak{B}_\alpha$ and $\mathcal{M}_{\alpha+1}$.

**Lemma 2.3.** Let $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ be atomless and countable and let $\bar{M}$ be a countable subset of

\[
\{(M, \phi(\tau, x, y)) \mid \tau \in A^{<\omega}, \phi(\tau, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}] \text{ qf, } \phi(\tau, x, y) \text{ small in } A \text{ and } M \text{ is maximally homogeneous for } \phi(\tau, x, y) \text{ in } A\}.
\]

Then for any $(a, b)_\alpha \in \mathcal{P}(A)$ there is a $u \in \langle a, b \rangle_{\mathcal{P}(\omega)}$ such that:

1. $u \notin A$.
2. $[A \cup \{u\}]^{\mathcal{P}(\omega)}$, the subalgebra generated by $A \cup \{u\}$ in $\mathcal{P}(\omega)$, is atomless.
3. For any $(M, \phi(\tau, x, y)) \in \bar{M}$ the set $M$ is maximally homogeneous for $\phi(\tau, x, y)$ also in $[A \cup \{u\}]^{\mathcal{P}(\omega)}$.

Now using Lemma 2.3 and $\diamondsuit$, we can construct our $\mathfrak{B}$. Let $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence. Let $\langle a_\xi \mid \xi \in \omega_1 \rangle$ be an enumeration of $\mathcal{P}(\omega)$ in which each element of $\mathcal{P}(\omega)$ appears $\omega_1$ times.

In step $\alpha + 1$, let $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha \cup \{\{a_\xi \mid \xi \in S_\alpha\}, \phi(\tau, x, y)\} \mid \{a_\xi \mid \xi \in S_\alpha\}$ is a maximally homogeneous set for $\phi(\tau, x, y)$ in $\mathfrak{B}_\alpha$ and $\phi(\tau, x, y)$ is small in $\mathfrak{B}_\alpha$ and $\tau \in B_\alpha$. Apply Lemma 2.3 with $\mathfrak{A} = \mathcal{M}_\alpha$ and $\bar{M} = \mathcal{M}_{\alpha+1}$ to get an $x_\alpha$. Define $B_{\alpha+1}$ as $[B_\alpha \cup \{x_\alpha\}]^{\mathcal{P}(\omega)}$. Let $\mathfrak{B} = \bigcup \{\mathfrak{B}_\alpha \mid \alpha \in \omega_1\}$. Take the $x_\alpha$ so that $\mathfrak{B} \models \forall x (x \neq 0 \rightarrow Q_1 y y \subseteq x)$. Then it is easy to see that for any $\phi(\tau, x, y)$ which is small in every $\mathfrak{B}_\alpha$ with $\tau \in B_\alpha$, we have
3. Large homogeneous sets. The aim of this section is to define a mapping

\[
\text{big} : \bigcup_{r \in \omega} \mathcal{L}_{w_1}[\tau_{BA}](\zeta, x, y) \rightarrow \bigcup_{r \in \omega} \mathcal{L}_{w_1}[\tau_{BA}](\zeta),
\]

\[
\phi(\zeta, x, y) \mapsto \text{big}(\phi(\zeta, x, y))(\zeta),
\]

such that for every \( \phi(\zeta, x, y) \in \mathcal{L}_{w_1}[\tau_{BA}] \)

\[
(\ast) \quad \mathfrak{B} \models \forall \zeta \, (Q^2_{xy} \phi(\zeta, x, y) \iff \text{big}(\phi(\zeta, x, y))(\zeta)).
\]

Then \( \Phi_2 \) will be

\[
\{ \phi(\tau, x, y) | \text{big}(\phi(\tau, x, y))(\tau) \text{ is valid in any atomless Boolean algebra} \}.
\]

In order to simplify the notation we tacitly assume that always the variables \( x \) and \( y \) are intended to be quantified by \( Q^2_{xy} \).

Let \( \mathfrak{A} \) be any atomless Boolean algebra. Since \( \mathfrak{A} \) admits the elimination of \( \exists \) it is enough to define \( \text{big} \) for quantifierfree \( \phi(\zeta, x, y) \in \mathcal{L}_{w_1}[\tau_{BA}] \).

For any \( \tau \in \mathcal{A} \) and qf \( \phi(\tau, x, y) \) there is a qf \( \psi(\tau', x, y) \) such that \( \tau' \) is an (injective) enumeration of the atoms of the subalgebra generated by \( \tau \), and \( \mathfrak{A} \models \forall xy \, (\psi(\tau', x, y) \iff \phi(\tau, x, y)) \). Also if \( \phi(\tau, x, y) \) is a disjunction \( \lor (\phi(\tau, x, y) \land \psi_i(\tau)) \) then knowing \( \chi_i = \text{big}(\phi(\tau, x, y) \land \psi_i(\tau))(\tau) \) we can define \( \text{big}(\phi(\tau, x, y))(\tau) \) to be \( \lor \chi_i \). Hence it suffices to define \( \text{big}(\phi(\tau, x, y))(\tau) \) only for those qf \( \phi(\tau, x, y) \) that imply that \( \{ z_0, \ldots, z_{r-1} \} \) is the set of atoms in the subalgebra generated by \( \{ z_0, \ldots, z_{r-1} \} \).

If \( H \) is an uncountable homogeneous set for \( \phi(\zeta, x, y) \), then there is an \( \mathcal{L}_{w_1} \)-type \( t(\zeta, x) \) over \( \zeta \) and an uncountable \( H_1 \subseteq H \) such that every element of \( H_1 \) has the \( \mathcal{L}_{w_1} \)-type \( t(\zeta, x) \) over \( \zeta \). Hence it is enough to define \( \text{big} \) for the \( \phi(\zeta, x, y) \) with the above mentioned property and the additional property that there is an \( \mathcal{L}_{w_1} \)-type \( t(\zeta, x) \) over \( \zeta \) (independent of the assignment \( \zeta \) of \( \zeta \), because we consider only \( \zeta \) that are atoms in the subalgebra generated by \( \zeta \)) such that

\[
\mathfrak{A} \models \forall xy \, (\phi(\zeta, x, y) \iff (\phi(\zeta, x, y) \land t(\zeta, x) = \text{tp}(x/\zeta) \land t(\zeta, y) = \text{tp}(y/\zeta))).
\]

