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Size levels for arcs

by

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Abstract. We determine the size levels for any function on the hyperspace of an arc as follows. Assume Z is a continuum and consider the following three conditions: 1) Z is a planar AR; 2) cut points of Z have component number two; 3) any true cyclic element of Z contains at most two cut points of Z. Then any size level for an arc satisfies 1)–3) and conversely, if Z satisfies 1)–3), then Z is a diameter level for some arc.

1. Introduction. Let X be a continuum and let $C(X)$ denote the hyperspace of subcontinua of X [N_1, p. 1]. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0, +\infty)$ such that $\mu(\{x\}) = 0$ and if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$. Much work has been done in the study of Whitney levels (point inverses) of Whitney maps. More generally, let us call a continuous function $\sigma : C(X) \to [0, +\infty)$ a *size map* provided that $\sigma(\{x\}) = 0$ and, if $A \subset B$, $\sigma(A) \leq \sigma(B)$. For example, the diameter map is a size map which is not in general a Whitney map. Point inverses of size maps are called *size levels*. Whitney levels are continua [EN, p. 1032] and, by the same proof as in [EN, p. 1032], size levels are continua. Whitney levels for arcs are arcs (or degenerate). In this paper we shall determine the size levels for any size function on the hyperspace of an arc as follows:

THEOREM. *Assume Z is a continuum and consider the following three conditions*:

1.1. *Z is a planar AR.*

1.2. *Cut points of Z have component number two.*

1.3. *Any true cyclic element of Z contains at most two cut points of Z.*

Then any size level for an arc satisfies 1.1–1.3 *and conversely*, *if Z satisfies* 1.1–1.3, *then Z is a diameter level for some arc.*

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We prove this theorem in Sections 3–6. In Section 7, we give an example of a Peano continuum which has a diameter level which is not locally connected. This is in contrast to the case of Whitney levels which, for Peano continua, must again be Peano continua [N_2, Th. 3].

2. Some general terminology and notation. Most of our terminology and notation is standard or will be explained later. We note here the following general definitions.

A *Peano continuum* is a locally connected continuum (*continuum* meaning nonempty, compact, connected metric space). The notion of a cyclic element may be found in [K] or [W].

If S is a connected space and $p \in S$, then the *component number of* p (*in* S) is the cardinality of the set of all components of $S - \{p\}$. We say p is a *cut point of* S provided that $S - \{p\}$ is not connected, i.e., the component number of p in S is ≥ 2.

The symbol $\overline{A}$ denotes the closure of A. If $H \subset X$, then $\mathrm{Bd}(H)$ denotes the (topological) boundary of H in X, i.e., $\mathrm{Bd}(H) = \overline{H} \cap \overline{X - H}$.

An arc A in X with end points p and q is called a *free arc* (*in* X) provided that $A - \{p, q\}$ is open in X.

We shall let $|A|$ denote the cardinality of a set A.

3. Proof of necessity of 1.1–1.3. Let I be the unit interval with any metric d. Then $C(I)$ is a 2-cell (see [N_1]). The geometric representation we use for $C(I)$ is the set Γ given by ($I^2 = I \times I$)

$$\Gamma = \{(x, y) \in I^2 : y \geq x\}$$

where (x, y) represents the subarc $[x, y]$ of I when $x \neq y$ and (x, x) represents the set $\{x\}$. A point in a size level L_t will be thought of interchangeably as a subcontinuum of I or as a point in Γ.

We define a map π: $C(I) \to D$, where D is the diagonal of I^2, by

$$\pi(x, y) = ((x + y)/2, (x + y)/2)\,.$$

Also, for each $z \in I$, let

$$F_z = \{(x, y) \in \Gamma : (x + y)/2 = z\} = \pi^{-1}(z, z)\,.$$

Let $y_t = \min\{y \in I : (0, y) \in L_t\}$ and $x_t = \max\{x \in I : (x, 1) \in L_t\}$. Let $\pi_t = \pi | L_t$.

3.1. Box Lemma. *Let* $z \in [0, 1]$. *If* $p, q \in F_z \cap L_t$ *where* $p = (x, y)$ *and* $q = (x', y')$, *then* $[x, x'] \times [y', y] \subset L_t$.

Proof. Let $r \in [x, x']$ and $s \in [y', y]$. Then we see immediately that

$$[x', y'] \subset [r, s] \subset [x, y]\,.$$

Thus, since $p, q \in L_t$, clearly $[r, s] \in L_t$. This proves 3.1.

3.2. Lemma. *The map* $\pi_t : L_t \to D$ *is monotone* (*but not necessarily onto* D). *In fact*, *for any* $(z,z) \in \pi_t(L_t)$, $\pi_t^{-1}(z,z)$ *is a one-point set or an arc.*

Proof. By 3.1, $\pi_t^{-1}(z,z)$ is a connected subset of $F_z \cap L_t$.

3.3. Lemma. *For each* $(x,y) \in L_t$, $y_t \le y$ *and* $x \le x_t$.

Proof. If $y \le y_t$, then $[x,y] \subset [0,y] \subset [0,y_t]$. Thus, since $[x,y]$, $[0,y_t] \in L_t$, $[0,y] \in L_t$. Hence, $y = y_t$. If $x_t \le x$, then $[x,y] \subset [x,1] \subset [x_t,1]$. Thus, since $[x,y]$, $[x_t,1] \in L_t$, $[x,1] \in L_t$. Hence, $x_t = x$.

