Algebras of Borel measurable functions

by

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Abstract. We show that, for each $0 < \alpha < \omega_1$, in the $\alpha$th class in the Baire classification of Borel measurable real functions defined on some uncountable Polish space there is a function which cannot be expressed as a countable union of functions which are (on their domains) in the $\alpha$th class in Sierpiński’s classification. This, in particular, solves positively a problem of Kempisty who asked whether for $1 < \alpha < \omega_1$ the $\alpha$th Baire and Sierpiński classes are different. We also show that, for every $\alpha$, in the $\alpha$th class of Sierpiński’s classification there is a function which cannot be expressed as a countable union of functions each of which is on its domain in one of the two $\alpha$th classes of Young’s classification (we refer here to the classical numbering of Baire’s, Young’s and Sierpiński’s classes and not to the one used in the paper).

1. Introduction. In [CM] and [CMPS] the following diagram was considered:

\[
\begin{array}{ccc}
B_\alpha & \rightarrow & B_{\alpha+1} \\
\downarrow & & \uparrow \\
L_\alpha & \leftarrow & U_\alpha \\
\end{array}
\]

where $B_\alpha$ is the $\alpha$th class in the Baire classification of real functions defined on $[0, 1]$ and $L_\alpha$ and $U_\alpha$ are the classes of limits of, respectively, nondecreasing and nonincreasing sequences of functions from $B_\alpha$; the arrows stand for proper inclusions. It was shown there that in every class of (1) there is a function which cannot be expressed as a union of countably many partial functions from lower classes. In the present paper, considering the algebra $L_\alpha + U_\alpha$ of all algebraic sums of functions from $L_\alpha$ and $U_\alpha$, we add to (1) the following diagram (cl stands for closure in the uniform convergence

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topology):

\[ L_\alpha \cup U_\alpha \rightarrow L_\alpha + U_\alpha \rightarrow \text{cl}(L_\alpha + U_\alpha) = B_{\alpha+1} \]

(the equality in (2) was proved by Sierpiński in [S2] for \( \alpha = 0 \) and the proof remains the same for all \( \alpha < \omega_1 \)). In fact, we consider algebras of bounded functions and then the diagram (2) gets more subtle (b stands for the bounded functions in the given class):

\[ bL_\alpha \cup bU_\alpha \rightarrow bL_\alpha + bU_\alpha \]

\[ \rightarrow \text{cl}(bL_\alpha + bU_\alpha) \rightarrow b(\text{cl}(L_\alpha + U_\alpha)) = bB_{\alpha+1}. \]

Again we show that in every class displayed in (3) there exists a function which cannot be expressed as a sum of countably many partial functions from lower classes. This, in particular, implies that the second inclusion from (2), \( L_\alpha + U_\alpha \subset \text{cl}(L_\alpha + U_\alpha) = B_{\alpha+1} \), is proper. This solves a problem of Kempisty [Ke] (for \( \alpha = 0 \) the inclusion was shown to be proper by Sierpiński [S1]).

We work in a more general setting enabling us to obtain, for example, analogous results for functions measurable with respect to the projective classes \( \Sigma^1_\alpha \).

2. Notation, definitions and basic facts. We use standard set-theoretical notation. \( \mathbb{N} \) is the set of positive integers, \( \mathbb{R} \) the set of reals, and \( P(A) \) the family of all subsets of a set \( A \). If \( A \) is fixed and \( \mathcal{A} \subseteq P(A) \) then \( \mathcal{A}^c = \{ A \setminus B : B \in \mathcal{A} \} \). \( \mathcal{A}^*_0 \) will stand for all countable intersections of elements of \( \mathcal{A} \). A family \( \mathcal{A} \subseteq P(A) \) is a partition of \( A \) if \( \bigcup \mathcal{A} = A \) and for all \( X, Y \in \mathcal{A}, X \neq Y \), we have \( X \cap Y = \emptyset \). If \( \mathcal{A} \subseteq P(A) \) and \( X \subseteq A \), then \( \mathcal{A}|_X = \{ Y \cap X : Y \in \mathcal{A} \} \). We denote by \( r(\mathcal{A}) \) the ring of sets generated by \( \mathcal{A} \), i.e. the smallest family containing \( \mathcal{A} \) and closed under taking complements and finite unions. Suppose that \( \mathcal{A} \subseteq P(A) \) is a family of sets. We say that \( \mathcal{A} \) is a \( \sigma \)-class if \( \{ \emptyset, A \} \subseteq \mathcal{A} \) and \( \mathcal{A} \) is closed under finite intersections and countable unions. If \( \mathcal{A} \subseteq P(A) \), then we denote by \( \mathcal{A}' \) the minimal \( \sigma \)-class containing \( r(\mathcal{A}) \). The symbol \( \chi_A \) will denote the characteristic function of \( A \). The domain of a function \( f \) will be denoted by \( \text{dom} \ f \) and its range by \( \text{Rg} \ f \). If \( A \) and \( B \) are sets, then \( A^B \) is the set of all functions with domain \( A \) and range contained in \( B \). If \( f \in A^B \) and \( C \subseteq A \), then \( f|_C \) denotes the restriction of \( f \) to \( C \). We write \( \Delta^\mathcal{A} \) for the set of all partial functions from \( A \) to \( B \), i.e. \( \Delta^\mathcal{A} = \{ f \in C^B : C \subseteq A \} \). Let \( f \) be a real function defined on some set \( A \); then \( \inf f = \inf \{ f(x) : x \in A \} \), \( \sup f = \sup \{ f(x) : x \in A \} \) and if \( f \) is bounded \( \| f \| = \sup | f | \). If \( \mathcal{H} \) is any class of real functions, we denote by \( b\mathcal{H} \) the class of all bounded functions from \( \mathcal{H} \), and by \( \text{cl} \mathcal{H} \) the class of all uniform limits of functions from \( \mathcal{H} \). For \( \mathcal{G} \subseteq \mathbb{Z}^\mathbb{R} \) and \( \mathcal{H} \subseteq \mathbb{Z}^\mathbb{R} \) let \( \mathcal{G} + \mathcal{H} = \{ g + h : g \in \mathcal{G} \text{ and } h \in \mathcal{H} \} \). Let \( A \subseteq \mathbb{R} \).
We denote by \( VB(A) \) the family of all real functions of bounded variation on \( A \), and by \( C(A) \) the continuous functions on \( A \).

We write \( \mathcal{N} \) and \( \mathcal{C} \) for the spaces \( ^{\omega}\mathbb{N} \) and \( ^{\omega}\{0,1\} \), respectively, with the product topology. The first space is homeomorphic to the irrational numbers and the second to the Cantor set.

