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Two-to-one maps on solenoids and Knaster continua

by

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Abstract. It is shown that 2-to-1 maps cannot be defined on certain solenoids, in particular on the dyadic solenoid, and on Knaster continua.

There are many results in which the non-existence of exactly 2-to-1 (continuous) maps defined on a given space (with values in Hausdorff spaces) is shown, for instance, the non-existence of exactly 2-to-1 maps on Euclidean n-cubes (Harrold (1939) for $n = 1$, Civin (1943) for $n \leq 3$ and Chernavskiĭ (1962) for arbitrary n). In the paper by Mioduszewski (1961) a method was developed of detecting 2-to-1 maps defined on some singular spaces and it was proved that such a map cannot be defined on the Knaster simplest indecomposable continuum. Applying this method, we shall show that 2-to-1 maps cannot be defined on certain solenoids, in particular on the dyadic solenoid, and on arbitrary Knaster continua. We find this search interesting in view of the yet unsolved problem raised by I. Rosenholtz (1974) concerning the existence of 2-to-1 maps defined on indecomposable chainable continua to which the Knaster continua belong. An example of 2-to-1 map defined on a chainable decomposable continuum was given by J. Heath (1989).

The search for a 2-to-1 map f defined on a space X reduces to the search for the involution ϕ associated with f defined by $f^{-1}(f(x)) = \{x, \phi(x)\}$. Although this involution is not necessarily continuous, it is semicontinuous in the sense that if x_n converges to x, the accumulation points of $\phi(x_n)$ lie in $\{x, \phi(x)\}$. If X enjoys a kind of regular structure, the search for ϕ can be reduced to the search for a certain continuous involution on X associated with ϕ.

We show that every non-identity continuous involution on a solenoid is homotopic either to the involution $y = a \cdot x^{-1}$ for some element a of the solenoid (the notation refers to the group structure), or to the involution

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$y = (-1) \cdot x$, where -1 is the solution of the equation $x^2 = 1$ different from 1 (if such a solution exists).

1. Continuous involutions on solenoids. A *solenoid* is the limit of the inverse sequence

$$S^1 \xleftarrow{z^{i_1}} S^1 \xleftarrow{z^{i_2}} S^1 \longleftarrow \ldots$$

where $i_1, i_2, \ldots$ is a sequence of natural numbers, and S^1 is the circle $|z| = 1$ on the complex plane. The *dyadic solenoid*, obtained by taking $i_1 = i_2 = \ldots = 2$, is particularly well-known. Solenoids are topological groups with multiplication generated by multiplication on S^1; for details see e.g. Eilenberg and Steenrod (1952), Chapter VIII, Exercises E. The equation $x^2 = 1$ on S has at most one solution different from 1 since the coordinates of each solution are 1 or -1. If infinitely many of $i_1, i_2, \ldots$ are even, then there is only the solution 1; in the opposite case there are two solutions, and the one which is different from 1 is denoted by -1. In the first case the homomorphism $y = x^2$ is 1-to-1 (with kernel $\{1\}$) and it is a homeomorphism, and in the second case it is exactly 2-to-1 (with kernel $\{-1, 1\}$).

LEMMA 1. *Any involution on a solenoid S is homotopic either to the map $y = x$, or to $y = a \cdot x^{-1}$, where a is an element of S, or to $y = (-1) \cdot x$.*

P r o o f. Let $g : S \to S$ be an involution. The map $a^{-1} \cdot g$, where $a = g(1)$, takes 1 into 1, and hence it is homotopic to a unique homomorphism h of S; this is a consequence of a much more general theorem of Scheffer (1972) stating that a continuous map of a locally compact connected commutative topological group into itself taking 1 into 1 is homotopic to a unique homomorphism.