3.4. Lemma. *Each of the points* $(0,y_t)$ *and* $(x_t,1)$ *are noncut points of* L_t.

Proof. Note that $\pi_t(L_t)$ is the arc in D with end points $\pi_t(0,y_t)$ and $\pi_t(x_t,1)$. Also note that

$$\pi_t^{-1}(\pi_t(0,y_t)) = \{(0,y_t)\} \quad \text{and} \quad \pi_t^{-1}(x_t,1) = \{(x_t,1)\}\,.$$

Hence, the lemma follows easily from 3.2 and from 2.2 of [W, p. 138].

3.5. Lemma. *A point* $p = (x,y)$ *is a cut point of* L_t *if and only if* $p \ne (0,y_t)$, $p \ne (x_t,1)$, *and* $\pi_t^{-1}(\pi_t(p)) = \{p\}$.

Proof. Assume $p \ne (0,y_t)$, $p \ne (x_t,1)$, and $\pi_t^{-1}(\pi_t(p)) = \{p\}$. Then the line Q of slope -1 in Γ through $\pi(p)$ separates $(0,y_t)$ and $(x_t,1)$ in Γ by 3.3. Thus, since

$$Q \cap L_t = \pi_t^{-1}(\pi_t(p)) = \{p\}\,,$$

we see that p is a cut point of L_t. To prove the other half of the lemma, assume p is a cut point of L_t. By 3.4, $p \ne (0,y_t)$ and $p \ne (x_t,1)$. Let $\pi_t(L_t) = A$. Recall that A is an arc with end points $\pi(0,y_t)$ and $\pi(x_t,1)$. Thus, since $p \ne (0,y_t)$ and $p \ne (x_t,1)$, $A - \{\pi_t(p)\}$ has exactly two components A_1 and A_2. By 3.2 (and by 2.2 of [W, p. 138]), $\pi_t^{-1}(A_1)$ and $\pi_t^{-1}(A_2)$ are connected. By the Box Lemma 3.1, any point in $\pi_t^{-1}(\pi_t(p)) - \{p\}$ would be a limit point of both $\pi_t^{-1}(A_1)$ and $\pi_t^{-1}(A_2)$. Hence, if

$$\pi_t^{-1}(\pi_t(p)) - \{p\} \ne \emptyset\,,$$

then $L_t - \{p\}$ would be connected, in contradiction to our assumption that p is a cut point of L_t. Therefore, $\pi_t^{-1}(\pi_t(p)) = \{p\}$. This completes the proof of 3.5.

3.6. Lemma. *Each* L_t *is a retract of* $C(I)$.

Proof. Let $A = \{(x,y) \in \Gamma :$ the straight line $Q(x,y)$ in Γ through (x,y) of slope -1 intersects $L_t\}$. For any $(x,y) \in A$, let $h(x,y)$ denote the highest point of $Q(x,y) \cap L_t$ (i.e., the point of $Q(x,y) \cap L_t$ with largest y-coordinate), and let $l(x,y)$ denote the lowest point of $Q(x,y) \cap L_t$ (i.e., the point of $Q(x,y) \cap L_t$ with smallest y-coordinate). It is easy using the Box Lemma to see that if $\{(x_i,y_i)\}_{i=1}^{\infty}$ is a sequence in A converging to $(x,y) \in A$, then

$\{h(x_i, y_i)\}_{i=1}^{\infty}$ converges to $h(x, y)$ and $\{l(x_i, y_i)\}_{i=1}^{\infty}$ converges to $l(x, y)$. Thus, h and l are continuous on A. We define a retraction r from Γ onto L_t as follows. Fix $(x, y) \in \Gamma$, and let Q denote the straight line in Γ through (x, y) of slope -1. If $(x, y) \in L_t$, let $r(x, y) = (x, y)$. If $(x, y) \notin L_t$ and $Q \cap L_t \neq \emptyset$, then we see by 3.2 that (1) $y >$ second coordinate of $h(x, y)$ or (2) $y <$ second coordinate of $l(x, y)$; if (1) holds, let $r(x, y) = h(x, y)$, and if (2) holds, let $r(x, y) = l(x, y)$. Finally, if $Q \cap L_t = \emptyset$, let $r(x, y) = (0, y_t)$ if $y \leq -x + y_t$ and let $r(x, y) = (x_t, 1)$ if $y \geq -x + x_t + 1$. Thus, we have defined a function r on all of $C(I)$ to L_t. The continuity of r follows from the continuity of h and l on A. Therefore, r is our desired retraction from Γ onto L_t.

3.7. Lemma. *Cut points of L_t have component number two.*

Proof. This lemma follows immediately from 3.2 and 3.5 by using 2.2 of [W, p. 138].

3.8. Lemma. *Assume H is a true cyclic element of L_t, and assume p and q are cut points of L_t such that $p, q \in H$ and $p \neq q$. Let J be the arc in D with end points $\pi_t(p)$ and $\pi_t(q)$ (note $\pi_t(p) \neq \pi_t(q)$ by 3.5). Then $H = \pi_t^{-1}(J)$.*

Proof. Since H is connected and $p, q \in H$, clearly $\pi_t(H) \supset J$. Suppose there exists $r \in H$ such that $\pi_t(r) \notin J$. Then one of $\pi_t(p)$ or $\pi_t(q)$, say $\pi_t(q)$, separates the other point from $\pi_t(r)$ in D. Hence, by 3.5, q separates p from r in L_t. Therefore, q is a cut point of H, in contradiction to H being a cyclic element. Hence, $\pi_t(H) \subset J$. This completes the proof of 3.8.