Let \( Z \) be any set. Let \( \mathcal{F} \subseteq \mathcal{Z} \mathcal{R} \) and \( \mathcal{G} \subseteq \mathcal{Z} \mathcal{R} \). We denote by \( \text{dec}(\mathcal{F}, \mathcal{G}) \) the least cardinal \( \kappa \) such that for every \( f \in \mathcal{F} \) one can find a family \( \{g_\alpha : \alpha < \kappa\} \subseteq \mathcal{G} \) such that \( \{\text{dom } g_\alpha : \alpha < \kappa\} \) is a partition of \( Z \) and \( f = \bigcup \{g_\alpha : \alpha < \kappa\} \). We shall only use this definition when it makes sense, i.e., when such subfamilies of \( \mathcal{G} \) exist.

Suppose \( A \) is a \( \sigma \)-class. We denote by \( \overline{M}_A \) the family of all functions \( f \in \mathcal{Z} \mathcal{R} \) such that \( f^{-1}((-\infty, c)) \in A \) for every \( c \in \mathbb{R} \). Similarly, \( \overline{M}_A \) is the family of all \( f \in \mathcal{Z} \mathcal{R} \) such that \( f^{-1}((c, \infty)) \in A \) for every \( c \in \mathbb{R} \). Note that if \( f \in \overline{M}_A \) if and only if \( -f \in \overline{M}_A \). We put \( M_A = \overline{M}_A \cap \overline{M}_A \). We denote by \( M_{B,A}, \overline{M}_{B,A} \) and \( \overline{M}_{B,A} \) the functions from \( M_A, \overline{M}_A \) and \( \overline{M}_A \), respectively, for which \( Rg \subseteq B \). Note that if \( A \) is a \( \sigma \)-class and \( B \) is closed then \( M_{B,A}, \overline{M}_{B,A} \) and \( \overline{M}_{B,A} \) are complete metric spaces (in the uniform convergence topology) and the same is true for \( bM_A, \overline{bM}_A \) and \( b\overline{M}_A \).

Let \( R\overline{M}_{B,A} = \bigcup \{\overline{M}_{B}(A,X) : X \in P(Z)\}, R\overline{M}_{B,A} = \bigcup \{\overline{M}_{B}(A|X) : X \in P(Z)\}, R\overline{M}_{B,A} + \overline{M}_{B,A} = \bigcup \{\overline{M}_{B}(A|X) + \overline{M}_{B}(A|X) : X \in P(Z)\} \) and \( R(\text{cl}(b\overline{M}_{B,A} + b\overline{M}_{B,A})) = \bigcup \{\text{cl}(b\overline{M}_{B}(A|X) + b\overline{M}_{B}(A|X)) : X \in P(Z)\} \).

We use standard notation from Descriptive Set Theory. For example, \( \Sigma^0_n \) denotes the \( n \)th additive (multiplicative, resp.) class in the hierarchy of Borel sets, and \( \Sigma^0_n \) is the \( n \)th projective class in the hierarchy of projective sets.

For \( X \) a Polish space and \( \alpha < \omega_1 \), let \( B_\alpha(X) = \{f \in X^\mathbb{R} : f^{-1}(G) \in \Sigma^0_{\omega_1+n}(X) \text{ for each } G \text{ open in } \mathbb{R}\} \). If \( X = \mathbb{R} \) we write briefly \( B_\alpha(\mathbb{R}) = B_\alpha \).

We have \( B_\alpha(X) = M_{\Sigma^0_{\omega_1+n},\alpha}(X) \). We also write \( L_\alpha(X) \) and \( U_\alpha(X) \) to denote \( M_{\Sigma^0_{\omega_1+n},\alpha}(X) \) and \( M_{\Sigma^0_{\omega_1+n},\alpha}(X) \), respectively. For \( X = \mathbb{R} \) we write \( L_\alpha(\mathbb{R}) = L_\alpha \) and \( U_\alpha(\mathbb{R}) = U_\alpha \). Obviously, \( L_\alpha(X) \) and \( U_\alpha(X) \) are the classes of lower and upper semicontinuous functions on \( X \) with values in \( \mathbb{R} \).

Remark. In the classical notation the class \( B_\alpha \) for \( \alpha < \omega \) and \( B_{\alpha+1} \) for \( \alpha \geq \omega \) is called the \( \alpha \)th class in the Baire classification and the classes \( L_\alpha \), \( U_\alpha \) (\( L_\alpha + U_\alpha \)) for \( \alpha < \omega \) have the number \( \alpha + 1 \) and for \( \alpha \geq \omega \) the number \( \alpha \) in Young’s (Sierpiński’s, resp.) classification (compare for instance [L]).

We say that a class \( A \) has the reduction property if for any \( A, B \in A \) there are \( A^*, B^* \in A \) such that \( A^* \subseteq A \) and \( B^* \subseteq B \), \( A^* \cap B^* = \emptyset \) and \( A^* \cup B^* = A \cup B \). Note that if \( 1 < \alpha < \omega_1 \), then \( \Sigma^0_\alpha \) has the reduction property. The same is true of \( \Sigma^0_n \), \( n \in \mathbb{N} \). Moreover, if \( Z \) is zero-dimensional, then \( \Sigma^0_\beta(Z) \) also has the reduction property.

Following the idea used in [Mo] we deal in this paper with a certain fixed
family $T$ of Polish spaces such that:

(i) if $X \subseteq Z \in T$ and $X$ is a closed subset of $Z$, then also $X \in T$;
(ii) $T$ is closed under finite Cartesian products;
(iii) $\mathcal{N}, \mathbb{R} \in T$.

As in [Mo] the idea is to include in $T$ any Polish space one wants to consider.

Now assume that to each $Z \in T$ we have assigned a certain family $\mathcal{A}(Z)$ of subsets of $Z$. Denote by $\mathcal{A}$ the collection of all these families. We say that $\mathcal{A}$ is closed under continuous substitutions if for each $X,Y \in T$ and for every continuous function $f \in ^YX$ we have $f^{-1}(A) \in \mathcal{A}(X)$ for every $A \in \mathcal{A}(Y)$. We shall call $\mathcal{A}$ a hereditary $\sigma$-class if $\mathcal{A}$ is closed under continuous substitutions and if for each $Z \in T$ the following two conditions are satisfied:

(I) $\mathcal{A}(Z)$ is a $\sigma$-class;

(II) $\mathcal{A}(Z)|X = \mathcal{A}(X)$ for each closed $X \subseteq Z$.

Obviously, $\Sigma_0^n$, $\alpha < \omega_1$, and $\Sigma^1_n$, $n \in \mathbb{N}$, are examples of hereditary $\sigma$-classes.

Let $X$ and $Y$ be any sets. For $A \subseteq X \times Y$ and $x \in X$ let $A_x = \{y \in Y : (x,y) \in A\}$. If $\mathcal{A} \subseteq \mathcal{P}(Y)$, a set $A \subseteq X \times Y$ is called a universal set for $\mathcal{A}$ if $A = \{A_x : x \in X\}$. Recall that if $X$ and $Y$ are Polish spaces and $X$ is uncountable, then for any $\alpha < \omega_1$ there is a universal set for $\Sigma_0^\alpha(Y)$ in the class $\Sigma_0^\alpha(X \times Y)$, and the same is true for the classes $\Sigma^1_n$, $n \in \mathbb{N}$ (see [Mo]).