Now, g and $a \cdot h$ are homotopic. Therefore, the map $a \cdot h(a) \cdot h \circ h = (a \cdot h) \circ (a \cdot h)$ is homotopic to the identity. Its value at 1, $a \cdot h(a)$, can be joined to 1 with an arc. Hence, $h \circ h$ is homotopic to $a \cdot h(a) \cdot h \circ h$, and, in consequence, to the identity. But, by the result of Scheffer quoted above, homotopic homomorphisms are equal. Thus, h is an involution.

Let us consider the map dual to h in the sense of Pontryagin duality (see Pontryagin 1966, Chapter VI). It is an involution on the subgroup of the rationals generated by the fractions $1/1, 1/i_1, 1/(i_1 \cdot i_2), 1/(i_1 \cdot i_2 \cdot i_3), \ldots$ (see Hewitt and Ross, 1963, (25.3)). Each homomorphism of that subgroup into itself is multiplication by a rational. If h is an involution, it is multiplication by a square root of 1, i.e. by 1 or -1. It follows that h is either the identity or the map $y = x^{-1}$.

The involution g is thus homotopic either to $y = a \cdot x$ or to $y = a \cdot x^{-1}$, where $a = g(1)$.

In the first case the map $y = a^2 \cdot x$, being the composition of $y = a \cdot x$ with itself, is homotopic to the identity. This means that a^2, the value of that map at 1, lies on the arc component of 1, which implies that a lies either on the arc component of 1 or on that of -1. It follows that g is homotopic either to $y = x$ or to $y = (-1) \cdot x$.

This finishes the proof.

A homomorphism multiplied by a constant will be called an *affine map*.

R e m a r k. We have $g(a^2 \cdot b^{-1}) = (g(a))^2 \cdot (g(b))^{-1}$ for each affine map g.

Two maps $g, g' : X \to X$ are called *conjugate* if $g \circ h = h \circ g'$ for some homeomorphism $h : X \to X$.

To each arc component of S we assign an *orientation*. It is given by a 1-to-1 map of the reals onto the arc component such that in the composition of that map with the projection of the solenoid S onto the first factor of the inverse sequence, $S \to S^1$, the argument of the value is an increasing function on the reals.

LEMMA 2. *Homotopic involutions on a solenoid S are conjugate if one of them is an affine map.*

P r o o f. Let $g, g' : S \to S$ be homotopic involutions and let g be affine.

The map $k(x) = (g(x))^{-1} \cdot g'(x)$ is homotopic to the constant map 1. It follows that the image of k is a subcontinuum contained in the arc component of 1, and hence in an arc passing through 1. This arc is the one-to-one image under $y = x^2$ of an arc lying in the arc component of 1. The formula $(\psi(x))^2 = k(x)$ defines a continuous map ψ with values in the arc component of 1. It follows that ψ is homotopic to the constant 1.

Let $h(x) = g(x) \cdot \psi(x)$. Then h is homotopic to g. Since g is a homeomorphism, h maps different arc components into different arc components.

We have $(h(x))^2 = (g(x))^2 \cdot (\psi(x))^2 = (g(x))^2 \cdot k(x) = g(x) \cdot g'(x)$. The map h is one-to-one on arc components. To see this note that its square has this property. Indeed, g and g' are homotopic homeomorphisms, since $g' = k \circ g$, where $k : S \to S$ has values in an arc contained in the arc component of 1. Therefore, on each arc component, g and g' are either both orientation-preserving or both orientation-reversing. Hence, $h^2 = g \cdot g'$ is monotone on each arc component. In consequence, h is one-to-one on S. This means that h is a homeomorphism.

We have $(g(h(x)))^2 = (g(h(x)))^2 \cdot (g(g(x)))^{-1} \cdot g(g(x)) = g((h(x))^2 \cdot (g(x))^{-1}) \cdot x = g(g'(x)) \cdot x$ (as g is affine; see the Remark above) and $(h(g'(x)))^2 = g(g'(x)) \cdot g'(g'(x)) = g(g'(x)) \cdot x$. Hence, $(g(h(x)))^2 = (h(g'(x)))^2$. It follows that the map $g(h(x)) \cdot (h(g'(x)))^{-1}$ is equal to the constant map 1 or -1. But the maps $\alpha(x) = g(h(x))$ and $\beta(x) = h(g'(x))$ are homotopic since g, g' and h are homotopic. Therefore, the map $g(h(x)) \cdot (h(g'(x)))^{-1}$ is homotopic to the constant map 1, hence it cannot be equal to -1. In consequence, $g(h(x)) = h(g'(x))$. So, g and g' are conjugate.

