The Vietoris system in strong shape and strong homology

by

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Abstract. We show that the Vietoris system of a space is isomorphic to a strong expansion of that space in the Steenrod homotopy category, and from this we derive a simple description of strong homology. It is proved that in ZFC strong homology does not have compact supports, and that enforcing compact supports by taking limits leads to a homology functor that does not factor over the strong shape category. For compact Hausdorff spaces strong homology is proved to be isomorphic to Massey's homology.

0. Introduction. In 1952 C. H. Dowker [5] proved that the cohomology groups of a topological space based on the Čech and the Vietoris constructions are isomorphic. However, since the Čech groups work very well in the context of *co*homology, the Vietoris approach was abandoned until 1973 when it was taken up by Porter [23, 24, 25] and three years later by Edwards and Hastings [6, p. 250]. The last two authors observed that only the Vietoris system has the rigidity properties necessary for the construction of exact (generalized) *homology* theories for compact metric spaces with properties similar to Steenrod–Sitnikov homology. In case of ordinary homology the domain of definition was extended to arbitrary topological spaces by Miminoshvili in [22] and independently by Lisica and Mardešić in [11, 12, 13], using resolutions instead of one of the two classical systems; the resulting theory is known as strong homology. A similar development went on in shape theory: The Čech system served its purpose in ordinary shape theory [18, p. 49] but had to be replaced by resolutions in strong shape theory [4, 10]. The disadvantage of resolutions is their lack of naturality.

In Section 1 of this paper we are going to prove that resolutions may be replaced by the Vietoris system in the context of strong shape theory. This will allow us to give a rather nice and simple presentation of strong homology, making this subject accessible for non-specialists in strong shape theory. We describe this form of homology in Section 2 and give self-contained proofs of the Eilenberg–Steenrod axioms. For compact Hausdorff spaces this approach is close to Massey's homology, and we give a proof of isomorphism in Section 3. There we also consider strong homology with compact supports. It has some advantages over the usual form of strong homology (it occurs in Sitnikov's duality theorem, and the question whether the homology functors in negative dimensions are trivial is decidable), but also some disadvantages (most notably: it does not factor over the strong shape category). We show that strong homology in its usual form does not have compact supports (in ZFC).

1. The Vietoris system. We consider the set M_X of all continuous pseudometrics (hereafter simply called *metrics*) on a given topological space X; M_X is directed if we define $d \leq d' \Leftrightarrow \forall x, y \in X : d(x,y) \leq d'(x,y)$. The *Vietoris nerve* of a metric d is the simplicial complex whose n-simplices are the finite sets $A \subseteq X$ with d-diameter $\delta_d(A)$ at most 1. By $N(d)$ we denote the Vietoris nerve with the *metric* topology [18, p. 301]. When d varies in M_X the polyhedra $N(d)$ form an inverse system $\mathbf{V}(X)$ with inclusions as bonding maps, which is called the *Vietoris system* of X. The Vietoris system has the interesting property of depending strictly functorially on X: Every continuous map $f : X \to Y$ induces a morphism $\mathbf{V}(f) : \mathbf{V}(X) \to \mathbf{V}(Y)$ in **pro-Top** (even more: in **inv-Top**) composed of the nondecreasing index map $f^* : M_Y \to M_X$, $f^*d(x,x') := d(f(x), f(x'))$, and the simplicial maps $f_* : N(f^*d) \to N(d)$ mapping every vertex x onto $f(x)$. This property distinguishes the Vietoris system from related constructions working only on the homotopy level (such as the Čech system) or being natural only up to coherent homotopy (such as resolutions), and makes it particularly useful for strong shape theory. There one needs to assign a *strong ANR-expansion* [15, Def. 1] to every space to obtain a functor $\mathbf{L} : \mathbf{HTop} \to \mathbf{H(pro\text{-}Top)}$ into the *Steenrod homotopy category* [4, Def. 1.11]; the strong shape category **ssh** is then defined by $\mathrm{Ob}(\mathbf{ssh}) := \mathrm{Ob}(\mathbf{HTop})$ and $\mathbf{ssh}(X,Y) := \mathbf{H(pro\text{-}Top)}(\mathbf{L}(X), \mathbf{L}(Y))$. A strong ANR-expansion of a space X consists of an inverse system of ANR-spaces $\mathbf{L}(X) = \{ f_\lambda^\mu : P_\mu \to P_\lambda \mid \mu \geq \lambda \in \Lambda \}$ and a morphism $\{ g_\lambda \} : X \to \mathbf{L}(X) \in \mathbf{pro\text{-}Top}$ satisfying the following two conditions:

(a) Any map $h : X \to Y$ into an ANR-space Y factors over some g_λ up to homotopy.

(b) For any two maps $h_0, h_1 : P_\lambda \to Y$ into an ANR-space and any

homotopy $H : h_0 g_\lambda \simeq h_1 g_\lambda$ there exist an index $\mu \geq \lambda$ and a homotopy $G : h_0 f_\lambda^\mu \simeq h_1 f_\lambda^\mu$ with $G(g_\mu \times \mathrm{id}) \simeq H$ relative $X \times \dot{I}$.

The main result of this section is to show that $\mathbf{L}(X)$ is isomorphic to the Vietoris system $\mathbf{V}(X)$ in the Steenrod homotopy category.

R e m a r k. The original construction of the Vietoris system in [23] used open coverings $\mathcal{U}$ of X instead of metrics as index set and specified the simplices of $N(\mathcal{U})$ to be the $\mathcal{U}$-small, finite subsets of X. Using the fact that each open covering $\mathcal{U}$ of a paracompact space X admits a metric d such that the open balls of radius 1 with respect to d are $\mathcal{U}$-small the reader may convince himself that in the paracompact case the two constructions are equivalent. For general spaces open coverings have to be replaced by normal ones.

DEFINITION 1. Let d be a metric on a space X and $N(d)$ the Vietoris nerve of d. A map $q : X \to N(d)$ is *barycentric* if $\delta_d(\{x\} \cup q^{-1}(\mathrm{st}\ x)) \leq 1/3$ for each $x \in X$.

In the formula above the letter x denotes once a point of X and secondly the corresponding vertex of $N(d)$.

PROPOSITION 1. *Every metric d on an arbitrary space X admits a barycentric map $q : X \to N(d)$. If $q_0, \ldots, q_n : X \to N(d)$ is a finite collection of barycentric maps, then for fixed $x \in X$ the points $q_0(x), \ldots, q_n(x) \in N(d)$ lie in a single simplex, so that we can define a map $\Gamma : X \times \Delta^n \to N(d)$ by $\Gamma(x, t_0, \ldots, t_n) := \sum_{k=0}^n t_k q_k(x)$.*

P r o o f. To construct a barycentric map $q : X \to N(d)$ we take a locally finite partition of unity $\{\varphi_\iota \mid \iota \in J\}$ on X with $\delta_d(\varphi_\iota^{-1}]0,1]) \leq 1/6$ for every index ι and choose $x_\iota \in X$ with $\varphi_\iota(x_\iota) > 0$. For $x \in X$ we denote by $J_x \subseteq J$ the finite set of all indices ι with $\varphi_\iota(x) > 0$. If ι and κ belong to J_x we have $d(x, x_\iota), d(x, x_\kappa) \leq 1/6$ and therefore $d(x_\iota, x_\kappa) \leq 1/3 \leq 1$. Hence $\{x_\iota \mid \iota \in J_x\}$ spans a simplex of $N(d)$ and we can define

$$q(x) := \sum_\iota \varphi_\iota(x) x_\iota \,.$$

We claim that this map q is barycentric. If $q(y)$ belongs to st x, then there must be an index ι with $x_\iota = x$ and $\varphi_\iota(y) > 0$. By construction we have $\varphi_\iota(x_\iota) > 0$ and therefore $d(y, x) \leq 1/6$. If y' is another point of $q^{-1}(\mathrm{st}\ x)$ then $d(y, y') \leq d(y, x) + d(y', x) \leq 1/3$ and therefore the map q is barycentric.