Let $F \in _X\mathbb{R}$ and $(x,y) \in X \times Y$. We put $F_x(y) = F(x,y)$. A function $F \in _X\mathbb{R}$ is called a universal function for a class $\mathcal{H} \subseteq \mathcal{P}(Y)$ if $\mathcal{H} = \{F_x : x \in X\}$.

We shall use the following known facts. Theorems 2.A and 2.B were formulated in [CMPS, Cor. 2.2 and Cor. 2.4] in a weaker form but, in fact, they are exactly the theorems proved there.

**Theorem 2.A.** If $\mathcal{A}$ is a $\sigma$-class of subsets of $Z$ with the reduction property, then for every countable family of functions $\mathcal{H} \subseteq \mathcal{M}\mathcal{A}$ there exists $g \in \mathcal{M}\mathcal{A}$ such that $\inf |f - g| > 0$ for every $f \in \mathcal{H}$.

**Theorem 2.B.** If $\mathcal{A}$ is a $\sigma$-class of subsets of $Z$ and $f \in \mathcal{M}\mathcal{A}$, then the set $\{g \in \mathcal{M}\mathcal{A} : \inf |f - g| > 0\}$ is open and dense in $\mathcal{M}\mathcal{A}$.

**Theorem 2.C ([CM, Th. 2.1]).** If $Z \in T$ and $\mathcal{A}$ is a hereditary $\sigma$-class such that $\mathcal{A}(Z)$ has a universal set in $\mathcal{A}(\mathcal{C} \times Z)$, then there exists a universal function for $\mathcal{M}\mathcal{A}(Z)$ in $\mathcal{M}\mathcal{A}(\mathcal{C} \times Z)$.

**Theorem 2.D** (see, for instance, [CM, Prop. 1.1]). If $n \in \mathbb{N}$ and $\mathcal{A}$ is a $\sigma$-class then $f \in \mathcal{F}\mathcal{M}_{[-n,n]}\mathcal{A}$ if and only if there exists $f^* \in \mathcal{M}_{[-n,n]}\mathcal{A}$ such that $f = f^*|\text{dom } f$. A similar result holds for functions from $\mathcal{M}_{[-n,n]}\mathcal{A}$.
Now to formulate Theorem 2.E ([H, XIV, p. 277]) we introduce some notation used in [H].

A family $F$ of real functions defined on a common domain $D$ will be called an ordinary function system if

(i) every real function which is constant on $D$ is in $F$;

(ii) the maximum and minimum of two functions from $F$ is in $F$;

(iii) the sum, difference, product, and quotient (with nowhere vanishing denominator) of two functions from $F$ is in $F$.

An ordinary function system $F$ is called complete if it also satisfies the following condition:

(iv) the limit of a uniformly convergent sequence of functions from $F$ is in $F$.

Let $A$ and $B$ be two families of functions. The function $f$ is said to be of class $(A, B)$ if for each $c \in \mathbb{R}$ the set $f^{-1}((c, \infty))$ is in $A$ and the set $f^{-1}([c, \infty))$ is in $B$ ([H, p. 267]).

Let $F$ be a given family of functions defined on a common domain. Let $f$ range over $F$, and let $g$ and $h$ range over all real functions which are pointwise limits of, respectively, nondecreasing and nonincreasing sequences of functions from $F$. Then the sets of the form $f^{-1}((c, \infty))$, $g^{-1}((c, \infty))$, $h^{-1}([c, \infty))$ will be called $N$, $P$, $Q$ sets, respectively ([H, p. 270]). Countable intersections of $N$ sets will be called $N_\delta$ sets. $P$ and $Q$ will stand for the families of all $P$ and $Q$ sets respectively.

The functions forming the least complete ordinary function system over $F$ will be called $v$ functions ([H, VII, p. 272]).

The following theorem was proved in [H, XIV, p. 277].

**Theorem 2.E.** Let $F$ be an ordinary function system. If $Q_0$ is a $Q$ set, then each function $\phi : Q_0 \to \mathbb{R}$ which is of class $(P|Q_0, Q|Q_0)$ can be extended to a function of class $(P, Q)$, that is (see [H, VII, p. 272]), to a $v$ function.

We now derive a corollary we shall use in the sequel.

**Corollary 2.F.** If $A$ is a $\sigma$-class of subsets of some set $Z$ and if $\phi \in M(A'|S)$ where $S \in A_\delta$, then $\phi$ can be extended to some $\phi^* \in M.A.'$

**Proof.** Notice that all $B \in A$ are $N$ sets for the ordinary function system $M.A.'$, because $\chi_B \in M.A.'$. Thus the sets from $A_\delta$ are $N_\delta$ sets and therefore $Q$ sets ([H, VI, p. 271]). Thus, by Theorem 2.E, the function $\phi$ can be extended to a $v$ function $\phi^*$. But, as $M.A'$ is a complete ordinary function system ([H, III, p. 268]), every $v$ function is in $M.A'$. ■
3. Algebras of measurable functions

**Lemma 3.1.** Let \( \mathcal{A} \) be a \( \sigma \)-class of subsets of \( Z \). Let \( h \in \mathcal{MA}, |\text{Rg} h| < \aleph_0 \), \( v \in \mathcal{MA} \). Then for each \( \varepsilon > 0 \) there exists \( g \in \mathcal{MA} [-1,1] \) such that \( |g| < \varepsilon \) and \( \inf |g + h - v| > 0 \).

**Proof.** Let \( \text{Rg} h = \{\alpha_1, \ldots, \alpha_n\}, \alpha_1 < \ldots < \alpha_n \). Let \( A_i = \{z \in Z : h(z) = \alpha_i\}, i \leq n \). Assume \( \varepsilon < 1 \). By Theorem 2.B for each \( i \leq n \) there exists \( g_i \in \mathcal{MA} [-1,1] \) and \( \inf |g_i| = \alpha_i - v| > 0 \) and \( \inf |g_i - \varepsilon(n-i)/(2n)| < \varepsilon/(2n) \). Observe that sup \( g_i, i \leq 1, \ldots, n - 1 \). Define \( g(z) = g_i(z) \) for \( z \in A_i, i \leq n \). To see that \( g \in \mathcal{MA} [-1,1] \) we check that \( g^{-1}(\{a,1\}) \in \mathcal{A} \) for each \( a \in [0,1] \). Assume first that

\[
g^{-1}(\{a,1\}) = \bigcup_{i=1}^{k-1} A_i \cup g^{-1}(\{a,1\}) \bigcup_{i=1}^{k-1} A_i \cup (B \cap A_k)
\]

for some \( B \in \mathcal{A} \). Further, we have

\[
\bigcup_{i=1}^{k-1} A_i \cup (B \cap A_k) = \left( \bigcup_{i=1}^{k-1} A_i \right) \cup \left( \bigcup_{i=1}^{k-1} A_i \right) \in \mathcal{A}.
\]

If (1) is not satisfied one can easily see that either

\[
g^{-1}(\{a,1\}) = \bigcup_{i=1}^{k} A_i \in \mathcal{A}
\]

for some \( k \leq n \), or \( g^{-1}(\{a,1\}) = \emptyset \). \( \blacksquare \)

We now apply Lemma 3.1 to prove the following:

**Lemma 3.2.** Let \( w \in \mathcal{MA} \). Then the set \( \{ (l, u) \in (\mathcal{MA} [-1,1], \mathcal{MA} \times \mathcal{MA} [-1,1] : \inf |u + l - w| > 0 \} \) is residual in \( \mathcal{MA} [-1,1] \times \mathcal{MA} [-1,1] \), in fact open and dense.