Corollary. *An involution on a solenoid S which is not the identity is conjugate either to $y = x^{-1}$, or to $y = (-1) \cdot x$.*

Proof. Let g be an involution on S. By Lemma 1, it is homotopic either to $y = x$, to $y = a \cdot x^{-1}$, or to $y = (-1) \cdot x$. The three maps are affine involutions. By Lemma 2, g is conjugate to one of them. The first map is excluded by the assumption that g is not the identity. If g is conjugate to $y = a \cdot x^{-1}$, then it is conjugate to $y = x^{-1}$. Indeed, $y = a \cdot x^{-1}$ is conjugate to $y = x^{-1}$ by means of the homeomorphism $y = b \cdot x$, where b is a point of S such that $b^2 = a^{-1}$. This completes the proof.

2. Semicontinuous involutions on solenoids. Let f be an exactly 2-to-1 map from a solenoid S onto a Hausdorff space $f(S)$. We associate with f the involution ϕ defined by $f^{-1}(f(x)) = \{x, \phi(x)\}$. The involution ϕ is not necessarily continuous. However, as was shown by Mioduszewski (1961), this involution, when restricted to any arc of S, has at most one discontinuity point and becomes continuous if we change the value $\phi(x)$ to x at this point (in fact, this is true not only for solenoids but for spaces which are locally bundles of arcs).

Let $\bar{\phi}$ be the modification of ϕ obtained by taking $\bar{\phi}(x) = x$ for x being a discontinuity point of ϕ restricted to an arc having x in its interior. Then $\bar{\phi}$ is also an involution, and it is continuous when restricted to arcs. The set of fixed points of $\bar{\phi}$ (i.e. the set of points at which ϕ and $\bar{\phi}$ differ) coincides with the set of discontinuity points of ϕ restricted to any arc with the considered point in its interior.

In particular, $\bar{\phi}$ maps arcs homeomorphically onto arcs. If such an arc and its image meet, their union is an arc on which $\bar{\phi}$ is an involution with a unique fixed point.

A value y of f is said to be a *value of openness* if for each x such that $f(x) = y$ the value y lies in the interior of the image of each neighborhood of x.

Theorem 1. *The involution $\bar{\phi}$ is continuous on S.*

Proof. Let $x \in S$. We show that $\bar{\phi}$ is continuous at x.

Consider the case $\bar{\phi}(x) \neq x$. Suppose $\bar{\phi}$ is not continuous at x. From the Baire theorem it follows that the set of values of openness of f is a dense G_δ-subset of the image. Since f is exactly 2-to-1 (more generally, for f of finite constant multiplicity) the set of values of openness is open. Then the preimage of any sufficiently small neighborhood of a value of openness is the union of two open sets, each mapped homeomorphically by f onto that neighborhood. In consequence, $\bar{\phi}$ is continuous on each of these two open sets and maps each of them onto the other.

Let U be an open non-empty subset of S on which ϕ is continuous. The arc component of x, being dense in S, meets U. There is at most one fixed point on that arc component. There are two arcs which are disjoint except for having the common endpoint x and which both reach U. Hence, a point a such that $\bar{\phi}(a) = a$ can only appear in one of these two arcs.