3.9. Lemma. *Each true cyclic element of L_t contains at most two cut points of L_t.*

Proof. The lemma follows immediately from 3.8.

By 3.6, 3.7 and 3.9, we have proved the necessity of 1.1–1.3.

4. Preliminary lemmas about cyclic elements

4.1. Lemma. *Assume Z satisfies 1.1 and 1.2. If v is the vertex of a simple triod K in Z, then v is a point of a true cyclic element of Z.*

Proof. Using 1.2 in the case when v is a cut point of Z, we see that there is an arc in $Z - \{v\}$ irreducible between points of two different legs of K. Hence, using 1.1, we see that v is a point of a two-cell and, therefore, v belongs to a true cyclic element of Z.

4.2. Lemma. *Assume Z satisfies 1.1 and 1.2. If p and q are distinct cut points of Z, then either (1) p and q are end points of a free arc, (2) p and q*

are in the same true cyclic element of Z, *or* (3) *there is a point* w *of a true cyclic element of* Z *such that* w *separates* p *and* q *in* Z.

Proof. Let A be an arc in Z from p to q. Assume (1) is false. Then, by 4.1, there is a true cyclic element T of Z such that

$$T \cap (A - \{p, q\}) \neq \emptyset .$$

Assume (2) is false. Then at least one of p or q, say q, is not an element of T. Hence, there is a point w such that

$$w \in \mathrm{Bd}(T) \cap (A - \{p, q\}) .$$

Now, suppose (3) is false, i.e., suppose p and q are in the same component of $Z - \{w\}$. Then there is an arc B from p to q such that $w \notin B$. It is easy to see that there is an arc $E \subset A \cup B$ such $E \cap T$ is not connected. This contradicts 3.4 and 3.5 of [W, p. 69].

4.3. Lemma. *Assume* Z *satisfies* 1.1 *and* 1.2 *and that* p *and* q *are distinct cut points of* Z *such that no point of any true cyclic element of* Z *separates* p *and* q *in* Z. *If* A *is an arc in* Z *from* p *to* q *then either* A *is contained in a true cyclic element of* Z *or* A *is a free arc in* Z.

Proof. Assume A is not a free arc in Z. Then (1) of 4.2 is false since if there were a free arc F in Z from p to q, $A \cup F$ would be a simple closed curve containing a free arc and, therefore, $A \cup F$ would be a retract of Z which would contradict 1.1. Thus, (2) of 4.2 must hold. Hence, by 3.4 and 3.5 of [W, p. 69], A is contained in a true cyclic element of Z.

4.4. Lemma. *Assume* Z *is a Peano continuum satisfying* 1.3. *If* T *is a true cyclic element of* Z, *then each point of* $\mathrm{Bd}(T)$ *is a cut point of* Z (*thus*, $|\mathrm{Bd}(T)| \leq 2$).

Proof. Suppose $p \in \mathrm{Bd}(T)$ such that p is a noncut point of Z. Then, by the local arcwise connectedness of Z and 1.3, there is an arc A from a point $x \in Z - T$ to a point $q \in T$ (near p) so that $A \cap T = \{q\}$ and q is a noncut point of Z. Now, there is an arc B in $Z - \{q\}$ from x to a point r in T. Clearly, $A \cup B$ contains an arc E such that $|E \cap T| = 2$. This contradicts 3.4 and 3.5 of [W, p. 69]. This proves 4.4.

4.5. Lemma. *Assume* Z *is nondegenerate and satisfies* 1.1–1.3. *Then there exist distinct points* p_l *and* p_r *of* Z *such that every cut point separates* p_l *and* p_r.

Proof. Assume there is a cut point of Z. Then, by 8.2 of [W, p. 77], Z has at least two nodes N_l and N_r. Let $p_l \in N_l$ and $p_r \in N_r$ such that p_l and p_r are noncut points of Z. Suppose there is a cut point c of Z such that c does not separate p_l and p_r in Z. Note that $c \notin N_l \cup N_r$ since c is a cut point of Z and cannot be the boundary point of N_l or N_r. Then there is an arc A in $Z - \{c\}$ from p_l to p_r. Since p_l and p_r are either end points of Z or non-boundary points of nodal sets, $p_l \notin N_r$ and $p_r \notin N_l$. Thus,

$$\mathrm{Bd}(N_l) \cap A \neq \emptyset \neq \mathrm{Bd}(N_r) \cap A\,. \tag{1}$$

Hence

$$\mathrm{Bd}(N_l) \cap A = \mathrm{Bd}(N_l) \quad \text{and} \quad \mathrm{Bd}(N_r) \cap A = \mathrm{Bd}(N_r)\,. \tag{2}$$

Since $c \notin N_l \cup N_r$, there is an arc B from c to a point w of A such that $A \cap B = \{w\}$. By (2),

$$(B - \{w\}) \cap (N_l \cup N_r) = \emptyset\,.$$

If $w \in \mathrm{Bd}(N_l) \cup \mathrm{Bd}(N_r)$, then at least two of $N_l - \mathrm{Bd}(N_l)$, $N_r - \mathrm{Bd}(N_r)$, and c are, by 1.2, in the same component of $Z - \{w\}$ and, therefore, it follows that at least one of N_l or N_r has a boundary point not in A, which is a contradiction to (2). Assume

$$w \notin \mathrm{Bd}(N_l) \cup \mathrm{Bd}(N_r)\,.$$

Then, by 4.1, w is a point of some true cyclic element H of Z. Recalling that c is a cut point of Z, it follows easily that either w does not satisfy 1.2 or H does not satisfy 1.3. This proves 4.5.