The second part of Proposition 1 follows from Lemma 1, applied to $x_0 = \ldots = x_n := x$:

LEMMA 1. *If $q_0, \ldots, q_n : X \to N(d)$ is a finite collection of barycentric maps and if $\{x_0, \ldots, x_n\} \subseteq X$ has diameter at most $1/3$, then there is*

a simplex of $N(d)$ containing the vertices $x_0, \ldots, x_n$ as well as the points $q_0(x_0), \ldots, q_n(x_n)$.

P r o o f. Let $y_0^k, \ldots, y_{n(k)}^k$ denote the vertices of the open simplex of $N(d)$ containing $q_k(x_k)$. By Definition 1 we have $d(x_k, y_m^k) \leq 1/3$, therefore the finite subset of X formed by all the points x_k and y_m^k has diameter at most 1 and spans a simplex of $N(d)$. ∎

We observe that for $d' \geq d$, $N(d')$ is a subcomplex of $N(d)$, and for every barycentric map $q : X \to N(d')$ the composed map $X \to N(d') \hookrightarrow N(d)$ is also barycentric. We denote by N_X the Vietoris nerve of the trivial metric 0; N_X contains all other Vietoris nerves.

LEMMA 2. *For every metric d on a space X and every open covering $\mathcal{U}$ of $N(d)$ there is a metric $d' \geq d$ on X such that the inclusion map $N(d') \hookrightarrow N(d)$ is homotopic to a map γ which maps each simplex of $N(d')$ to a $\mathcal{U}$-small subset of $N(d)$.*

P r o o f. We may suppose that $\mathcal{U}$ consists of balls with respect to the canonical linear metric on $N(d)$. If $\{\psi_\iota\}$ is a locally finite partition of unity on $N(d)$ subordinate to a star refinement of $\mathcal{U}$, then $\varrho(x, y) := \sum_\iota |\psi_\iota(x) - \psi_\iota(y)|$ is a metric on $N(d)$ such that every subset with diameter at most 1 is $\mathcal{U}$-small. We choose a barycentric map $q : X \to N(d)$ and set $d'(x, y) := \varrho(q(x), q(y)) + 3d(x, y)$. If the points $x_0, \ldots, x_n$ span a simplex of $N(d')$ then in particular $\delta_d\{x_0, \ldots, x_n\} \leq 1/3$, and Lemma 1 tells that the points $x_0, \ldots, x_n, q(x_0), \ldots, q(x_n)$ belong to a single simplex of $N(d)$. Therefore we can define a homotopy $H : N(d') \times I \to N(d)$ by

$$H(z, s) := sz + (1 - s) \sum_{x \in X} \varphi_x(z) q(x).$$

Here $\varphi_x(z)$ means the barycentric coordinate of the point $z \in N(d')$ with respect to the vertex x. Then $H_1 = \mathrm{id}$, and we claim that $\gamma := H_0$ has the required property. Indeed, if the points $x_0, \ldots, x_n$ span a simplex σ of $N(d')$, then $\varrho(q(x_k), q(x_m)) \leq 1$, and therefore the set $\{q(x_0), \ldots, q(x_n)\}$ is $\mathcal{U}$-small. Since $\gamma(\sigma)$ is the convex hull of this set it is also $\mathcal{U}$-small. ∎