Let L be an arc with x in its geometric interior, meeting U and such that $\bar{\phi}(a) \neq a$ for a in L. Let K be an arc in the geometric interior of L also having these properties. Since x is by assumption a discontinuity point of $\bar{\phi}$, there exists a sequence x_n converging to x such that $\bar{\phi}(x_n)$ converges to a point different from $\bar{\phi}(x)$. By the continuity of f, $\bar{\phi}(x_n)$ converges to the point in $f^{-1}(f(x))$ different from $\bar{\phi}(x)$, i.e. to x. All but finitely many points of the sequence x_n lie outside L, as $\bar{\phi}$ is continuous when restricted to L.

Let L_n be a sequence of arcs with topological limit L and such that x_n lies in the geometric interior of L_n. Take arcs K_n in such a way that each K_n lies in the geometric interior of L_n, x_n lies in the geometric interior of K_n and the sequence K_n converges topologically to K.

A point a such that $\bar{\phi}(a) = a$ can only appear on finitely many of the arcs K_n. Namely, if $\bar{\phi}(a) = a$ for a in K_n, then at one of the ends of L_n, f assumes the same value as at some other point of L_n. In other words, f sticks an end of L_n to another point of L_n. There are points of K_n lying between these two points: a is one of them. Suppose that there exists infinitely many arcs L_n having these properties. Then f has the same values at one of the ends of L and at some point of L such that the segment of L determined by these points meets K. Thus, these would be two different points of L at which f has the same value. It follows that $\bar{\phi}$ has a fixed point on L. But this is impossible as $\bar{\phi}$ is fixed point free on L by assumption.

So, we can assume that $\bar{\phi}(a) \neq a$ for a in all arcs K_n.

The set $f^{-1}(f(K_n))$ then consists of two disjoint arcs, namely K_n and $K'_n = \phi(K_n)$. By continuity of f, the set of accumulation points of the sequence of arcs K'_n is contained in $f^{-1}(f(K))$, which is the union of two disjoint arcs K and $K' = \bar{\phi}(K)$. The points $\bar{\phi}(x_n)$ lie on K'_n and converge to a point of K, namely to x. Hence, the set of accumulation points of K'_n is contained in K. In consequence, $\bar{\phi}$ is discontinuous everywhere on K. But $\bar{\phi}$ is continuous at some points of K, namely those in U. A contradiction.

Consider now the case $\bar{\phi}(x) = x$. Suppose that $\bar{\phi}$ is discontinuous at x. This means that there exists a sequence x_n converging to x such that $\bar{\phi}(x_n)$ converges to the point of $f^{-1}(f(x))$ different from $\bar{\phi}(x) = x$, i.e. to $\phi(x)$.

Since $\bar{\phi}(x) = x$, there exists an arc K having x in its geometric interior and such that $\bar{\phi}(K) = K$. Then x is a unique fixed point of $\bar{\phi}$ on K. This is a consequence of the fact that the fixed points of $\bar{\phi}$ coincide with the discontinuity points of ϕ restricted to arcs, and that there is at most one such point on any arc.

All but finitely many x_n lie outside K, since $\bar{\phi}$ restricted to K is continuous.

Let K_n be arcs topologically convergent to K and such that x_n lies in the geometric interior of K_n.

The set of accumulation points of the arcs $K_n' = \bar{\phi}(K_n)$ is contained in $f^{-1}(f(K)) = K \cup \{\phi(x)\}$. But $\bar{\phi}(x_n) \in K_n'$ converges to $\phi(x) \notin K$. Since K_n' has points arbitrarily close to $\phi(x)$ if n is sufficiently large the set of accumulation points of K_n' is a single point $\phi(x)$ (which is isolated in the set $f^{-1}(f(K))$). In consequence, all points of K are discontinuity points of $\bar{\phi}$. This contradicts the fact that $\bar{\phi}$ is continuous at each $x' \in K$ for which $\bar{\phi}(x') \neq x'$, i.e. at all the points different from x, as was shown in the previous case.