For each Z satisfying 1.1 through 1.3, we shall construct an arc A in the plane with the max metric d,

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, |y_2 - y_1|\}\,,$$

such that Z is a diameter level for A. In the next two sections we shall use the set C defined as follows:

$$\begin{aligned} C = \{c \in Z : {}& c \text{ is a cut point of } Z \text{ and either} \\ & c \text{ is a point of a true cyclic element of } Z \text{ or} \\ & c \text{ is an end point of a maximal free arc of } Z\}\,. \end{aligned}$$

We shall first consider the case where the number of true cyclic elements is finite.

5. Finite case. For convenience we first note the following special case:

Case 1: *Z is a point, an arc or a 2-cell.* Let $I = \{(x, y) : 0 \leq x \leq 1, y = 0\}$. Then $L_0(I)$ is an arc and $L_1(I)$ is a point. If Z is a 2-cell, then let

$$A = ([1/2, 1] \times \{0\}) \cup (\{1\} \times [0, 1]) \cup ([1, 3/2] \times \{1\})\,.$$

Then $L_1(A)$ is homeomorphic to Z.

Next, we consider the following case:

Case 2: *Z has at least one and at most a finite number of true cyclic elements and Z contains a cut point which is a point of a true cyclic element of Z.* We see that Case 2 covers all the finite cases that were not considered in Case 1 as follows. Suppose that Z does not satisfy the conditions of Case 2. Then either Z contains no true cyclic element or Z contains no cut point which is a point of a true cyclic element of Z. If Z contains no true cyclic element, then, by [W, p. 89] and 1.1, Z must be a dendrite or a point. Assume Z is nondegenerate. Then, by 1.2, [W, Th. 6.1, p. 54] and [W, Th. 1.1(ii), p. 88], Z has exactly two noncut points. Hence, Z is an arc. If Z contains a true cyclic element T such that $Z-T$ is nonempty, then, by [W, 2.1, p. 66], T contains a cut point of Z. Consequently, by our assumption, $Z=T$ and thus by 1.1 and [K, p. 534], Z is a 2-cell.

Assume that Z is a continuum satisfying 1.1–1.3 and the conditions of this case. Then the set C is finite and we let n be the cardinality of C. We can label the points of C by $c_1,\ldots,c_n$, using 4.5 and the following scheme. Let c_1 be the element of C such that $C\cap N_l=\emptyset$ where N_l is the component of $Z-\{c_1\}$ containing p_l. In general for $i=2,\ldots,n$, let c_i be the point of C such that $C\cap N_i=\{c_1,\ldots,c_{i-1}\}$ where N_i is the component of $Z-\{c_i\}$ containing p_l. We shall represent Z by an $(n+1)$-tuple of points $(x_1,\ldots,x_{n+1})$ as follows. Let $p_l=c_0$ and $p_r=c_{n+1}$. For a given $i=1,\ldots,n-1$, we let $x_i=2$ if c_{i-1} and c_i are points of the same true cyclic element of Z and we let $x_i=1$ if c_{i-1} and c_i are the end points of a maximal free arc in Z. Note that by 4.2, this defines x_i for each i.

Let t be the number of true cyclic elements in Z, and let f be the number of maximal free arcs in Z; thus t is the number of 2's and f is the number of 1's in $(x_1,\ldots,x_{n+1})$. We define numbers v and h as follows. First, let k be the greatest integer less than or equal to $t/2$. Then let $v=k+f$, let $h=k+f$ if $t=2k$, and let $h=k+f+1$ if $t=2k+1$.

Let $Y=([0,1]\times[0,1])\cup([1,2]\times[1,2])\cup([2,3]\times[2,3])$. We shall construct an arc A in Y such $L_2(A)$ is homeomorphic to Z. First, we construct $n+1$ arcs such that the union of these arcs is an arc in $[0,1]\times[0,1]$ with end points $(0,0)$ and $(1,1)$. Let $e_0^1=(0,0)$. If $x_1=1$, let $e_1^1=(1/h,1/v)$. If $x_1=2$, let $e_1^1=(1/h,0)$. Let $A^1=e_0^1e_1^1$, the convex arc from e_0^1 to e_1^1. We define A^k for $k=2,\ldots,n+1$ by induction. Assume we have defined $e_0^je_1^j$. We let $e_0^{j+1}=e_1^j$. If $x_{j+1}=1$, let $e_1^{j+1}=e_0^{j+1}+(1/h,1/v)$. If $x_{j+1}=2$ and the number of coordinates which are 2 in $(x_1,\ldots,x_j)$ is even, let $e_1^{j+1}=e_0^{j+1}+(1/h,0)$. If $x_{j+1}=2$ and the number of coordinates which are 2 in $(x_1,\ldots,x_j)$ is odd, let $e_1^{j+1}=e_0^{j+1}+(0,1/v)$. Now, let $A^{j+1}=e_0^{j+1}e_1^{j+1}$.