Since barycentric maps are not uniquely determined we cannot hope to build a morphism $X \to \mathbf{V}(X)$ in **pro-Top** out of them. In fact, such a morphism would give rise to a limit map $q : X \to \lim \mathbf{V}(X) = \bigcap_{d \in M_X} N(d)$ barycentric with respect to any metric d. If X is completely regular, then the limit space is the set X with the discrete topology, and q would be the identity map, which is absurd if X is not discrete. As remedy we perform the following construction: Let $\mathcal{M}_X$ denote the simplicial complex whose vertices are all metrics on X, a finite collection of metrics spanning a simplex if and only if it is linearly ordered by the natural order relation; by abuse of

notation we denote the geometric realization of $\mathcal{M}_X$ by the same symbol. For any metric d, $\mathcal{M}(d)$ is the full subcomplex spanned by all metrics $d' \geq d$. Λ denotes the directed set of all finite subcomplexes of $\mathcal{M}_X$, ordered by inclusion, and for $K \in \Lambda$ we define X_K to be the function space

$$X_K := \{\omega : K \to N_X \mid \forall d : \omega(K \cap \mathcal{M}(d)) \subseteq N(d)\}$$

with the compact-open topology. By induction on the number of simplices of K one can show that X_K is an ANR-space. These spaces constitute an inverse system $\mathbf{V}'(X) := \{f_K^L : X_L \to X_K \mid K \subseteq L \in \Lambda\}$, whose bonding maps are the appropriate restriction maps. $\mathbf{V}'(X)$ depends functorially on X.

PROPOSITION 2. *In the Steenrod homotopy category* $\mathbf{V}'(X)$ *is naturally isomorphic to the Vietoris system* $\mathbf{V}(X)$.

P r o o f. The subset $\Lambda_0 \subseteq \Lambda$ formed by all finite, full subcomplexes $K \subseteq \mathcal{M}_X$ which contain a largest vertex d_K is cofinal, and so is the image of the nondecreasing index map $\Lambda_0 \to M_X$, $K \mapsto d_K$. For each $K \in \Lambda_0$ let $i_K : N(d_K) \hookrightarrow X_K$ be the natural embedding onto the set of constant mappings. The system of these maps determines a level morphism $\mathbf{i} : \mathbf{V}(X) \to \mathbf{V}'(X)$ in **pro-Top**, and since each i_K is a strong deformation retraction, $\mathbf{i}$ is a trivial cofibration. ∎

For every $d \in M_X$ we choose a fixed barycentric map $q_d : X \to N(d)$, and for every simplex σ of $\mathcal{M}_X$, given as a sequence of metrics $d_0 \leq \ldots \leq d_n$ with corresponding barycentric maps $q_k : X \to N(d_k) \subseteq N(d_0)$, we denote by $\Gamma_\sigma : X \times \Delta^n \to N(d_0)$ the map described in Proposition 1. There is a family of maps $q_K : X \to X_K$ such that for $K = \sigma$ the map q_K is the adjoint of Γ_σ and such that for $K \subseteq L$ the condition $f_K^L q_L = q_K$ holds. This defines a morphism $\mathbf{q} : X \to \mathbf{V}'(X)$ in **pro-Top**, which is natural up to homotopy.

PROPOSITION 3. $\mathbf{q} : X \to \mathbf{V}'(X)$ *is a strong expansion.*

From Propositions 2 and 3 we derive the main result:

THEOREM 1. *For every space X the Vietoris system $\mathbf{V}(X)$ is naturally isomorphic in the Steenrod homotopy category to a strong ANR-expansion of X.*