**Proof.** Let \( u \in \mathcal{MA} [-1,1], l \in \mathcal{MA} [-1,1] \), \( \varepsilon > 0 \). Let \( n \in \mathbb{N} \) and \( n > \varepsilon^{-1} \). Define \( A_i = u^{-1}(\{(i-1)n^{-1}, in^{-1}\}) \) for \( i \in \{-n+1, \ldots, n-1\} \), \( A_n = u^{-1}(\{(n-1)n^{-1}, 1\}) \) and \( h = \sum_{i=n+1}^{n} (i-1)n^{-1} \chi_i \). Obviously, \( h \in \mathcal{MA} [-1,1], \|u - h\| < \varepsilon \) and \( |\text{Rg} h| < \aleph_0 \). Let \( l' = \max(\min(l, 1-\varepsilon), -1 + \varepsilon) \) and \( -l' + w = v \). The functions \( h \) and \( v \) satisfy the conditions of the hypothesis of Lemma 3.1 and, by that lemma, there exists \( g \in \mathcal{MA} [-1,1] \) such that \( |g| < \varepsilon \) and \( \delta = \inf |g + h + l' - w| > 0 \). Of course \( (l' + g, h) \in \mathcal{MA} [-1,1] \times \mathcal{MA} [-1,1] \) and for any pair \( (\tilde{l}, \tilde{u}) \in \mathcal{MA} [-1,1] \times \mathcal{MA} [-1,1] \) such that \( |\tilde{u} - h| < \delta/2 \) and \( |\tilde{l} - (l' + g)| < \delta/2 \) we have \( |\tilde{u} + \tilde{l} - w| > 0 \). \( \blacksquare \)
We also need the following dual lemma.

**Lemma 3.3.** Let \( w \in \mathcal{M}_\alpha \). Then the set \( \{ (l, u) \in \mathbb{M}_{[-1,1]} \mathcal{A} \times \mathbb{M}_{[-1,1]} \mathcal{A} : \inf |u + l - w| > 0 \} \) is residual in \( \mathbb{M}_{[-1,1]} \mathcal{A} \times \mathbb{M}_{[-1,1]} \mathcal{A} \), in fact open and dense. \( \blacksquare \)

From Lemmas 3.2 and 3.3 and the Baire category theorem we derive the following corollary.

**Corollary 3.4.** If \( \mathcal{A} \) is a \( \sigma \)-class of subsets of \( Z \), then for every countable family \( \mathcal{H} \subseteq \mathcal{M}_\alpha \mathcal{A} \cup \mathbb{M}_\alpha \mathcal{A} \) there exists \( f \in \mathbb{M}_{[-1,1]} \mathcal{A}(Z) + \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) such that \( f(t) \neq g(t) \) for every \( g \in \mathcal{H} \) and every \( t \in Z \). \( \blacksquare \)

We are now able to prove our first decomposition theorem. The scheme of the proof is, in fact, the same as for Theorem 3.2 of [CMPS].

**Theorem 3.5.** Let \( \mathcal{A} \) be a hereditary \( \sigma \)-class, and let \( Z \in T \) be uncountable and such that \( \mathcal{A}(Z) \) has a universal set in \( \mathcal{A}(C \times Z) \). Then there exists \( f \in \mathbb{M}_{[-1,1]} \mathcal{A}(Z) + \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) such that there is no countable partition of \( Z \), \( Z = \bigcup \{ Z_n : n \in \mathbb{N} \} \), such that \( f|Z_n \in \mathbb{M}(\mathcal{A}(Z)|Z_n) \cup \mathbb{M}(\mathcal{A}(Z)|Z_n) \) for every \( n \in \mathbb{N} \). In other words,

\[
\text{dec}(\mathbb{M}_{[-1,1]} \mathcal{A}(Z) + \mathbb{M}_{[-1,1]} \mathcal{A}(Z), R \mathbb{M}(\mathcal{A}(Z)) \cup R \mathbb{M}(\mathcal{A}(Z))) > \aleph_0.
\]

**Proof.** Let \( C \subseteq Z \) be homeomorphic to \( C \). Let \( F \in \mathbb{M}_{[-1,1]} \mathcal{A}(C \times Z) \) and \( G \in \mathbb{M}_{[-1,1]} \mathcal{A}(C \times Z) \) be universal functions for \( \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) and \( \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \), respectively. Let \( \pi = (\pi_1, \pi_2, \ldots) : C \to \mathbb{N} \) be a fixed homeomorphism. For every \( n \in \mathbb{N} \) let \( f_n \in \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) and \( g_n \in \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) be such that \( f_n(t) = F(\pi_n(t), t) \) and \( g_n(t) = G(\pi_n(t), t) \) for every \( t \in C \). By Corollary 3.4 there exists \( f \in \mathbb{M}_{[-1,1]} \mathcal{A}(Z) + \mathbb{M}_{[-1,1]} \mathcal{A}(Z) \) such that \( f(t) \neq g_n(t) \) and \( f(t) \neq f_n(t) \) for each \( t \in Z \).

Now assume that \( f = \bigcup \{ h_k : k \in \mathbb{N} \} \) and \( h_k \in R \mathbb{M}(\mathcal{A}(Z)) \cup R \mathbb{M}(\mathcal{A}(Z)) \) for each \( k \in \mathbb{N} \). Let \( h_k^* \in \mathcal{M}_\alpha \mathcal{A}(Z) \cup \mathbb{M}_\alpha \mathcal{A}(Z) \) be an extension of \( h_k \) (see Theorem 2.D). There exists \( c \in C \) such that for every \( k \in \mathbb{N} \) and for every \( t \in Z \) either \( h_k^*(t) = F(\pi_k(c), t) \) or \( h_k^*(t) = G(\pi_k(c), t) \). Thus \( f(c) \in \{ f_k(c) : k \in \mathbb{N} \} \cup \{ g_k(c) : k \in \mathbb{N} \} \), which is impossible. \( \blacksquare \)

**Corollary 3.6.** If \( Z \) is an uncountable Polish space, then for any \( \alpha < \omega_1 \)

\[
\text{dec}(\mathbb{L}_\alpha(Z) + U_\alpha(Z), R \mathbb{L}_\alpha(Z) \cup R U_\alpha(Z)) > \aleph_0. \quad \blacksquare
\]