It is clear that $\bigcup_{i=1}^{n+1}A^i$ is an arc. We show that $\bigcup_{i=1}^{n+1}A^i\subset[0,1]\times[0,1]$ with end points $(0,0)$ and $(1,1)$ as follows. If $t=2k$ then

$$\begin{aligned}e_1^{n+1}&=(0,0)+f\left(\frac{1}{h},\frac{1}{v}\right)+k\left(\frac{1}{h},0\right)+k\left(0,\frac{1}{v}\right)\\&=\left(\frac{f+k}{h},\frac{f+k}{v}\right)=(1,1)\,.\end{aligned}$$

If $t = 2k + 1$ then

$$e_1^{n+1} = (0,0) + f\left(\frac{1}{h}, \frac{1}{v}\right) + (k+1)\left(\frac{1}{h}, 0\right) + k\left(0, \frac{1}{v}\right)$$
$$= \left(\frac{f+k+1}{h}, \frac{f+k}{v}\right) = (1,1)\,.$$

In both cases, $e_1^n = (1,1)$. Therefore, $(0,0)$ and $(1,1)$ are the end points of $B = \bigcup_{i=1}^{n+1} A^i$ and hence $B \subset [0,1] \times [0,1]$. Now, let

$$A = B \cup \{(x,x) : x \in [1,2]\} \cup \{(x+2, y+2) : (x,y) \in B\}\,.$$

By inspection we can see that $L_2(A)$ is homeomorphic to Z (recall that we are using the max metric d).

For use later on, note the following two facts.

5.1. In the construction above, we could have chosen the first horizontal arc to have been vertical (and make appropriate changes for the other arcs).

5.2. We shall use the following terminology. An *admissible staircase* is an arc B constructed as above (consisting of horizontal, vertical and diagonal segments) but perhaps in a rectangle or in a square other than $[0,1] \times [0,1]$.

6. Infinite case. In this case the set C associated with X is countably infinite. We construct an arc B in $I \times I$ and an arc A by

$$A = B \cup \{(x,x) : x \in [1,2]\} \cup \{(x+1, y+1) : (x,y) \in B\}$$

and show that $L_2(A)$ is homeomorphic to Z.

Pick $c_1 \in C$. Then $X - \{c_1\} = U_0/U_\infty$ where $p_r \in U_0$ and $p_l \in U_\infty$.

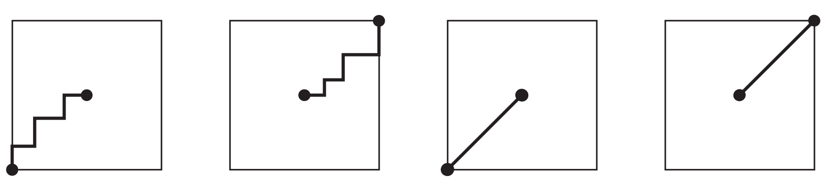
We begin the construction of B in (a)–(g) below.

(a) *$U_0 \cap C$ is nonempty and finite.* Then let A_1 be an admissible staircase in $[0,1/2] \times [0,1/2]$ with end points $(0,0)$ and $(1/2,1/2)$.

(b) *$U_\infty \cap C$ is nonempty and finite.* Let A_1 be an admissible staircase in $[1/2,1] \times [1/2,1]$ with end points $(1/2,1/2)$ and $(1,1)$.

(c) *$U_0 \cap C = \emptyset$ and $U_0 \cup \{c_1\}$ is a free arc.* Let $A_1 = (0,0)(1/2,1/2)$.

(d) *$U_\infty \cap C = \emptyset$ and $U_\infty \cup \{c_1\}$ is a free arc.* Let $A_1 = (1/2,1/2)(1,1)$.



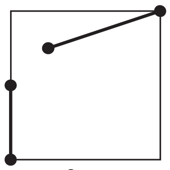
a b c d

(e) *$U_0 \cap C = \emptyset$ and $U_0 \cup \{c_1\}$ is a true cyclic element of Z.* Then let $A_1 = \{0\} \times [0,1/2]$. Now pick $c_2 \in C - \{c_1\}$. Let $X - \{c_2\} = V_0/V_\infty$. We consider the following subcases.

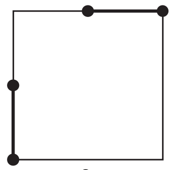
(e_1) *$V_\infty \cap C = \emptyset$ and $V_\infty \cup \{c_2\}$ is a free arc.* Let $A_2 = (1/4, 3/4)(1, 1)$.

(e_2) *$V_\infty \cap C = \emptyset$ and $V_\infty \cup \{c_2\}$ is a true cyclic element of Z.* Then let $A_2 = [1/2, 1] \times \{1\}$.

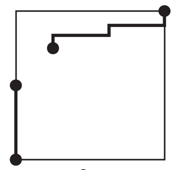
(e_3) *$V_\infty \cap C$ is nonempty and finite.* Then let A_2 be an admissible staircase in $[1/4, 1] \times [3/4, 1]$ with end points $(1/4, 3/4)$ and $(1, 1)$.



e_1



e_2

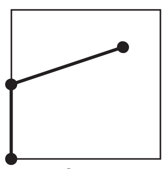


e_3

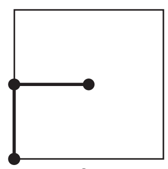
(e_4) *$V_0 \cap C = \{c_1\}$ and c_1 and c_2 are end points of the same maximal free arc.* Let $A_2 = (0, 1/2)(3/4, 3/4)$.

(e_5) *$V_0 \cap C = \{c_1\}$ and c_1 and c_2 are end points of the same true cyclic element of Z.* Let $A_2 = (0, 1/2)(1/2, 1/2)$.