We will call such formulas special. Finally, note that any \( \mathcal{L}_{w_2} \)-type \( t(\zeta, x, y) \) over \( \zeta \) is determined by the corresponding \( r \)-tuple of the quantifierfree types of \( x \cap c_i, y \cap c_i \) in \( \{ a \in A | a \subseteq c_i \} \), \( i < r \). For any such type there are 15 possibilities, and under the condition \( \text{tp}(x/\zeta) = \text{tp}(y/\zeta) \) there remain the 9 possibilities not marked with an * in the table below.
Rigid Boolean algebra

The possibilities for the quantifierfree types of \( x \cap c_i \), \( y \cap c_i \), \( i < r \), in \( \{ a \in A \mid a \subseteq c_i \} \)

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Let \( \phi^k(z_i, x \cap z_i, y \cap z_i) \) say “the \( L_{\omega \omega} \)-type of \( x \cap c_i, y \cap c_i \) over \( c_i \) has number \( k \)”, \( k = 0, \ldots, 14 \). The disjunction \( \phi^{012}(u, v, w) := \phi^0(u, v, w) \lor \phi^1(u, v, w) \lor \phi^2(u, v, w) \) will play an important role in the following.

**Definition 3.1.** Let \( \phi(z, x, y) \in L_{\omega \omega}[rBA] \) be quantifierfree and be of the special form as described above.

\[
\text{big}(\phi(z, x, y))(z) = \exists a \subset b \forall x \left( (a \subseteq x, y \subseteq b \land \bigwedge_{i < r} ((b \setminus a) \cap z_i) \neq 0 \rightarrow \phi^012(z_i, x \cap z_i, y \cap z_i) \right) \\
\rightarrow \phi(z, x, y).
\]

Equivalent to \( \text{big}(\phi(z, x, y))(z) \) is the formula

\[
\bigvee_{I_0 \cup I_1 \cup I_2 \cup I_3 = \{0, \ldots, r-1\}, I_0 \neq 0} \forall x y \left( \bigwedge_{i \in I_0} \phi^012(z_i, x \cap z_i, y \cap z_i) \land \bigwedge_{i \in I_1} x \cap z_i = y \cap z_i \neq 0, z_i \right)
\]
\[ 1 \land \bigwedge_{i \in I_2} x \cap z_i = y \cap z_i = 0 \]
\[ 1 \land \bigwedge_{i \in I_3} x \cap z_i = y \cap z_i = z_i \rightarrow \phi(z, x, y) \]

(\cup denotes the disjoint union) which will be useful for the easy direction of (\star):

**Lemma 3.2.** Let \( A \) be an atomless Boolean algebra. Let \( A \models \forall x \neq 0 Q_1 y \subseteq x \), and \( \phi(z, x, y) \) be as above. Then \( A \models \forall r \phi(z, x, y) \rightarrow (Q_1 y \phi(x, y)) \).

**Proof.** Let \( A \models \big( \forall \phi(z, x, y) \rightarrow (Q_1 y \phi(x, y)) \big) \).

Now for \( B \) as in Section 2, we shall prove the other direction of (\star). By the construction, it would suffice to show:

(\star\star) For any enumeration \( \bar{r} \) of the atoms in the subalgebra of \( B \) generated by \( z \), if \( B \models \neg \big( \forall \phi(z, x, y) \rightarrow (Q_1 y \phi(x, y)) \big) \), then \( \phi(c, x, y) \) is small in every \( B_\alpha \) with \( r \in B_\alpha \).

Unfortunately, this is true only for \( \phi(c, x, y) \) that do not forbid certain equalities of Boolean terms. We introduce some notation and then give a sketch of our proof of the hard direction of (\star).

We say briefly “\( \phi(z, x, y) \) is valid” or just “\( \phi \)” for “\( \phi(z, x, y) \) is valid in all atomless Boolean algebras if the assignment of \( z \) is an enumeration of the atoms in the subalgebra generated by \( z \). \( \phi(z, x, y) \) is satisfiable or consistent if \( \neg \phi(z, x, y) \) is not valid.

We will define two mappings \( s \) and \( enl \) from the set of all special \( \phi(z, x, y) \) into itself. The mapping \( s \) is a technical means used to prove \( enl(enl(s(\phi))) \rightarrow enl(s(\phi)) \) (Lemma 3.7) and \( \neg \big( s(\phi) \big) \rightarrow \neg \big( enl(s(\phi)) \big) \) (Lemma 3.8). Lemma 3.9 says that (\star\star) is true for formulas of the form \( enl(s(\phi)) \) for some
special \( \phi \). Hence we get from the construction and from 3.8

\[ \forall \mathcal{B} \vdash -\mathbf{big}(s(\phi))(\mathcal{C}) \rightarrow -Q_2^2xy\ \mathbf{enl}(s(\phi))(\mathcal{C},x,y), \]

whence \( s(\phi) \rightarrow \mathbf{enl}(s(\phi)) \) and the monotonicity of the quantifier \( Q_2^2 \) imply

\[ \forall \mathcal{B} \vdash -\mathbf{big}(s(\phi))(\mathcal{C}) \rightarrow -Q_2^2xy\ s(\phi)(\mathcal{C},x,y) \]

(Theorem 3.10). Using this result we prove by induction on \( \text{card}(\mathcal{R}(\phi)) \), simultaneously for all special formulas \( \phi \).

\[ \forall \mathcal{B} \vdash -\mathbf{big}(\phi)(\mathcal{C}) \rightarrow -Q_2^2xy\ \phi(\mathcal{C},x,y), \]

which will finish the proof of (s).

In order to simplify the notation, we often suppress the free variables \((\mathbf{z},x,y)\) or \((z_i,x \cap z_i,y \cap z_i)\).