**Corollary 3.7.** If \( Z \) is an uncountable Polish space, then for any \( n \in \mathbb{N} \)

\[
\text{dec}(\mathbb{M}_{\Sigma^1_n}(Z) + \mathbb{M}_{\Sigma^1_n}(Z), R \mathbb{M}_{\Sigma^1_n}(Z) \cup R \mathbb{M}_{\Sigma^1_n}(Z)) > \aleph_0. \quad \blacksquare
\]

We shall need the following lemma.
Lemma 3.8. If $A$ is a hereditary $\sigma$-class, $Z \in T$ and $A(Z)$ has a universal set in $A(C \times Z)$, then for every $n \in \mathbb{N}$ the class $\mathcal{M}_{[-n,n]} A(Z) + \mathcal{M}_{[-n,n]} A(C \times Z)$ has a universal function in $\mathcal{M}_{[-n,n]} A(C \times Z) + \mathcal{M}_{[-n,n]} A(C \times Z)$.

Proof. Let $\phi = (\phi_1, \phi_2) : C \to \mathcal{C}^2$ be any homeomorphism. Let $F \in \mathcal{M}_{[-n,n]} A(C \times Z)$ and $G \in \mathcal{M}_{[-n,n]} A(C \times Z)$ be universal functions for $\mathcal{M}_{[-n,n]} A(Z)$ and $\mathcal{M}_{[-n,n]} A(Z)$, respectively. Then $H(c, x) = F(\phi_1(c), x) + G(\phi_2(c), x)$ is a universal function for $\mathcal{M}_{[-n,n]} A(Z) + \mathcal{M}_{[-n,n]} A(Z)$.

Lemma 3.9. Let $A$ be a $\sigma$-class of subsets of some set $Z$. If $A \in r(A)$ then $\chi_A \in b\mathcal{M}A + b\mathcal{M}A$.

Proof. The family $\mathcal{S} = \{ A \in r(A) : \chi_A \in b\mathcal{M}A + b\mathcal{M}A \}$ is obviously closed under finite intersections and taking complements and at the same time $A \subseteq \mathcal{S}$. Thus $\mathcal{S} = r(A)$.

Lemma 3.10. Let $n \in \mathbb{N}$, let $A$ be a $\sigma$-class of subsets of some set $Z$ and $g \in \mathcal{M}_{[-N,N]} A + \mathcal{M}_{[-N,N]} A$, $N \in \mathbb{N}$. Then there exists $w \in b\mathcal{M}A + b\mathcal{M}A$ such that $\|g - w\| < 2^{-n+1}$ and

$$w = \sum_{i=-2^{n+1}}^{2^{n+1}} i \cdot 2^{-n} \chi_{A_i},$$

where the sets $A_i$ are pairwise disjoint and, for each $i$, $A_i \in r(A)$ and therefore $\chi_{A_i} \in b\mathcal{M}A + b\mathcal{M}A$.

Proof. Let $g = u + l$, $l \in \mathcal{M}_{[-N,N]} A$ and $u \in \mathcal{M}_{[-N,N]} A$. Let $B_i = l^{-1}((i \cdot 2^{-n}, (i+1) \cdot 2^{-n}))$ and $C_i = u^{-1}([i \cdot 2^{-n}, (i+1) \cdot 2^{-n}])$. The sets $B_i$ and $C_i$ belong to $r(A)$. Let

$$w = \sum_{i=-2^n}^{2^n} i \cdot 2^{-n} \chi_{B_i} + \sum_{i=-2^n}^{2^n} i \cdot 2^{-n} \chi_{C_i} = \sum_{j=-2^{n+1}}^{2^{n+1}} j \cdot 2^{-n} \chi_{A_j},$$

where $A_j = \bigcup\{B_i \cap C_k : i + k + 1 = j\}$. It follows from Lemma 3.9 that $w$ is the function we need.

Lemma 3.11. Let $A$ be a $\sigma$-class of subsets of some set $Z$. Let $f, g \in b\mathcal{M}A + b\mathcal{M}A$. Let $\varepsilon > 0$. Then there exists $h \in b\mathcal{M}A + b\mathcal{M}A$ such that $\|h - g\| < 3\varepsilon$ and $\inf |h - f| \geq \varepsilon/3$.

Proof. By Lemma 3.10 there exist $\phi = \sum_{i=1}^{N} c_i \chi_{A_i}$ and $\psi = \sum_{i=1}^{M} d_j \chi_{B_j}$ such that $A_i, B_j \in r(A)$, $i \leq N$, $j \leq M$, the sets $A_i$ are pairwise disjoint, the $B_j$ are pairwise disjoint, $\|f - \phi\| < \varepsilon/3$, and $\|g - \psi\| < \varepsilon/3$. Taking appropriate intersections we can assume that for each $j \leq M$ there exists
i \leq N$ such that $B_j \subseteq A_i$. Let $B_j \subseteq A_i$. Then we define $h$ on $B_j$ in the following way:

$$h|B_j = \begin{cases} 
\psi|B_j & \text{if } |d_j - c_i| \geq 2\varepsilon/3, \\
\psi|B_j + 2\varepsilon & \text{if } |d_j - c_i| < 2\varepsilon/3.
\end{cases}$$

**Lemma 3.12.** If $\mathcal{A}$ is a $\sigma$-class of subsets of $Z$ then for every countable family $\mathcal{G} \subseteq \text{cl}(\mathbb{M}_\mathcal{A} + b\mathbb{M}_\mathcal{A})$ there exists $g \in \text{cl}(\mathbb{M}_\mathcal{A} + b\mathbb{M}_\mathcal{A})$ such that \( \inf |f - g| > 0 \) for every $f \in \mathcal{G}$.

**Proof.** By Lemma 3.11 for any $f \in \mathcal{G}$ the family \( \{h \in \text{cl}(\mathbb{M}_\mathcal{A} + b\mathbb{M}_\mathcal{A}) : \inf |f - h| > 0 \}$ is residual in $\text{cl}(\mathbb{M}_\mathcal{A} + b\mathbb{M}_\mathcal{A})$. As the latter space is complete, the lemma follows by the Baire category theorem.