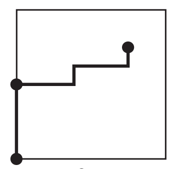
(e_6) *$V_0 \cap C - \{c_1\}$ is nonempty and finite.* Then let A_2 be an admissible staircase in $[0, 3/4] \times [1/2, 3/4]$ with end points $(0, 1/2)$ and $(3/4, 3/4)$.



e_4



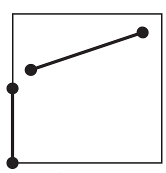
e_5



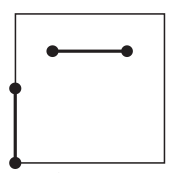
e_6

(e_7) *$V_0 \cap C$ and $V_\infty \cap C$ are both infinite.* Then we pick a point $c_3 \in C$ such that either (e_{7a}) c_2 and c_3 are the end points of a maximal free arc, or (e_{7b}) c_1 and c_2 are points of the same true cyclic element of Z. By the definition of C we can pick a point c_3 satisfying either (e_{7a}) or (e_{7b}).

If (e_{7a}) holds, then let $A_2 = (1/8, 5/8)(7/8, 7/8)$. If ($e_{7b}$) holds, then let $A_2 = [1/4, 3/4] \times \{3/4\}$.



e_{7a}



e_{7b}

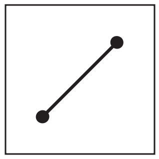
(f) *$U_\infty \cap C = \emptyset$ and $U_\infty \cup \{c_1\}$ is a true cyclic element of Z.* This case is similar to (e).

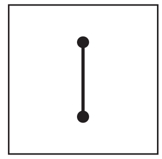
By (a)–(f) only one general case remains to be considered for the first step in this construction.

(g) *$U_0 \cap C$ and $U_\infty \cap C$ are infinite.* Then pick $c_2 \in C$ such that either

($\mathrm{g_a}$) c_1 and c_2 are end points of a maximal free arc, or ($\mathrm{g_b}$) c_1 and c_2 are points of the same true cyclic element of Z. By the definition of C we can pick a point c_2 satisfying either ($\mathrm{g_a}$) or ($\mathrm{g_b}$). We shall assume without loss of generality, by reindexing c_1 and c_2, that c_1 separates p_l and c_2 in Z.

If ($\mathrm{g_a}$) holds, then let $A_1 = (1/4, 1/4)(3/4, 3/4)$. If ($\mathrm{g_b}$) holds, then let $A_1 = \{1/2\} \times [1/4, 3/4]$.


$\mathrm{g_a}$

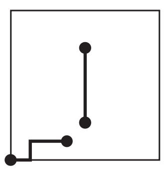

$\mathrm{g_b}$

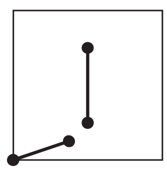
Then pick $c_3 \in C$ such that c_3 separates p_l and c_1 in Z. Let $X - \{c_3\} = V_0/V_\infty$ where $p_l \in V_0$ and $p_r \in V_\infty$. We consider the following seven subcases for the construction in $[0, 3/8] \times [0, 1/8]$.

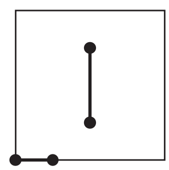
($\mathrm{g_{b1}}$) *$V_0 \cap C$ is nonempty and finite.* Then we let A_2 be an admissible staircase in $[0, 3/8] \times [0, 1/8]$ with end points $(0, 0)$ and $(3/8, 1/8)$.

($\mathrm{g_{b2}}$) *$V_0 \cap C = \emptyset$ and $V_0 \cup \{c_3\}$ is a maximal free arc in Z.* Then let $A_2 = (0, 0)(3/8, 1/8)$.

($\mathrm{g_{b3}}$) *$V_0 \cap C = \emptyset$ and $V_0 \cup \{c_3\}$ is a true cyclic element of Z.* Then let $A_2 = [0, 1/4] \times \{0\}$.


$\mathrm{g_{b1}}$

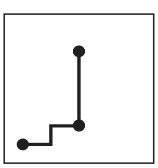

$\mathrm{g_{b2}}$

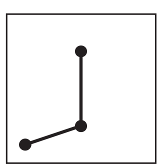

$\mathrm{g_{b3}}$

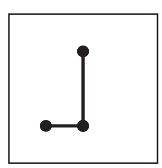
($\mathrm{g_{b4}}$) *$V_\infty \cap U_0 \cap C$ is nonempty and finite.* Then let A_2 be an admissible staircase in $[1/8, 1/2] \times [1/8, 1/4]$ with end points $(1/8, 1/8)$ and $(1/2, 1/4)$.

($\mathrm{g_{b5}}$) *$V_\infty \cap U_0 \cap C = \emptyset$ and c_3 and c_1 are end points of a maximal free arc in Z.* Then let $A_2 = (1/8, 1/8)(1/2, 1/4)$.

($\mathrm{g_{b6}}$) *$V_\infty \cap U_0 \cap C = \emptyset$ and c_3 and c_1 are points of the same true cyclic element of X.* Then let $A_2 = [1/4, 1/2] \times \{1/4\}$.


$\mathrm{g_{b4}}$

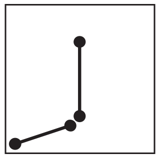

$\mathrm{g_{b5}}$


$\mathrm{g_{b6}}$

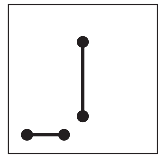
($\mathrm{g_{b7}}$) *$V_0 \cap C$ and $V_\infty \cap U_0 \cap C$ are both infinite.* Then pick $c_4 \in C$ such that either ($\mathrm{g'_{b7}}$) c_3 and c_4 are end points of a maximal free arc, or ($\mathrm{g''_{b7}}$) c_3

and c_4 are points of the same true cyclic element of Z. By the definition of C we can pick c_4 satisfying either ($\mathrm{g}'_{\mathrm{b}7}$) or ($\mathrm{g}''_{\mathrm{b}7}$).