P r o o f of P r o p o s i t i o n 3. 1) Suppose a map $h : X \to P$ into an ANR-space P is given; we construct a metric d on X and a map $g : N(d) \to P$ with $gq_d \simeq h$. P may be considered as a bounded subset of a normed vector space Y such that P is closed in its convex hull Y_0. There is a covering $\mathcal{U}$ of P consisting of open, convex subsets of Y_0 which are small enough to ensure the existence of a retraction map $r : \bigcup \mathcal{U} \to P$. As in the proof of Proposition 2 we construct a metric ϱ on P such that every subset of P

with diameter not exceeding 1 is $\mathcal{U}$-small. We set $d(x,y) := \varrho(h(x), h(y))$ and define $g' : N(d) \to \bigcup \mathcal{U}$ by

$$g'(z) := \sum_{x \in X} \varphi_x(z)h(x),$$

φ_x being the barycentric coordinate function with respect to the vertex x. Then g' is continuous because $h(X)$ is bounded. We set $g := rg' : N(d) \to P$. Let us consider a point $x \in X$ and denote by A_x the finite set of points $y \in X$ with $\varphi_y(q_d(x)) > 0$. Lemma 1 ensures that $\{x\} \cup A_x$ spans a simplex of $N(d)$, hence $\{h(x)\} \cup h(A_x)$ is $\mathcal{U}$-small and therefore the points $h(x)$ and $g'(q_d(x))$ are $\mathcal{U}$-near. Now we can define a homotopy $H : h \simeq gq_d$ by $H(x,s) := r(sg'(q_d) + (1-s)h(x))$.

2) We consider two maps $g_0, g_1 : N(d) \to P$ into an ANR-space P and a homotopy $H : g_0 q_d \simeq g_1 q_d$, these data being prescribed, and a metric $e \geq d$ at our disposal. Let $\chi_d^e : N(e) \hookrightarrow N(d)$ denote the natural inclusion map and let $\sigma \subseteq K$ denote the 1-simplex connecting d and e. The map $\Gamma_\sigma : X \times \Delta^1 \to N(d)$ may be considered as a homotopy $\Gamma_\sigma : q_d \simeq \chi_d^e q_e$. Now we claim that e can be chosen such that there is a homotopy $G : g_0 \chi_d^e \simeq g_1 \chi_d^e$ with $g_0 \Gamma_\sigma \circ G(q_e \times \mathrm{id}) \circ g_1 \Gamma_\sigma^{-1} \simeq H$.

To this end we take an embedding $P \subseteq Y$, a covering $\mathcal{U}$, a retraction map $r : \bigcup \mathcal{U} \to P$ and a metric ϱ on P as above; and in addition, taking into account Lemma 2 we assume that g_0 and g_1 map every simplex of $N(d)$ into a $\mathcal{U}$-small subset of P. Now we define a metric e on X by

$$e(x,y) := \sup_{t \in I} \varrho(H(x,t), H(y,t)) + 3d(x,y).$$

For abbreviation we set $q := q_d$, $q' := q_e$ and define

$$G(z,s) := \begin{cases} \varrho((1-3s)g_0(z) + 3s \sum_{x \in X} \varphi_x(z)g_0(q(x))), & s \leq 1/3, \\ \varrho(\sum_{x \in X} \varphi_x(z)H(x, 3s-1)), & 1/3 \leq s \leq 2/3, \\ \varrho((3s-2)g_1(z) + 3(1-s) \sum_{x \in X} \varphi_x(z)g_1(q(x))), & s \geq 2/3. \end{cases}$$

We observe that if $x_0, \ldots, x_n$ are the vertices of the open simplex carrying z, then $\delta_d\{x_0, \ldots, x_n\} \leq 1/3$. Lemma 1 ensures the existence of a simplex of $N(d)$ containing z and $q(x_0), \ldots, q(x_n)$, and therefore $g_0(z), g_0(q(x_0)), \ldots$ $\ldots, g_0(q(x_n))$ are $\mathcal{U}$-near points of P. This shows that $G(z,s)$ is well defined for $s \leq 1/3$, and the other cases are handled similarly.