**Definition 3.3** (The mapping \( s \)). For \( R \subseteq r = \{0,1,\ldots,r-1\} \) and for \( \chi(z_i,x \cap z_i,y \cap z_i) \in L_{\omega\omega}[\mathcal{R}_A] \) we define

\[ s_R(\chi(z_i,x \cap z_i,y \cap z_i)) := \begin{cases} \chi(z_i,x \cap z_i,y \cap z_i) & \text{if } i \notin R \text{ or } \\
012(z_i,x \cap z_i,y \cap z_i) \rightarrow \chi(z_i,x \cap z_i,y \cap z_i) & \text{is valid; } \\
\chi(z_i,x \cap z_i,y \cap z_i) \land x \cap z_i \neq y \cap z_i & \text{else. } \\
\end{cases} \]

Let \( S = \{ \bigwedge_{i \in r} \chi_{w,i}(z_i,x \cap z_i,y \cap z_i) \mid w \in W \} \) be a finite set such that for all \( w \in W \) the conjunction \( \bigwedge_{i \in r} \chi_{w,i}(z_i,x \cap z_i,y \cap z_i) \) is satisfiable and \( \bigwedge_{i \in r} \chi_{w,i}(z_i,x \cap z_i,y \cap z_i) \rightarrow \phi(\mathbf{z},x,y) \) is valid, and such that for any satisfiable conjunction \( \delta = \bigwedge_{i \in r} \chi'_i(z_i,x \cap z_i,y \cap z_i) \) such that \( \delta \rightarrow \phi(\mathbf{z},x,y) \) is valid there is a \( w \in W \) with \( \bigwedge_{i \in r} \chi'_i(z_i,x \cap z_i,y \cap z_i) \rightarrow \bigwedge_{i \in r} \chi_{w,i}(z_i,x \cap z_i,y \cap z_i) \). We will call such a set \( S \) a set of representatives for \( \phi \). Given such a set, let \( R = R(\phi) \) and define

\[ s(\phi(\mathbf{z},x,y)) = \bigvee_{w \in W \cap \mathcal{R}} \bigwedge_{i \in r} s_R(\chi_{w,i}(z_i,x \cap z_i,y \cap z_i)). \]

If \( \models \exists xy \phi(\mathbf{z},x,y) \), then let \( s(\phi(\mathbf{z},x,y)) \) be any inconsistent formula.

A brief reflection shows that \( s(\phi) \) is well defined up to logical equivalence: Let \( S' = \{ \bigwedge_{i \in r} \chi'_\mathbf{w,i}(z_i,x \cap z_i,y \cap z_i) \mid w' \in W' \} \) be another set of representatives for \( \phi \).

For \( \bigvee_{w' \in W'} \bigwedge_{i \in r} s_R(\chi'_\mathbf{w,i}) \rightarrow \bigvee_{w \in W} \bigwedge_{i \in r} s_R(\chi_{\mathbf{w,i}}) \), it suffices to show that for each \( w' \in W' \) there is some \( w \in W \) such that \( \bigwedge_{i \in r} s_R(\chi'_\mathbf{w,i}) \rightarrow \bigwedge_{i \in r} s_R(\chi_{\mathbf{w,i}}) \). Let \( w' \in W' \) be given. Since \( S \) is a set of representatives for \( \phi \) there is a \( w \in W \) such that \( \bigwedge_{i \in r} \chi'_{\mathbf{w,i}} \rightarrow \bigwedge_{i \in r} \chi_{\mathbf{w,i}} \), which is equivalent to
\(X_{w,i} \rightarrow \chi_{w,i} \text{ for } i < r. \) Immediately from the definition of \(s_R, \) if \(X_{w,i} \rightarrow \chi_{w,i}, \) then \(s_R(X_{w,i}) \rightarrow s_R(\chi_{w,i}). \) Hence \( \bigwedge_{i < r} s_R(X_{w,i}) \rightarrow \bigwedge_{i < r} s_R(\chi_{w,i}). \)

The other direction follows by symmetry.

**Remark.** \(s(\phi)\) may be unsatisfiable, e.g. for \(\phi = (x \cap z_0 = y \cap z_0 \land x \cap z_1 \subset y \cap z_1) \lor (x \cap z_0 \subset y \cap z_0 \land x \cap z_1 = y \cap z_1) \land \bigwedge_{i=0,1} x \cap z_i \neq z_i, 0 \land \bigwedge_{i=0,1} y \cap z_i \neq z_i, 0 \land z_0 \cap z_1 = 0 \lor z_0 \cup z_1 = 1.\)

**Definition 3.4 (The mapping \(enl.\)**) For \(\chi(z_i, x \cap z_i, y \cap z_i) \in \mathcal{L}_{\omega \omega}[\tau_{BA}]\) we define

\[
\begin{align*}
enl(\chi(z_i, x \cap z_i, y \cap z_i)) := \\
\begin{cases}
\chi(z_i, x \cap z_i, y \cap z_i) \\
\lor (x \cap z_i = (-y) \cap z_i \land \exists x \chi(z_i, x \cap z_i, y \cap z_i)) \\
\lor \exists y \chi(z_i, x \cap z_i, y \cap z_i) \\
&\text{if } \phi^{012}(z_i, x \cap z_i, y \cap z_i) \rightarrow \chi(z_i, x \cap z_i, y \cap z_i) \\
&\text{is not valid;}
\end{cases}
\end{align*}
\]

\(\) otherwise.

Let \(\{\bigwedge_{i < r} X_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W\}\) be a set of representatives for \(\phi.\) Then set

\[
enl(\phi^R(z, x, y)) = \bigvee_{w \in W, i < r} \bigwedge \enl(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i)).
\]

If \(\vDash \exists x y \phi(\bar{z}, x, y),\) then let \(\enl(\phi^R(z, x, y))\) be any inconsistent formula.

From the fact that \(X_{w,i} \rightarrow \chi_{w,i}\) implies \(\enl(X_{w,i}) \rightarrow \enl(\chi_{w,i}),\) we conclude by an analogous consideration as above that \(\enl(\phi)\) is well-defined.

In order to apply Lemmas 2.2 and 2.3 we may replace \(\enl(\phi^R(z, x, y))\) by an equivalent (with respect to the theory of atomless Boolean algebras) \(qf\) formula.

The next two lemmas collect some properties of \(s\) and \(enl\) that will be useful in the proofs of 3.7 and of 3.8.

**Lemma 3.5.** Let \(\chi_s(z_i, x \cap z_i, y \cap z_i), s = 0, 1,\) be \(qf\) and \(R \subseteq r.\)

(i) \(\enl(\chi_0) \lor \enl(\chi_1) \rightarrow \enl(\chi_0 \lor \chi_1).\)

(ii) \(s_R(0) \lor s_R(1) \rightarrow s_R(0 \lor 1).\)

For (iii), (iv) and (v), assume additionally that \(\chi_s(z_i, x \cap z_i, y \cap z_i), s = 0, 1,\) determine the same 1-type \(l(z_i, x \cap z_i)\) of \(x \cap z_i\) over \(z_i\) and of \(y \cap z_i\) over \(z_i.)\)
(iii) Assume that, for $s = 0, 1$, if not $\phi^{012}(z_i, x \cap z_i, y \cap z_i) \rightarrow \chi_s(z_i, x \cap z_i, y \cap z_i)$, then $\chi_s(z_i, x \cap z_i, y \cap z_i) \rightarrow x \cap z_i \neq y \cap z_i$. Then $(\text{enl}(\chi_0) \land \text{enl}(\chi_1)) \rightarrow \text{enl}(\chi_0 \land \chi_1)$.