**Theorem 3.13.** Let $\mathcal{A}$ be a hereditary $\sigma$-class on $T$. Let $\mathcal{A}(Z)$, for some uncountable $Z \in T$, have a universal set in $\mathcal{A}(C \times Z)$. Then there exists a function $f \in \text{cl}(\mathbb{M}_\mathcal{A}(Z) + b\mathbb{M}_\mathcal{A}(Z))$ for which there is no countable partition $Z = \bigcup\{Z_m : m \in \mathbb{N}\}$ such that $f|Z_m \in \mathbb{M}_\mathcal{A}(Z_m) + \mathbb{M}_\mathcal{A}(Z_m)Z_m$ for each $m \in \mathbb{N}$. In other words,

$$\text{dec}(\text{cl}(\mathbb{M}_\mathcal{A}(Z) + b\mathbb{M}_\mathcal{A}(Z)), R(\mathbb{M}_\mathcal{A}(Z) + \mathbb{M}_\mathcal{A}(Z))) > \aleph_0.$$

**Proof.** Let $C \subseteq Z$ be homeomorphic to $C$. By Lemma 3.8 for each $n \in \mathbb{N}$ there exists $G_n \in \mathbb{M}_{[-n,n]}\mathcal{A}(C \times Z) + \mathbb{M}_{[-n,n]}\mathcal{A}(C \times Z)$ which is a universal function for $\mathbb{M}_{[-n,n]}\mathcal{A}(Z) + \mathbb{M}_{[-n,n]}\mathcal{A}(Z)$. Let $\pi = (\pi_1, \pi_2, \ldots) : C \to \mathbb{C}$ be a fixed homeomorphism. Let $g_n(t) = G_n(\pi_n(t), t)$ for every $t \in C$. It is easy to see that $g_n \in \mathbb{M}_{[-n,n]}\mathcal{A}(Z) + \mathbb{M}_{[-n,n]}\mathcal{A}(Z)$. By Lemma 3.12 there exists $f \in \text{cl}(\mathbb{M}_\mathcal{A}(Z) + b\mathbb{M}_\mathcal{A}(Z))$ such that $f(t) \neq g_n(t)$ for each $t \in Z$ and for each $n \in \mathbb{N}$.

Assume there is a partition $Z = \bigcup\{Z_m : m \in \mathbb{N}\}$ such that $f|Z_m \in \mathbb{M}_\mathcal{A}(Z_m) + \mathbb{M}_\mathcal{A}(Z_m)Z_m$ for each $m \in \mathbb{N}$. Let $f|Z_m = l_m + u_m$ where $l_m \in \mathbb{M}_\mathcal{A}(Z_m)Z_m$ and $u_m \in \mathbb{M}_\mathcal{A}(Z_m)Z_m$. Let $Z_{m,n} = \{x \in Z_m : |l_m(x)| \leq n \text{ and } |u_m(x)| \leq n\}$. Of course $\bigcup\{Z_{m,n} : n \in \mathbb{N}\} = Z_m$. Let $l_{m,n} \in \mathbb{M}_{[-n,n]}\mathcal{A}(Z)$ and $u_{m,n} \in \mathbb{M}_{[-n,n]}\mathcal{A}(Z)$ be extensions of $l_m|Z_{m,n}$ and $u_m|Z_{m,n}$, respectively (see Theorem 2.D). There exists $c \in C$ such that for each pair $m, n \in \mathbb{N}$ there exists $i(m,n) \in \mathbb{N}$ such that $l_{m,n}(t) + u_{m,n}(t) = G_{i(m,n)}(\pi_{i(m,n)}(c), t)$ for each $t \in \mathbb{N}$. Let $c \in Z_{m,n}$ for some $m, n \in \mathbb{N}$. Then $f|Z_{m,n} = l_{m,n}(c) + u_{m,n}(c) = g_{i(m,n)}(c)$, which is a contradiction.

**Corollary 3.14.** If $Z$ is an uncountable Polish space, then for any $\alpha < \omega_1$

$$\text{dec}(\text{cl}(\mathbb{L}_\alpha(Z) + b\mathbb{U}_\alpha(Z)), R(\mathbb{L}_\alpha(Z) + \mathbb{U}_\alpha(Z))) > \aleph_0.$$

**Corollary 3.15.** If $Z$ is an uncountable Polish space, then for any $n \in \mathbb{N}$

$$\text{dec}(\text{cl}(\mathbb{M}_\alpha\mathbb{S}_n^1(Z) + b\mathbb{M}_\alpha\mathbb{S}_n^1(Z)), R(\mathbb{M}_\alpha\mathbb{S}_n^1(Z) + \mathbb{M}_\alpha\mathbb{S}_n^1(Z))) > \aleph_0.$$


LEMMA 3.16. Let $\mathcal{A}$ be a $\sigma$-class of subsets of some set $Z$. If $f \in \text{cl}(b\mathcal{M}A + b\overline{\mathcal{M}}A)$, then for given $n \in \mathbb{N}$ and $\delta > 0$ there exists $v \in b\mathcal{M}A + b\overline{\mathcal{M}}A$ such that $\|f-v\| < 2^{-n+1}$. $\text{Rg} v \subseteq \{i \cdot 2^{-n} : i \in \mathbb{Z}\}$ and $v(x) < v(y)$ implies $f(x) < f(y) + \delta$ for all $x, y \in Z$.

Proof. Let $m \in \mathbb{N}$, $m > n + 1$ and $2^{-m+2} < \delta$. Let $g \in b\mathcal{M}A + b\overline{\mathcal{M}}A$ and $\|f-g\| < 2^{-m}$. By Lemma 3.10 there exists

$$w = \sum_{i=-M}^{M} i \cdot 2^{-m-1} \chi_{A_i},$$

where $M \in \mathbb{N}$, $\|g-w\| < 2^{-m}$, the sets $A_i$ are pairwise disjoint and, for each $i$, $A_i \in r(\mathcal{A})$ and thus $\chi_{A_i} \in b\mathcal{M}A + b\overline{\mathcal{M}}A$. Then $\|w-f\| < 2^{-m+1}$. Let $v(x) = [2^n w(x)] \cdot 2^{-n}$. Obviously $\text{Rg} v \subseteq \{i \cdot 2^{-n} : i \in \mathbb{Z}\}$ and $\|f-v\| < 2^{-n+1}$. If $v(x) < v(y)$ then $w(x) < w(y)$, whence $f(x) < f(y) + 2^{-m+2} < f(y) + \delta$. ■

THEOREM 3.17. Let $\mathcal{A}$ be a $\sigma$-class of subsets of some set $Z$. Then each $f \in \text{cl}(b\mathcal{M}A + b\overline{\mathcal{M}}A)$ can be expressed as the superposition $f = g \circ h$ where $h \in b\mathcal{M}_{[-1,1]}A + b\overline{\mathcal{M}}_{[-1,1]}A$ and $g \in C([-2,2]) \cap \text{VB}([-2,2])$.

Proof. Let $f \in \text{cl}(b\mathcal{M}A + b\overline{\mathcal{M}}A)$. Let $\|f\| < N \in \mathbb{N}$. The function $f$ is the uniform limit $f = \lim_{n \to \infty} v_n$ of functions $v_n \in b\mathcal{M}A + b\overline{\mathcal{M}}A$. By Lemma 3.16 we can assume that $\text{Rg} v_n \subseteq \{i/2^n : i \in \mathbb{Z}\}$, $\|f-v_n\| < 2^{-n+1}$ and $\sum_{i>n} \alpha_i$. Let $v_n(x) < v_n(y)$ implies $f(x) < f(y) + \delta_n$,

$\delta_n = (N + 1)^{-1} \cdot 2^{-2n-2}$. Let $v_n = l_n + u_n$, where $l_n \in b\mathcal{M}A$ and $u_n \in b\overline{\mathcal{M}}A$. Let $v(x) = (v_1(x), v_2(x), \ldots)$. Let $s$ be the function defined on $\text{Rg} v$ as $s(v_1(x), v_2(x), \ldots) = \lim_{n \to \infty} v_n(x)$. Of course $f = s \circ v$.