If ($\mathrm{g}'_{\mathrm{b}7}$) holds, then let $A_2 = (1/16, 1/16)(7/16, 3/16)$. If ($\mathrm{g}''_{\mathrm{b}7}$) holds, then let $A_2 = [1/8, 3/8] \times \{1/8\}$.



$\mathrm{g}'_{\mathrm{b}7}$



$\mathrm{g}''_{\mathrm{b}7}$

Construct an arc A_3 in $[1/2, 1] \times [3/4, 1]$ in a manner similar to the construction of A_2 in cases ($\mathrm{g}_{\mathrm{b}1}$) through ($\mathrm{g}_{\mathrm{b}7}$).

In each of the cases above there are one, two, three, or four squares determined by the arcs and the points $(0, 0)$ and $(1, 1)$. The procedure done in (a) through (g) can be repeated in each of these squares thereby leaving a finite number of squares of at most half the diameter. Thus, an inductive argument results in a set whose closure is the required arc B. The fact that B is an arc can be shown by first showing that it is connected and then showing that for any point in it other than $(0, 0)$ and $(1, 1)$, the horizontal line in the plane through that point or the vertical line in the plane through that point intersects B only in that point—thus $(0, 0)$ and $(1, 1)$ are the only two noncut points of B (see Th. 1 of [K, p. 179]).

7. Examples

Example 1. We describe a diameter level for a metric on a 2-cell which in not locally connected. The example is similar to one in [P]. Let $i = 0, 1, \ldots$ For $t \in [3/2^{i+2}, 1/2^i]$, let $S_t = \{(x, y, t) : x^2 + y^2 = (1 + 1/2^i - t)^2\}$. For $t \in [1/2^{i+1}, 3/2^{i+2})$, let $S_t = \{(x, y, t) : x^2 + y^2 = (1 - 1/2^{i+1} + t)^2\}$. Let $Y = \{(x, y, 1) : x^2 + y^2 < 1\}$ and $S_0 = \{(x, y, 0) : x^2 + y^2 = 1\}$. Let

$$X = \Big[\bigcup_{t\in[0,1]} S_t\Big] \cup Y\,.$$

We will show that any arc in L_2 from $S_{1/2^i}$ to $S_{1/2^{i+1}}$, where $i \geq 1$, must be of diameter larger than or equal to $\sqrt{2}$.

Let f be a homeomorphism from $[0, 1]$ to a subset of L_2 such that $f(0) = S_{1/2^i}$ and $f(1) = S_{1/2^{i+1}}$. Using cylindrical coordinates, we define a projection $\pi : \bigcup_{t\in[0,1]} S_t \to S_{1/2^{i+1}}$ by $\pi(\theta, r, h) = (\theta, 1, 1/2^{i+1})$ where $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 1\frac{1}{4}$, and $0 \leq h \leq 1$.

We say that two lines $\pi^{-1}(p)$ and $\pi^{-1}(p')$ are *antipodal lines* if p and p' are antipodal points on S_{i+1}. We note the following observation which will be used in our exposition.

(*) If $\pi^{-1}(p)$ and $\pi^{-1}(p')$ are antipodal lines and if $q \in \pi^{-1}(p)$ such that $q \notin (\bigcup_{i=0}^{\infty} S_{1/2^i}) \cup S_0$, then $d(q, \pi^{-1}(p')) > 2$.

Let $t' = \min\{t \in [0,1] : f(t)$ contains antipodal points on $S_{1/2^{i+1}}\}$. There are two cases to consider.

Case 1: $t' = 1$. Let $0 < \varepsilon < 1/2^{i+2}$. Let $\delta > 0$ such that if $t \in (1-\delta, 1]$ then $H_d(f(1), f(t)) < \varepsilon$. Let $t_0 \in (1-\delta, 1)$. So, $f(t_0) \subset \bigcup_{p \in S_{1/2^{i+1}}} B(p, \varepsilon)$. Clearly, $H_d(S_{1/2^{i+1}}, f(t_0)) \geq H_d(S_{1/2^{i+1}}, \pi f(t_0))$. The set $\pi f(t_0)$ must be a connected subset of $S_{1/2^{i+1}}$ which properly contains a semicircle, since $f(t_0)$ is connected, π is continuous, and $H_d(S_{1/2^{i+1}}, \pi f(t_0)) < \sqrt{2}$. Let p and p' be antipodal points of $\pi(f(t_0))$. Then $\{p, p'\} \not\subseteq f(t_0)$ since $t_0 < 1$. Let $q \in \pi^{-1}(p) \cap f(t_0)$ and $q' \in \pi^{-1}(p') \cap f(t_0)$. By (*), $d(q, q') > 2$. Hence, $\operatorname{diam} f(t_0) > 2$. Consequently, $t' \neq 1$.