(iv) $(s_R(\chi_0) \land s_R(\chi_1)) \rightarrow s_R(\chi_0 \land \chi_1)$.

(v) Assume that $\chi_s \rightarrow x \cap z_i = y \cap z_i$ for $s = 0, 1$ if $i \notin R$. Then for any $i < r$ the formula

$$(\text{enl}(s_R(\chi_0))(z_i, x \cap z_i, y \cap z_i) \land \text{enl}(s_R(\chi_1))(z_i, x \cap z_i, y \cap z_i))$$

$$(\text{enl}(s_R(\chi_0 \land \chi_1)))(z_i, x \cap z_i, y \cap z_i)$$

is valid.

Proof. (i), (ii) $\chi_s \rightarrow \chi_0 \lor \chi_1$ implies $\text{enl}(\chi_s) \rightarrow \text{enl}(\chi_0 \lor \chi_1)$ and $s_R(\chi_s) \rightarrow s_R(\chi_0 \lor \chi_1)$.

(iii) Define

$\phi_\pm(z_i, x \cap z_i, y \cap z_i) := x \cap z_i = y \cap z_i \land t(z_i, x \cap z_i)$ and

$\phi_\mp(z_i, x \cap z_i, y \cap z_i) := x \cap z_i = (y) \cap t(z_i, x \cap z_i) \land t(z_i, y \cap z_i)$.

Case 1: $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $\text{enl}(\chi_0) \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_\pm \lor \phi_\mp) \land (\chi_1 \lor \phi_\pm \lor \phi_\mp) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_\pm \lor \phi_\mp = \text{enl}(\chi_0 \land \chi_1)$.

Case 2: Not $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $\text{enl}(\chi_0 \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_\pm \lor \phi_\mp) \land (\chi_1 \lor \phi_\pm \lor \phi_\mp) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_\pm \lor \phi_\mp = \text{enl}(\chi_0 \land \chi_1)$.

Case 3: $\phi^{012} \rightarrow \chi_0$ and not $\phi^{012} \rightarrow \chi_1$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $\text{enl}(\chi_0 \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_\pm \lor \phi_\mp) \land (\chi_1 \lor \phi_\pm \lor \phi_\mp) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_\pm \lor \phi_\mp = \text{enl}(\chi_0 \land \chi_1)$.

Since by the assumption of (iii), $\phi_\pm \land \chi_1$ is not satisfiable, the latter formula is equivalent to $(\chi_0 \land \chi_1) \lor \phi_\mp = \text{enl}(\chi_0 \land \chi_1)$.

(iv) Assume $i \in R$, otherwise $s_R$ does not change $\chi_0 \land \chi_1 \land \chi_0 \lor \chi_1$.

Case 1: $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $s_R(\chi_0) \land s_R(\chi_1) = \chi_0 \land \chi_1 = s_R(\chi_0 \land \chi_1)$.

Case 2: E.g. not $\phi^{012} \rightarrow \chi_0$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $s_R(\chi_0) \land s_R(\chi_1) = (\chi_0 \land \chi_1) \land (\chi_0 \land \chi_1) = s_R(\chi_0 \land \chi_1)$.

(v) For $i \in R$, the assumptions for (iii) are true for $\psi_s = s_R(\chi_s)$. Hence by (iii) and (iv),

$$(\text{enl}(s_R(\chi_0))(z_i, x \cap z_i, y \cap z_i) \land \text{enl}(s_R(\chi_1))(z_i, x \cap z_i, y \cap z_i))$$

$$\rightarrow \text{enl}(s_R(\chi_0 \land \chi_1))(z_i, x \cap z_i, y \cap z_i)$$

For $i \notin R$, we have $\chi_s \rightarrow x \cap z_i = y \cap z_i$ for $s = 0, 1$ and hence $\text{enl}(s_R(\chi_0)) \land \text{enl}(s_R(\chi_1)) = (\chi_0 \lor \phi_\pm) \land (\chi_1 \lor \phi_\pm) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_\pm = \text{enl}(s_R(\chi_0 \land \chi_1))$.

Lemma 3.6. Let $\phi$ be special and satisfiable, $R = R(\phi)$, and let $\{\Lambda_{i < r} \chi_{w,i} \mid w \in W\}$ be a set of representatives for $\phi$.

(i) For any $\Lambda_{i < r} \chi_{r} \rightarrow \bigvee_{w \in W} \Lambda_{i < r} s_R(\chi_{w,i})$, there is a $w \in W$ such that $\Lambda_{i < r} \chi_{r} \rightarrow \Lambda_{i < r} s_R(\chi_{w,i})$. 
(ii) \( \text{enl}(s(\phi)) \leftrightarrow \bigvee_{w \in W} \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})). \)

(iii) For any \( \bigwedge_{i < r} \chi_i, \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})), \) there is a \( w \in W \) such that \( \bigwedge_{i < r} \chi_i^n \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})). \)

**Proof.** We will first prove (iii). Then the proof of (i) which is similar but easier will be clear. Let \( \bigwedge_{i < r} \chi_i, x \cap z_i, y \cap z_i \) be consistent, otherwise one can take any \( w \in W. \)

For \( i < r \) there is an \( n_i, 0 < n_i < 15, \) and there are \( \hat{\chi}_{i,0}, \ldots, \hat{\chi}_{i,n_i-1} \in \{ \phi^0, \ldots, \phi^{14} \} \) such that

\[
\bigwedge_{i < r} \chi_i(z_i, x \cap z_i, y \cap z_i) \leftrightarrow \bigwedge_{i < r} (\hat{\chi}_{i,0} \lor \ldots \lor \hat{\chi}_{i,n_i-1})(z_i, x \cap z_i, y \cap z_i).
\]

We will show the claim by induction on \( \prod_{i < r} n_i. \)