Now, let $\alpha_n$, $n \in \mathbb{N}$, satisfy the following conditions:

1. $\alpha_n > 0$, $n \in \mathbb{N}$;
2. $\alpha_n \sup |l_n| < 2^{-n}$ and $\alpha_n \sup |u_n| < 2^{-n}$;
3. $\alpha_n 2^{-n} > 2 \sum_{i>n} \alpha_i (\sup |l_i| + \sup |u_i|)$.

Let

$$\phi(v(x)) = \sum_{n=1}^{\infty} \alpha_n v_n(x) = \sum_{n=1}^{\infty} \alpha_n l_n(x) + \sum_{n=1}^{\infty} \alpha_n u_n(x).$$

The convergence of the series follows from 2°. From 3° it follows that $\phi$ is 1-1. We have $f = s \circ v = (s \circ \phi^{-1}) \circ (\phi \circ v)$. Of course $\phi \circ v \in \mathcal{M}_{[-1,1]}A + \overline{\mathcal{M}}_{[-1,1]}A$ and we put $h = \phi \circ v$.

Now we show that $\varphi = s \circ \phi^{-1}$ can be extended to a function $g \in \text{VB}([-2,2]) \cap \delta([-2,2])$. To this end it is enough to show that $\varphi$ can be extended to a continuous function $\tilde{\varphi}$ on $\text{cl}(\text{dom} \varphi)$ and that $\varphi$ is of bounded variation on its domain.
Let $s_k = \sum_{n=1}^{\infty} \alpha_n v_n(x_k)$ and $t_k = \sum_{n=1}^{\infty} \alpha_n v_n(x'_k)$ and $s_k \not\equiv q \not\equiv t_k$. We shall show that the sequences $\varphi(s_k)$ and $\varphi(t_k)$ converge to the same limit $\varphi(q)$. For each $n \in \mathbb{N}$ there is some $k(n)$ such that $v_n(x_k) = v_n(x'_k)$ for $k > k(n)$. Indeed, as $\varphi$ preserves the lexicographic order on $v(Z)$ the sequence $v_1(x_k)$ is nondecreasing and, as $|Rg v_1| < \aleph_0$, it is constant for $k \geq n_1$, for some $n_1 \in \mathbb{N}$. Then for $k \geq n_1$ the sequence $v_2(x_k)$ is nondecreasing and is constant for $k \geq n_2$ for some $n_2 \geq n_1$. Inductively we prove that for each $m \in \mathbb{N}$ the sequence $v_m(x_k)$ is constant for $k \geq n_m$ for some $n_m \geq n_{m-1}$. Similarly, putting $n'_0 = 0$, by induction we show that for each $m \in \mathbb{N}$ the sequence $v_m(x_k)$ is nonincreasing for $k \geq n'_m$ and constant for $k \geq n'_m$ for some $n'_m \geq n_{m-1}$.

If $v_1(x_{n_1}) < v_1(x'_{n'_1})$ then for all $k > \max(n_1, n'_1)$ we would have $v_1(x'_k) = v_1(x'_{n'_1}) > v_1(x_{n_1}) = v_1(x_k)$ and by 3°, $t_k - s_k > \varepsilon$ for some fixed $\varepsilon > 0$, which is a contradiction. Inductively $v_m(x_{n_m}) = v_m(x'_{n'_m})$ for each $m \in \mathbb{N}$.

As the sequence $v_m$ is uniformly convergent to $f$ and for any $m \in \mathbb{N}$ we have $v_m(x_k) = v_m(x'_k)$ for $k \geq \max(n_m, n'_m)$, the sequences $f(x_k) = \varphi(s_k)$ and $f(x'_k) = \varphi(t_k)$ are convergent and $f(x_k) - f(x'_k) \to 0$. Thus $\varphi$ can be extended to a continuous function $\tilde{\varphi}$ defined on $\text{cl}(\text{dom } \varphi)$.

Now we show that $\varphi$ is of bounded variation on its domain. Let $t_1 < \ldots < t_m$, where $t_i = \sum_{n=1}^{\infty} \alpha_n v_n(x_i) = \varphi(x_i)$, $i \leq m$. We shall estimate the sum $\sum_{i \in I}(\varphi(t_i) - \varphi(t_{i+1}))$, where $I \subseteq \{1, \ldots, m-1\}$ is the set of all $i$ for which $\varphi(t_i) - \varphi(t_{i+1}) > 0$. Let $A_n = \{i \in I : \min\{j : v_j(x_i) < v_j(x_{i+1})\} = n\}$. We have

$$\sum_{i \in I}(\varphi(t_i) - \varphi(t_{i+1})) = \sum_{n=1}^{\infty} \sum_{i \in A_n \cap I} (\varphi(t_i) - \varphi(t_{i+1})).$$

For $i \in A_n$ we have, by $(\ast)$, $\varphi(t_i) - \varphi(t_{i+1}) = f(x_i) - f(x_{i+1}) < \delta_n$, whence

$$\sum_{i \in I}(\varphi(t_i) - \varphi(t_{i+1})) < \sum_{n=1}^{\infty} 2(N + 1) \cdot 2^n \delta_n < 1.$$

Thus $\varphi$ is of bounded variation. ■

By [Ma, Th. 2] we have the following converse theorem:

**Theorem 3.18.** If $\mathcal{R}$ is any algebra of functions such that $\text{cl } \mathcal{R} = \mathcal{R}$ then for any $f \in b\mathcal{R}$ and any function $g$ continuous on a closed interval containing $\text{Rg } f$ we have $g \circ f \in \mathcal{R}$. ■

**Remark.** Corollary 3.14 and Theorem 3.17 show that Theorem 14 in [L] is false. That there was a mistake in its proof in [L] was already noticed by A. Lindenbaum himself in [L, corr.].

**Lemma 3.19.** Let $\mathcal{A}$ be a $\sigma$-class of subsets of some set $Z$. Let $X \subseteq Z$. Then
every \( f \in \text{cl}(bM(A(Z)|X) + b\overline{M}(A(Z)|X)) \) can be extended to a function \( f^* \in \text{cl}(bM(A(Z)) + b\overline{M}(A(Z))) \).

**Proof.** By Theorem 3.17, \( f = h \circ g \) where \( h \in C(R) \) and \( g \in bM(A(Z)|X) + b\overline{M}(A(Z)|X) \). The function \( g \) can be extended to some \( g^* \in bM(A(Z)) + b\overline{M}(A(Z)) \). Then \( f^* = h \circ g^* \in \text{cl}(bM(A(Z)) + b\overline{M}(A(Z))) \) by Theorem 3.18.