Case 2: $t' < 1$. Let q and q' be antipodal points of $f(t') \cap S_{1/2^{i+1}}$. Let $p \in S_{1/2^{i+1}}$ such that $\pi^{-1}(p) \cap f(t') = \emptyset$. To see that such a point p exists, we make the following observations. If $f(t') \subset S_{1/2^{i+1}}$ then it is clear there exists such a p since $t' \neq 1$ and $f(t') \neq S_{1/2^{i+1}}$. If $f(t') \cap (X - S_{1/2^{i+1}}) \neq \emptyset$ then let $p' \in f(t') \cap (\bigcup_{t \in J} S_t)$ where $J = (1/2^{i+2}, 1/2^{i+1}) \cup (1/2^{i+1}, 1/2^i)$. Clearly, such a point p' exists since if $f(t')$ is connected, $f(t') \not\subseteq S_{1/2^{i+1}}$, and $f(t') \cap S_{1/2^{i+1}} \neq \emptyset$ then $f(t') \not\subseteq Y \cup (\bigcup_{t \in I} S_t) \cup S_{1/2^{i+1}}$, where $I = [0, 1/2^{i+1}] \cup [1/2^i, 1]$. Let p be the point on $S_{1/2^{i+1}}$ which is antipodal to $\pi(p')$. By (*), $d(p, \pi^{-1}(\pi(p')) > 2$, so $\pi^{-1}(p) \cap f(t') = \emptyset$. Let $0 < \varepsilon < \min\{1/2^{i+2}, d(f(t'), \pi^{-1}(p))/2\}$. Pick $\delta > 0$ such that if $t \in (t'-\delta, t']$ then $H_d(f(t), f(t')) < \varepsilon$. Let $t_0 \in (t'-\delta, t')$. Note that $\pi^{-1}(p) \cap f(t_0) = \emptyset$, since $d(\pi^{-1}(p), f(t')) > 2\varepsilon$ and $H_d(f(t')), f(t_0)) < \varepsilon$.

Let $a \in f(t_0) \cap B(q, \varepsilon)$ and $b \in f(t_0) \cap B(q', \varepsilon)$. Note that $\pi(a) \in B(q, \varepsilon)$ and $\pi(b) \in B(q', \varepsilon)$. Let a' be the point which is antipodal to $\pi(a)$. If $\pi(a) \neq a$ then $\pi^{-1}(a') \cap f(t_0) = \emptyset$ by (*) since $a \notin (\bigcup_{i=0}^{\infty} S_{1/2^i}) \cup S_0$. If $\pi(a) = a$ then $a' \notin f(t_0)$ since $t_0 < t'$. Hence, $f(t_0) \cap \pi^{-1}(a') = \emptyset$ by (*). In either case $f(t_0) \cap \pi^{-1}(a') = \emptyset$. Let b' be the point which is antipodal to $\pi(b)$. Similarly, we see that $f(t_0) \cap \pi^{-1}(b') = \emptyset$.

Since $f(t_0)$ is connected and $\pi(f(t_0)) \subset S_0 - \{p, b', a'\}$ we have either $\pi(f(t_0)) \subsetneq \widehat{a'p}$ where $\widehat{a'p}$ is the arc on S_0 which does not contain b', or $\pi(f(t_0)) \subsetneq \widehat{pb'}$ where $\widehat{pb'}$ is the arc on S_0 which does not contain a', or $\pi(f(t_0)) \subseteq \widehat{b'a}$ where $\widehat{b'a}$ is the arc on S_0 which does not contain p. Let $\widehat{qq'}$ be the arc on S_0 which does not contain p. So, $\widehat{b'a'} \subset \widehat{qq'} \cup B(q, \varepsilon) \cup B(q', \varepsilon)$ and $H_d(\widehat{b'a'}, S_0) > \sqrt{2} - \varepsilon$. Consequently, if $\pi(f(t_0)) \subset \widehat{b'a'}$ then $H_d(S_0, f(t_0)) \geq H_d(S_0, \pi(f(t_0))) \geq H_d(S_0, \widehat{b'a'}) > \sqrt{2} - \varepsilon$. Similarly, $\widehat{a'p} \cup \widehat{pb'} \subset (S_0 - \widehat{qq'}) \cup B(q, \varepsilon) \cup B(q', \varepsilon)$ and $H_d(\widehat{a'p} \cup \widehat{pb'}, S_0) > \sqrt{2} - \varepsilon$. So, if $\pi(f(t_0)) \subset \widehat{a'p} \cup \widehat{pb'}$

then $H_d(S_0, f(t_0)) \geq H_d(S_0, \pi(f(t_0))) \geq H_d(S_0, \widehat{a'p} \cup \widehat{pb'}) > \sqrt{2} - \varepsilon$. Hence, for every $\varepsilon > 0$, $\operatorname{diam} f([0,1]) > \sqrt{2} - \varepsilon$. Consequently, $\operatorname{diam} f([0,1]) \geq \sqrt{2}$. Hence, we see that L_2 is not locally connected.

EXAMPLE 2. We describe a one-dimensional Peano continuum X which has a non-locally connected diameter level. Let

$$X = \bigcup_{i=0}^{\infty} S_{1/2^i} \cup \Big[\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^i} \bigcup_{k=1}^{2} L_{i,j,k}\Big] \cup S_0$$

where $S_{1/2^i}$ is a circle as defined in Example 1 for $i = 0, 1, \ldots$ and S_0 is as defined in Example 1. For each pair (i, j), $L_{i,j,1}$ is the straight line segment joining the points $x_{i,j}$ and $y_{i,j}$ and $L_{i,j,2}$ is the straight line segment joining the points $y_{i,j}$ and $z_{i,j}$ where

$$\begin{aligned} x_{i,j} &= ((j-1)\pi 2^{1-i}, 1, 2^{1-i}), \\ y_{i,j} &= ((j-1)\pi 2^{1-i}, 1 + 2^{-i-1}, 3 \cdot 2^{-i-1}). \\ z_{i,j} &= ((j-1)\pi 2^{1-i}, 1, 2^{-i}). \end{aligned}$$

By a proof similar to the one in the first example, we see that L_2 of $C(X)$ is not locally connected.

We remark that when X is a one-dimensional absolute retract then all the levels for size maps are locally connected.

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