**Case** \( \prod_{i < r} n_i = 1. \) Take an atomless Boolean algebra \( \mathfrak{A} \) and \( c \in A \) such that \( c \) is an enumeration of all the atoms in the generated subalgebra. Take \( a, b \in A \) such that \( \mathfrak{A} \models \bigwedge_{i < r} \chi_i(c, a \cap c_i, b \cap c_i). \) Then there is some \( w \in W \) with \( \mathfrak{A} \models \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i}(c, a \cap c_i, b \cap c_i))). \) Since \( \bigwedge_{i < r} \chi_i(z_i, x \cap z_i, y \cap z_i) \) defines an \( L_{\omega \omega} \)-2-type of \( (x, y) \) over \( z, \) we have \( \bigwedge_{i < r} \chi_i(z_i, x \cap z_i, y \cap z_i) \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i))). \)

**Induction step.** We consider the step from \( \prod_{i < r} n_i \) to \((n_0 + 1) \times \prod_{i < r} n_i \), the other cases are similar.

\[
\bigwedge_{0 < i < r} \chi_i(z_i, x \cap z_i, y \cap z_i) \leftrightarrow \bigwedge_{0 < i < r} (\hat{\chi}_{0,0} \lor \ldots \lor \hat{\chi}_{0,n_0})(z_i, x \cap z_i, y \cap z_i).
\]

By induction hypothesis there are \( w', w'' \in W \) such that

\[
\bigwedge_{0 < i < r} \chi_i \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w',i})),
\]

\[
(\hat{\chi}_{0,0} \lor \ldots \lor \hat{\chi}_{0,n_0}) \rightarrow \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w'',i})).
\]

Thus we have

\[
\left( \bigwedge_{0 < i < r} \chi_i \right) \lor \left( (\hat{\chi}_{0,0} \lor \ldots \lor \hat{\chi}_{0,n_0}) \land \bigwedge_{0 < i < r} \chi_i \right) \rightarrow
\]

\[
\left( \text{enl}(s_R(\chi_{w',0})) \lor \text{enl}(s_R(\chi_{w'',0}))) \land \bigwedge_{0 < i < r} \left( \text{enl}(s_R(\chi_{w',i})) \land \text{enl}(s_R(\chi_{w'',i}))) \right). \]

Note that in the last conjunction we get “and” and not only “or”, because

\[
\bigwedge_{0 < i < r} \chi_i \rightarrow \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w',i})) \land \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w'',i})),
\]

as the situation below any \( z_i \) is independent of the situation below the other \( z_j. \)
From 3.5(i), (ii) and (v) we get
\[
(\tilde{x}_{0,0} \land \bigwedge_{0<i<r} \chi'_i) \lor (\tilde{x}_{0,1} \lor \ldots \lor \tilde{x}_{0,n_0}) \land \bigwedge_{0<i<r} \chi'_i
\]
\[\to \text{enl}(s_R(\chi_{w',0} \lor \chi_{w'})) \land \bigwedge_{0<i<r} \text{enl}(s_R(\chi_{w',i} \land \chi_{w'})) .
\]

Since \(\{\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W\} \) is a set of representatives for \(\phi(z, x, y)\) and since \(w', w'' \in W\), we have \((\chi_{w',0} \lor \chi_{w'}(w) \land \bigwedge_{0<i<r} \chi_{w',i} \lor \chi_{w''}) \rightarrow \phi\) and there is a \(w \in W\) such that
\[
(\chi_{w',0} \lor \chi_{w'}(w)) \land \bigwedge_{0<i<r} (\chi_{w',i} \lor \chi_{w''}) \rightarrow \bigwedge_{i<r} \chi_{w,i}.
\]

For such a \(w\) we have
\[
\text{enl}(s_R(\chi_{w',0} \lor \chi_{w'}(w))) \land \bigwedge_{0<i<r} \text{enl}(s_R(\chi_{w',i} \lor \chi_{w'})) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i})),
\]
and thus the induction step is complete and (iii) is shown.

(ii) Assume \(s(\phi)\) is satisfiable, otherwise both sides are not satisfiable.

Let \(S = \{\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W\}\) be a set of representatives for \(\phi\), and \(S' = \{\bigwedge_{i<r} \chi'_{w,i}(w') \mid w' \in W'\} \) be a set of representatives for \(s(\phi) = \bigvee_{w \in W} \bigwedge_{i<r} s_R(\chi_{w,i})\) such that \(W' \supseteq \hat{W} := \{w \in W \mid \bigwedge_{i<r} s_R(\chi_{w,i})\) is satisfiable\) and \(\chi'_{w,i} = s_R(\chi_{w,i})\) for \(w \in \hat{W}\).

By definition, \(\text{enl}(\bigwedge_{i<r} \phi) = \bigvee_{w' \in W'} \bigwedge_{i<r} \text{enl}(\chi'_{w,i})\). By (i), for any \(w' \in W'\) there is some \(w \in W\) such that \(\bigwedge_{i<r} \chi'_{w,i} \rightarrow \bigwedge_{i<r} s_R(\chi_{w,i})\) and hence \(\bigwedge_{i<r} \text{enl}(\chi'_{w,i}) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). Thus \(\text{enl}(s(\phi)) \rightarrow \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). The other direction follows immediately from the choice of \(S'\) and the definition of \(\text{enl}\).

**Lemma 3.7.** Let \(\phi\) be a special formula. Then \(\text{enl}(\text{enl}(s(\phi))) \leftrightarrow \text{enl}(s(\phi))\).

**Proof.** Assume \(s(\phi)\) is satisfiable, otherwise both sides are not satisfiable.

Let \(S, W\) be as above and \(S'' = \{\bigwedge_{i<r} \chi''_{w',i}(w') \mid w'' \in W''\}\) be a set of representatives for \(\text{enl}(s(\phi))\). By definition, \(\text{enl}(\text{enl}(s(\phi))) = \bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi''_{w',i})\). For \(w'' \in W''\) we have \(\bigwedge_{i<r} \chi''_{w',i} \rightarrow \text{enl}(s(\phi))\), hence by 3.6(ii), \(\bigwedge_{i<r} \chi''_{w',i} \rightarrow \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). By 3.6(iii) there is some \(w \in W\) such that \(\bigwedge_{i<r} \chi''_{w',i} \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\), whence \(\bigwedge_{i<r} \text{enl}(\chi''_{w',i}) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). It is easy to check that for \(\phi(z_i, x \cap z_i, y \cap z_i)\) by definition
\[
\text{enl}(\text{enl}(\chi(z_i, x \cap z_i, y \cap z_i))) \rightarrow \text{enl}(\chi(z_i, x \cap z_i, y \cap z_i)).
\]

Therefore \(\bigwedge_{i<r} \text{enl}(\chi''_{w',i}) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\), and putting things together yields \(\bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi''_{w',i}) \rightarrow \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\), and, by 3.6(ii), \(\bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi''_{w',i}) \rightarrow \text{enl}(s(\phi))\).
The other direction is obvious.