**Theorem 3.20.** Let \( A \) be a hereditary \( \sigma \)-class on \( T \) and suppose \( \Sigma^0_1(X) \subseteq A(X) \) for every \( X \in T \). Let \( Z \in T \) and suppose \( A(Z) \) has a universal set in \( A(C \times Z) \). Then there is an \( F \in MA'(N \times Z) \) which is a universal function for \( \text{cl}(bM(A(Z)) + b\overline{M}(A(Z))) \).

**Proof.** By Lemma 3.8 there is a function \( H \in M[-1,1]A(C \times Z) + \overline{M}[-1,1]A(C \times Z) \) universal for \( M[-1,1]A(Z) + \overline{M}[-1,1]A(Z) \). Let \( \phi : N \to C \) be a continuous surjection ([Ku, 37, I, Th. 1]). Let \( G(w,x) = H(\phi(w),x) \) for \( w \in N \) and \( x \in Z \). Then \( G \in MA'(N \times Z) \) because \( H \in MA'(C \times Z) \) and, as is easy to see, \( A' \) is closed under continuous substitutions. Let \( \psi : N \to C([-2,2]) \) be a continuous surjection ([Ku, 37, I, Th. 1]). We write \( \psi_w(\cdot) \) for \( \psi(w) \) in the sequel. Let \( (\xi_1, \xi_2) \) be any homeomorphism from \( N \) onto \( N^2 \). Let \( F(w,x) = \psi_{\xi_1(w)}(G(\xi_2(w),x)) \). By Theorems 3.17 and 3.18, \( F \) is universal for \( \text{cl}(bM(A(Z)) + b\overline{M}(A(Z))) \).

We show that \( F \in MA'(N \times Z) \). Let \( \psi(w,s) = \psi_{\xi_1(w)}(s), w \in N, s \in [-2,2], \tilde{G}(w,x) = G(\xi_2(w),x) \) and \( \phi(w,x) = (w, \tilde{G}(w,x)) \). We have \( F(w,x) = \psi(\phi(w,x)) \). Then \( \tilde{G} \in MA'(N \times Z) \) because \( A' \) is closed under continuous substitutions. An easy argument shows that \( \psi \) is continuous. Finally, \( \Phi^{-1}(U) \in A'(N \times Z) \) for any open set \( U \subseteq N \times R \): indeed, as \( U = \bigcup_{i=1}^{\infty} V_i \times W_i \), where the \( V_i \) are open in \( N \) and \( W_i \) are open in \( R \), we have

\[
\Phi^{-1}(U) = \bigcup_{i=1}^{\infty} (V_i \times Z) \cap \tilde{G}^{-1}(W_i) \in A'(N \times Z).
\]

In the next theorem we add new assumptions on the hereditary \( \sigma \)-class \( A \) and the family \( T \). Namely, we assume that \( T \) satisfies the following stronger form of (i):

\((i^*)\) if \( X \subseteq Z \in T \) and \( X \), as a subspace of \( Z \), is completely metrizable by some metric \( \rho \), then \((X,\rho) \in T\).

We then assume that \( A \) satisfies for any \( Z \in T \) and \( X \subseteq Z \):

\((II^*)\) \( A(Z)|X = A(X) \) where \( X \) is considered with any metric \( \rho \) such that \((X,\rho) \in T \) is topologically a subspace of \( Z \).

Assume also \( \Sigma^0_1 \subseteq A \).
However, the conditions imposed on $A$ are not very restrictive as the classes $\Sigma^0_n$, $\Sigma^0_\infty$ still satisfy them.

**Theorem 3.21.** Let $A$ be a hereditary $\sigma$-class satisfying (II$^\ast$). Let $Z \in T$ be uncountable and suppose $A(Z)$ has a universal set in $A(C \times Z)$. Then there exists $g \in MA'(Z)$ for which there is no countable partition $Z = \bigcup\{Z_n : n \in \mathbb{N}\}$ such that $g|Z_n \in \text{cl}(bM(A(Z)|Z_n) + b\overline{M}(A(Z)|Z_n))$ for each $n \in \mathbb{N}$. In other words,

$$\text{dec}(MA'(Z), R(\text{cl}(bM(A(Z) + b\overline{M}(A(Z)))) > \aleph_0.$$

**Proof.** Let $A'$ be any subset of $Z$ homeomorphic to $N'$ ([Ku, 36, IV, Cor. 2]). Let $\varphi = (\varphi_1, \varphi_2, \ldots) : N' \to N$ be any homeomorphism. By Theorem 3.20 there exists a universal function $F \in MA'(N' \times Z)$. Let $F_n(s, x) = F(\varphi_n(s), x)$ for $s \in N'$ and $x \in Z$. Then $F_n \in MA'(N' \times Z)$ and thus $f_n : N' \to \mathbb{R}$ defined as $f_n(s) = F_n(s, s)$ belongs to $MA'(N')$ because, by our assumption on $A$, $A'(N' \times N') = A'(N' \times Z)(N' \times N')$. By Corollary 2.1 and the fact that $A'(N') = A'(Z)|N'$ and $N' \ni (A(Z))_s$, $f_n$ can be extended to a function $f_n^* \in MA'(Z)$. By Theorem 2.1 there exists $g \in MA'(Z)$ such that $g(x) \neq f_n^*(x)$ for each $x \in Z$ and $n \in \mathbb{N}$.

Now assume that $g = \bigcup\{g_n : n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$, $g_n \in \text{cl}(bM(A(Z)|\text{dom } g_n) + b\overline{M}(A(Z)|\text{dom } g_n))$. By Lemma 3.19 each $g_n$ has an extension $g_n^* \in \text{cl}(bM(A(Z) + b\overline{M}(A(Z)))$ for all $n \in \mathbb{N}$. There is an $\delta \in N'$ such that $F_n(s, x) = g_n^*(x)$ for each $n \in \mathbb{N}$. But then $f_n(s) = g_n(s)$ for each $n \in \mathbb{N}$ and, as $g(s) \in \{g_n(s) : n \in \mathbb{N}\}$, we obtain $g(s) = f_n^*(s)$ for some $n_0 \in \mathbb{N}$, which is a contradiction. $\blacksquare$

For any uncountable Polish space $Z$ we derive from Theorem 3.21 the following immediate corollaries.

**Corollary 3.22.**

$$\text{dec}(B_{\alpha + 1}(Z), R(\text{cl}(bL_\alpha(Z) + bU_\alpha(Z)))) > \aleph_0.$$

**Corollary 3.23.**

$$\text{dec}(M_{\Sigma^1_{\alpha + 1}}(Z), R(\text{cl}(bM_{\Sigma^1_\alpha}(Z) + b\overline{M}_{\Sigma^1_\alpha}(Z))) > \aleph_0.$$

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