COROLLARY 1. *Let S be the solenoid defined by a sequence $i_1, i_2, \ldots$ with i_n even for infinitely many n. Then there do not exist* 2*-to-*1 *continuous maps defined on S.*

Proof. Suppose that f is an exactly 2-to-1 map on S. By Theorem 1, the involution $\bar{\phi}$ induced by f is continuous, and, by the Corollary from Section 1, it is conjugate to the map $y = x^{-1}$, as in that case the solution -1 of the equation $x^2 = 1$ does not exist. In consequence, $\bar{\phi}$ has exactly one fixed point, as does $y = x^{-1}$. We get a contradiction, since f is 2-to-1 when restricted to the set of fixed points of $\bar{\phi}$.

COROLLARY 2. *There do not exist exactly* 2*-to-*1 *maps defined on the dyadic solenoid.*

By the *Knaster continuum related to a solenoid S* we mean the quotient space obtained from S after identification of all pairs x, x^{-1}. By the *endpoints* of that continuum we mean the images of points of S such that $x = x^{-1}$, i.e. the images of x such that $x^2 = 1$. There are two endpoints in the case when all but finitely many i_n are odd and one endpoint otherwise.

Remark. If all but finitely many i_n are odd then $y = x^2$ is exactly 2-to-1 on S. Another exactly 2-to-1 map on S is obtained by identifying all pairs x, x^{-1} and identifying the endpoints in the resulting Knaster continuum.

COROLLARY 3. *Let S be a solenoid such that in the sequence $i_1, i_2, \ldots$ defining S all but finitely many i_n are odd. If f is an exactly* 2*-to-*1 *map on S, then $f(S)$ is homeomorphic either to S or to the Knaster continuum related to S with its endpoints identified.*

Proof. By Theorem 1, the involution $\bar{\phi}$ induced by f is continuous, and by the Corollary from Section 1, it is conjugate either to $y = x^{-1}$ or to $y = (-1) \cdot x$.

In the first case $f(S)$ is homeomorphic to the space obtained from S by identifying all pairs x, x^{-1} and the points 1 and -1. In consequence, it is homeomorphic to the Knaster continuum with its endpoints identified.

In the second case $f(S)$ is homeomorphic to the space obtained from S by identifying all pairs x, $(-1)\cdot x$ by means of the map $y=x^2$ which maps S onto S. Thus, the image is homeomorphic to S.

3. Two-to-one maps on Knaster continua. Let K be the Knaster continuum related to the solenoid S given by a sequence $i_1, i_2, \ldots$ Then K is the limit of the inverse sequence

$$J \xleftarrow{T_{i_1}} J \xleftarrow{T_{i_2}} J \longleftarrow \ldots$$

where $J=[-1,1]$ and T_{i_n} are the Chebyshev polynomials serving as a standard model for open maps from J onto J of multiplicity i_n (the T_k are determined by the equation $\operatorname{Re} z^k = T_k(\operatorname{Re} z)$, and are given by $T_k(x) = \cos(k\cdot \arccos(x))$).

Note that for any two arcs containing a given endpoint of K, one of them is contained in the other, and that only endpoints of K have this property.

Let f be an exactly 2-to-1 map from K onto a Hausdorff space $f(K)$. Let ϕ be the involution associated with f by $f^{-1}(f(x)) = \{x, \phi(x)\}$. As for the case of solenoids, ϕ is not necessarily continuous, but when restricted to any arc of K, it has at most one discontinuity point and becomes continuous on that arc if we pass to the involution $\overline{\phi}$ by changing $\phi(x)$ to x at discontinuity points. The set of fixed points of $\overline{\phi}$ (i.e. the set of points at which $\overline{\phi}$ and ϕ differ) coincides with the set of discontinuity points of the involution ϕ restricted to any arc with the considered point in its interior.

In particular, $\overline{\phi}$ maps any arc homeomorphically onto an arc. If such an arc and its image meet, their sum is an arc on which $\overline{\phi}$ is an involution with a unique fixed point.