**Lemma 3.8.** \( \neg \text{big}(s(\phi)) \rightarrow \neg \text{big}(\text{enl}(s(\phi))) \) is valid for special \( \phi \).

**Proof.** Let \( \mathfrak{A} \) be any atomless Boolean algebra. Assume \( \mathfrak{A} \models \text{big}(\text{enl}(s(\phi(x, y))))(c) \). We show that \( \mathfrak{A} \models \text{big}(s(\phi(x, y)))(c) \). Since the 1-types of \( x \) and of \( y \) over \( c \) are determined by \( \mathfrak{A} \models \exists y \text{enl}(s(\phi(c, x, y))) \) and \( \mathfrak{A} \models \exists x \text{enl}(s(\phi(c, x, y))) \), there is just one pair \((I_2, I_3)\) such that

\[
\mathfrak{A} \models \bigvee_{(I_0, I_1) \mid I_0 \cup I_1 \cup I_2 \cup I_3 = \{0, \ldots, r-1\}, I_0 \neq 0} \left( \left( \bigwedge_{i \in I_0} \phi_{i_12}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \right) \land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \right) \rightarrow \text{enl}(s(\phi(c, x, y))) \right).
\]

Take \( I_0 \subseteq \) maximal such that

\[
\mathfrak{A} \models \forall xy \left( \left( \bigwedge_{i \in I_0} \phi_{i_12}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \right) \land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \right) \rightarrow \text{enl}(s(\phi(c, x, y))) \right).
\]

Let \( R = R(\phi) \) and \( \{\bigwedge_{i < r} \chi_{w, i} \mid w \in W\} \) be a set of representatives for \( \phi \). By 3.6(ii) and (iii) there is a \( w \in W \) such that

\[
\mathfrak{A} \models \forall xy \left( \left( \bigwedge_{i \in I_0} \phi_{i_12}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \right) \land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \right) \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w, i}(c_i, x \cap c_i, y \cap c_i))).
\]

We claim that also

\[
\mathfrak{A} \models \forall xy \left( \left( \bigwedge_{i \in I_0} \phi_{i_12}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \right) \land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \right) \rightarrow \bigwedge_{i < r} s_R(\chi_{w, i}(c_i, x \cap c_i, y \cap c_i))).
\]

Indeed, by the definition of \( \text{enl} \) we have for any \( s_R(\chi_{w, i}(z_i, x \cap z_i, y \cap z_i)) \):

For \( i \in I_0 \), if \( \phi_{i_12} \rightarrow \text{enl}(s_R(\chi_{w, i})) \), then \( \phi_{i_12} \rightarrow s_R(\chi_{w, i}) \). For \( i \in I_2 \), if \( x \cap z_i = y \cap z_i = 0 \rightarrow \text{enl}(s_R(\chi_{w, i})) \), then \( x \cap z_i = y \cap z_i = 0 \rightarrow s_R(\chi_{w, i}) \).
For \( i \in I_3 \), if \( x \cap z_i = y \cap z_i = z_i \rightarrow \text{enl}(s_R(\chi_{w,i})) \), then \( x \cap z_i = y \cap z_i = z_i \rightarrow s_R(\chi_{w,i}) \).

For \( i \in I_4 \) the formula \( x \cap z_i = y \cap z_i \neq 0, z_i \wedge \text{enl}(s_R(\chi_{w,i})) \wedge \neg s_R(\chi_{w,i}) \) is consistent only if \( \phi^{012} \rightarrow s_R(\chi_{w,i}) \). But then we could take \( I'_0 := I_0 \cup \{i\} \) and \( I'_1 = I_1 \setminus \{i\} \) and replace \((I_0, I_1)\) by \((I'_0, I'_1)\), which contradicts the maximality of \( I_0 \).

Now we are ready to prove (**) for special formulas of the form \( s(\phi) \).

**Lemma 3.9.** Let \( \phi \) be special and \( \bar{c} \in B \) be an \( r \)-tuple that consists of atoms in the generated subalgebra.

(i) If \( \neg \text{big}(\phi) \) and \( \text{enl}(\phi) \rightarrow \phi \) are valid, then for any \( \alpha \) with \( \bar{c} \in B_\alpha \) the relation \( \phi(\bar{c}, x, y) \) is small in \( B_\alpha \).

(ii) If \( \neg \text{big}(s(\phi)) \) is valid, then for any \( \alpha \) with \( \bar{c} \in B_\alpha \) the relation \( \text{enl}(s(\phi(\bar{c}, x, y))) \) is small in \( B_\alpha \).

**Proof.** (i) Let \( \mathfrak{B} := \neg \text{big}(\phi(\bar{z}, x, y))(\bar{c}) \) and \( \bar{c} \in B_\alpha \) be atoms in the generated subalgebra. Set \( B_\alpha := \mathfrak{A} \), and let \( M \neq \emptyset \) be a maximally homogeneous set for \( \phi(\bar{c}, x, y) \) in \( \mathfrak{A} \), and \( (a, b)_A \in \mathcal{P}(\mathfrak{A}) \), i.e. \( (a, b)_A \) is an interval in \( \mathfrak{A} \). Take \((a', b')_A \leq (a, b)_A\) such that there is just one \( i \in r \), say \( i_0 \), with \((b' \setminus a') \subseteq c_i \) and \( c_i \cap a' \neq 0 \) and \( b' \cap c_i \neq c_i \). We assume \( \mathfrak{B} \) (and also \( \mathfrak{A} \) and \( \mathcal{P}(\omega) \)) satisfy

\[
\forall x \in (a', b')_A \exists y \phi(\bar{c}, x, y) \wedge \exists y \phi(\bar{z}, y, x)(\bar{c}),
\]

for otherwise \((a', b')_A \in \mathcal{D}_A(M, \phi(\bar{c}, x, y), 1, 0)\).