THEOREM 2. *The involution $\overline{\phi}$ is continuous on K.*

Proof. Let $x \in K$. If x lies in an arc component containing no endpoint of K, then the argument for the continuity of $\overline{\phi}$ at x is the same as for the solenoid S. So, we can assume that x lies on the arc component containing an endpoint p of K.

The involution $\overline{\phi}$ has no fixed point on the arc component of x.

To prove this, suppose that a is such a fixed point. Then a is not an endpoint of K, since otherwise points of the arc component of x would be fixed points of $\overline{\phi}$. Let L be an arc with $\overline{\phi}(p)$ in its geometric interior; such an arc exists as $\overline{\phi}(p)$ is in the arc component of x and is different from p. Thus, $\overline{\phi}(L)$ is an arc with an endpoint of K in its geometric interior, which is impossible.

If x is not an endpoint of K then there exists an arc L with x in its geometric interior and meeting the open set of continuity points of $\bar{\phi}$. The involution $\bar{\phi}$ is fixed point free on L. Further argument for the continuity of $\bar{\phi}$ at x is the same as that given for the solenoid S.

It remains to consider the case when x is an endpoint of K.

Once again, suppose that $\bar{\phi}$ is discontinuous at x. Thus there exists a sequence x_n converging to x such that $\bar{\phi}(x_n)$ converges to a point different from $\bar{\phi}(x)$. By continuity of f, $\bar{\phi}(x_n)$ converges to the point in $f^{-1}(f(x))$ different from $\bar{\phi}(x)$, i.e. to x; recall that x is not a fixed point of $\bar{\phi}$ as $\bar{\phi}$ has no fixed point on the arc component of x. Let L be an arc having x as an end, and let a be the other end of L. As was shown in the preceding cases, $\bar{\phi}$ is continuous at non-endpoints of K; in particular, it is continuous at a. The point $\bar{\phi}(a)$ lies outside L as otherwise $\bar{\phi}$ will have a fixed point on L (between a and $\bar{\phi}(a)$). Let U be a neighborhood of a such that x does not belong to the closure of U, $\bar{\phi}(U)$ is disjoint from U, and the closure of $\bar{\phi}(U)$ is disjoint from L; the existence of such a U is a consequence of the continuity of $\bar{\phi}$ at a.

For $x_n \notin U$, let L_n be the arc component of x_n in $K \setminus U$. We can assume that all x_n lie outside L as $\bar{\phi}$ is continuous on L. Each L_n is an arc with ends on the boundary of U. In consequence, $\bar{\phi}(L_n)$ is in arc with ends in the closure of $\bar{\phi}(U)$. But $\bar{\phi}(x_n)$ converges to x, which is an endpoint of K. Therefore, the arc $\bar{\phi}(L_n)$ joining $\bar{\phi}(x_n)$ to the closure of $\bar{\phi}(U)$ meets U for sufficiently large n.

To see this, note that the component of x in $K \setminus U$ is disjoint from the closure of $\bar{\phi}(U)$. The same is true therefore for the components of points lying sufficiently close to x, in particular for x_n with sufficiently large n.

Thus, L_n meets $\bar{\phi}(U)$ for those n. But the set of accumulation points of L_n is contained in L, and we get a contradiction, as L is disjoint from the closure of $\bar{\phi}(U)$.

Corollary. *There do not exist* 2-*to*-1 *continuous maps defined on the Knaster continua.*

Proof. Suppose that f is an exactly 2-to-1 continuous map on a Knaster continuum K. By Theorem 2, the involution $\bar{\phi}$ induced by f is continuous. Fugate and McLean have shown (1981) that the set of fixed points of a continuous involution on a tree-like continuum is connected. Hence, the set of fixed points of $\bar{\phi}$ is a subcontinuum of K. It cannot be an arc, as $\bar{\phi}$ restricted to any arc has at most one fixed point (as noted at the beginning of Section 1). Thus, $\bar{\phi}$ has exactly one fixed point. We get a contradiction as f is 2-to-1 when restricted to the set of fixed points of $\bar{\phi}$.

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