Since \( \mathfrak{B} \models \neg \text{big}(\phi(\bar{c})) \), we have \((a', b')_A \cap M \neq (a', b')_A\). We fix a \( d \in (a', b')_A \setminus M \) and an \( m \in M \) such that \( \mathfrak{A} \models \neg \phi(\bar{c}, d, m) \vee \neg \phi(\bar{c}, m, d) \), say \( \mathfrak{A} \models \neg \phi(\bar{c}, d, m) \), and show that there is an \((a'', b'')_A \leq (a', b')_A\) such that for any \( x \in (a'', b'')_\mathcal{P}(\omega) \) we have \( x \in M \) or \( \mathcal{P}(\omega) \models \neg \phi(\bar{c}, x, m) \).

Then (i) will be proved, because such an \((a'', b'')_A\) is in \( \mathcal{D}_A(M, \phi(\bar{c}, x, y), 1, 0)\). Fix a set \( \{\chi_{w,i} \mid w \in W\} \) of representatives for \( \phi \).

**Claim.** \( d \cap c_{i_0} \neq c_{i_0} \setminus m \).

**Proof.** \( \phi(\bar{z}, x, y) = \bigvee_{w \in W} \bigwedge_{i < r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \), w.l.o.g. \( W = \{0, 1, \ldots, s - 1\} \). Hence \( \mathfrak{A} \models \bigwedge_{w \in W} \bigvee_{i < r} \neg \chi_{w,i}(c_i, d \cap c_i, m \cap c_i) \), say for \( w = 0, 1, \ldots, s^t - 1 \)

\[
\mathfrak{A} \models \bigwedge_{i < r, i \neq i_0} \neg \chi_{w,i}(c_i, d \cap c_i, m \cap c_i),
\]

and for \( w = s^t \), \( s^t + 1, \ldots, s - 1 \)

\[
\mathfrak{A} \models \bigwedge_{i < r, i \neq i_0} \chi_{w,i}(c_i, d \cap c_i, m \cap c_i) \wedge \neg \chi_{w,i_0}(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0}).
\]
We may assume $s > 0$ and $s' \leq s-1$, because otherwise $(a', b')_A \in D_A(M, \phi(c, x, y), 1, 0)$. Since
\[
\mathfrak{A} \models \forall x y \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \chi_{w, i}(c_i, x \cap c_i, y \cap c_i)
\right.
\left. \land \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \rightarrow \phi(c, y, x) \right),
\]
we have
\[
\mathfrak{A} \models \forall x y \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \chi_{w, i}(c_i, x \cap c_i, y \cap c_i)
\right.
\left. \land \left( \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \lor (x \cap c_{i_0} = (y) \cap c_{i_0})
\right.
\left. \land \exists x \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \land \exists y \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})) \right)
\right.
\left. \rightarrow \text{enl}(\phi(c, x, y)) \right).
\]
By the assumptions on $\phi(c, x, y)$ and on $\mathfrak{A}$ there is just one 1-type of $x \cap c_{i_0}$ over $c_{i_0}$ consistent with $\phi(c, x, y)$ such that for every $w \in W$ the formula $\exists y \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$ is implied by this type. The same holds for the 1-type of $y \cap c_{i_0}$ over $c_{i_0}$, which coincides with the 1-type of $x \cap c_{i_0}$ over $c_{i_0}$, and the formula $\exists x \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$. Since $m \cap c_{i_0}$ and $d \cap c_{i_0}$ have this 1-type, we get
\[
\mathfrak{A} \models \exists x \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0})
\left. \land \exists y \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, d \cap c_{i_0}, y \cap c_{i_0}) \right).
\]
Note that $\mathfrak{A} \models \neg \phi(c, d, m)$ and $\phi$ is equivalent to $\text{enl}(\phi)$. Therefore $d \cap c_{i_0} \neq c_{i_0} \setminus m$ and the claim is proved.

We now give $(a'', b'')_A$ case by case.

**Case 1**: $d \cap c_{i_0} \neq m \cap c_{i_0}$. Then
\[
\mathfrak{A} \models \bigvee_{i=0, 1, 2, 4, 8} \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0}).
\]
Assume that $\mathfrak{A} \models \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0})$.

If $i = 0$ or $i = 2$, take an $e'$ such that $0 \subset e' \subset c_{i_0} \cap m \cap (-d)$, and $(a'', b'')_A = (d, b' \setminus e')_A$. If $i = 1$ or $i = 8$, take $(a'', b'')_A = (a', d)_A$. Finally, if $i = 4$, take $(a'', b'')_A = (d, b')_A$. 

Then, in each subcase, for any \( x \in (a''', b''')_P(\omega) \) we have
\[
P(\omega) \models tp(x, m/\hat{c}) = tp(d, m/\hat{c}) \text{ and hence } P(\omega) \models \neg \phi(\hat{c}, x, m).
\]

Case 2: \( d \cap c_{i_0} = m \cap c_{i_0} \).

Subcase 2.1:
\[
\mathfrak{A} \models \exists x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \land \neg \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).
\]

Since \( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \) determines the \( L_\omega \)-1-type of \( y \cap c_{i_0} \) over \( c_{i_0} \), and \( m \) has the same one, we have
\[
\mathfrak{A} \models \exists x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \land \neg \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).
\]

There is an example \( d' \) for \( x \) with \( d' \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A \), because \( m \cap c_{i_0} = d \cap c_{i_0} \in (d' \cap c_{i_0}, b' \cap c_{i_0})_A \) and hence within the given 1-type of \( x \cap c_{i_0} \) over \( c_{i_0} \) the formula \( \phi(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \) can be realized with some \( x \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A \) for \( i = 0, 1, 2 \). We can argue with \( (d' \cap c_{i_0}) \cup (d \cap c_{i_0}) \) as with \( d \) in case 1 for \( i = 0, 1, 2 \).

Subcase 2.2:
\[
\mathfrak{A} \models \forall x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \rightarrow \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).
\]

Again we have
\[
\mathfrak{A} \models \forall x \left( \left( \bigwedge_{s' \leq w < s} \chi_{w,i}(c_{i}, x \cap c_{i}, y \cap c_{i}) \right) \land \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \rightarrow \phi(\hat{c}, x, y) \right).
\]

Since
\[
\phi^{012}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \rightarrow \bigvee_{s' \leq w < s} \chi_{w,i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}),
\]

by the definition of enl we have
\[
\forall x \left( \left( \bigwedge_{i < r, i \neq i_0} \text{enl} \left( \bigwedge_{s' \leq w < s} \chi_{w,i}(z_{i}, x \cap z_{i}, y \cap z_{i}) \right) \right) \land \left( \bigvee_{s' \leq w < s} \chi_{w,i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \right) \lor \left( x \cap z_{i_0} = y \cap z_{i_0} \right) \right) \land \exists x \left( \bigvee_{s' \leq w < s} \chi_{w,i_0}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \right) \land \exists y \left( \bigvee_{s' \leq w < s} \chi_{w,i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \right) \rightarrow \text{enl}(\phi(\hat{z}, x, y)).
\]
In $\mathfrak{A}$ we get
\[ \mathfrak{A} \models \forall xy \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \text{enl}(\chi_{w,i}(c_i, x \cap c_i, y \cap c_i)) \right) \]
\[ \land \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \lor \left( x \cap c_{i_0} = y \cap c_{i_0} \right) \right) \]
\[ \land \exists x \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right) \]
\[ \land \exists y \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right) \rightarrow \text{enl}(\phi(c, x, y)). \]

As in the first subcase, we get
\[ \mathfrak{A} \models \exists x \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \right) \land \exists y \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, d \cap c_{i_0}, y \cap c_{i_0}) \land d \cap c_{i_0} = m \cap c_{i_0} \right). \]

Putting things together yields $\mathfrak{A} \models \text{enl}(\phi(c, d, m))$ and hence $\mathfrak{A} \models \phi(c, d, m)$, a contradiction to the choice of $d$ and $m$.

(ii) By 3.8, $\neg \text{big}(s(\phi)) \rightarrow \neg \text{big}(\text{enl}(s(\phi)))$, and, by 3.7, $\text{enl}(\text{enl}(s(\phi))) \rightarrow \text{enl}(s(\phi))$ is valid. Therefore (ii) follows from (i) applied to $\text{enl}(s(\phi))$.

Lemma 3.9, the construction and the monotonicity of $Q_1$ yield:

Theorem 3.10. For any special $\phi$,
\[ \mathfrak{B} \models \forall z \left( \left( \text{"}z \text{ are the atoms in the generated subalgebra"} \land \neg \text{big}(s(\phi))(\hat{z}) \right) \rightarrow \neg Q_1^2 xy s(\phi(c, x, y)) \right). \]

Finally, we show how to get Theorem 3.10 for $\phi$ instead of $s(\phi)$.

Theorem 3.11. For any special $\phi$
\[ \mathfrak{B} \models \forall z \left( \left( \text{"}z \text{ are the atoms in the generated subalgebra"} \land \neg \text{big}(\phi)(\hat{z}) \right) \rightarrow \neg Q_1^2 xy \phi(c, x, y) \right). \]

Proof (by induction on $\text{card}(R(\phi))$). If $R(\phi) = \emptyset$, then $\phi(c, x, y) \rightarrow x = y$, and hence $\mathfrak{B} \models \neg Q_1^2 xy \phi(c, x, y)$.

Now assume $\mathfrak{B} \models \forall z \left( \left( \text{"}z \text{ are the atoms in the generated subalgebra"} \land \neg \text{big}(\psi)(\hat{z}) \right) \rightarrow \neg Q_1^2 xy \psi(c, x, y) \right)$ for all $\psi$ with $R(\psi) \subset R(\phi)$. We show $\mathfrak{B} \models Q_1^2 xy \phi(c, x, y) \rightarrow \text{big}(\phi)(\hat{c})$ for any $r$-tuple $\hat{c}$ that consists of atoms in the generated subalgebra. Assume $\mathfrak{B} \models Q_1^2 xy \phi(c, x, y)$ and let $H$ be an uncountable homogeneous set for $\phi(c, x, y)$ in $\mathfrak{B}$. By recursion on $i \leq r$ we define uncountable subsets $H^{(i)}$, $0 \leq i \leq r$. 

Set $H^{(0)} := H$. Assume $H^{(i)}$ is defined. We distinguish two cases:

**Case 1:** \( \{ x \cap c_i \mid x \in H^{(i)} \} \) is uncountable. Then take $H^{(i+1)} \subseteq H^{(i)}$ such that $H^{(i+1)}$ is uncountable and for any $x, y \in H^{(i+1)}$, if $x \neq y$, then $x \cap c_i \neq y \cap c_i$.

**Case 2:** \( \{ x \cap c_i \mid x \in H^{(i)} \} \) is countable. Then there is some $x \in H^{(i)}$ such that \( \{ y \in H^{(i)} \mid x \cap c_i = y \cap c_i \} \) is uncountable. Let $H^{(i+1)}$ be such a set.

For $i \notin R$, \( \{ x \cap c_i \mid x \in H^{(i)} \} \) is a singleton, and we are in case 2. Now consider $H^{(0)}, H^{(1)}, \ldots, H^{(r)}$. If for all $i \in R$ case 1 is true, then $H^{(r)}$ shows $\mathcal{B} \models Q^2x y (\phi(c, x, y))$. By 3.10, $\mathcal{B} \models \text{big}(s(\phi(c)))$. Since $s(\phi) \rightarrow \phi$, $\mathcal{B} \models \text{big}(\phi(c))$.

If there is some $i \in R$ with case 2 being true, fix such an $i$. Then $H^{(i+1)}$ shows $\mathcal{B} \models Q^2x y (\phi \land x \cap z_i = y \cap z_i)(c, x, y)$. Take $\psi = \phi \land x \cap z_i = y \cap z_i$. Then $\psi$ is also special. Since $\psi \rightarrow \phi$ and $i \notin R(\phi) \setminus R(\psi)$, we have $R(\psi) \subseteq R(\phi)$. By induction hypothesis, we conclude from $\mathcal{B} \models Q^2x y (\phi \land x \cap z_i = y \cap z_i)(c, x, y)$ that $\mathcal{B} \models \text{big}(\psi(c))$ and hence $\mathcal{B} \models \text{big}(\phi(c))$.

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**References**


MATHEMATISCHES INSTITUT
UNIVERSITAT BONN
BERINGSTR. 4
D-5300 BONN 1, GERMANY

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