We claim that G satisfies $G(q' \times \mathrm{id}) \simeq g_0 \Gamma_\sigma^{-1} \circ H \circ g_1 \Gamma_\sigma$. The left hand homotopy is given by

$$\begin{cases} \varrho((1-3s)g_0(q'(y)) + 3s \sum_{x \in X} \varphi_x(q'(y))g_0(q(x))), & s \leq 1/3, \\ \varrho(\sum_{x \in X} \varphi_x(q'(y))H(x, 3s-1)), & 1/3 \leq s \leq 2/3, \\ \varrho((3s-2)g_1(q'(y)) + 3(1-s) \sum_{x \in X} \varphi_x(q'(y))g_1(q(x))), & s \geq 2/3, \end{cases}$$

and the right hand one by

$$\begin{cases} g_0((1-3s)q'(y) + 3sq(y)), & s \leq 1/3, \\ H(y, 3s-1), & 1/3 \leq s \leq 2/3, \\ g_1((3s-2)q'(y) + 3(1-s)q(y)), & s \geq 2/3. \end{cases}$$

Now let $x_0, \ldots, x_n$ be the vertices of the open simplex of $N(e)$ carrying $q'(y)$. Since q' is barycentric we have $\delta_e\{y, x_0, \ldots, x_n\} \leq 2/3$ and hence $\delta_d\{y, x_0, \ldots, x_n\} \leq 2/9$, and Lemma 1 implies that the points $q(y), q'(y),$ $q(x_0), \ldots, q(x_n)$ belong to a common simplex of $N(d)$. This means that for $s \leq 1/3$ the points $g_0((1-3s)q'(y) + 3sq(y))$ and $(1-3s)g_0(q'(y)) +$ $3s\sum_{x \in X} \varphi_x(q'(y))g_0(q(x))$ are $\mathcal{U}$-near and hence can be joined by a linear path $\Phi(y, s, -)$ in $\bigcup \mathcal{U}$, and similarly for $1/3 \leq s \leq 2/3$ and $s \geq 2/3$. Then $r\Phi : X \times I^2 \to P$ is a 2-homotopy connecting $G(q' \times \mathrm{id})$ and $g_0 \Gamma_\sigma^{-1} \circ H \circ g_1 \Gamma$.

3) Suppose we are given $K \in \Lambda$, two maps $g_0, g_1 : X_K \to P \in \mathrm{ANR}$ and a homotopy $H : g_0 q_K \simeq g_1 q_K$; we have to construct $L \in \Lambda$ with $L \supseteq K$ and a homotopy $G : g_0 f_K^L \simeq g_1 f_K^L$ with $G(q_L \times \mathrm{id}) \simeq H$. We may assume that the complex K is full and contains a largest vertex d. Then the map $f_d^K : X_K \to X_d$ is a homotopy equivalence whose inverse $a : X_d \to X_K$ embeds X_d in X_K as a set of constant mappings and the homotopy $A : \mathrm{id} \simeq af_d^K$ is determined by a parameter transformation; the equation $f_d^K a = \mathrm{id}$ holds. Now we apply step 2 to the mappings $g_0 a, g_1 a : X_d = N(d) \to P$ and the homotopy $(g_0 A(q_K \times \mathrm{id}))^{-1} \circ H \circ (g_1 A(q_K \times \mathrm{id})) : g_0 a q_d \simeq g_1 a q_d$ and obtain a metric $e \geq d$ and a homotopy $G' : g_0 a \chi_d^e \simeq g_1 a \chi_d^e$ with $(g_0 A(q_K \times \mathrm{id}))^{-1} \circ H \circ (g_1 A(q_K \times \mathrm{id})) \simeq g_0 a \Gamma_\sigma \circ G'(q_e \times \mathrm{id}) \circ g_1 a \Gamma_\sigma^{-1}$. Let L be the full subcomplex generated by K and the vertex e. We define a homotopy $F : f_d^L \simeq \chi_d^e f_e^L$ by $F(\omega, s) := \omega(se + (1-s)d)$ and observe that $F(q_L \times \mathrm{id}) = \Gamma_\sigma$. Now the homotopy $G := g_0(A(f_K^L \times \mathrm{id}) \circ aF) \circ G'(f_e^L \times \mathrm{id}) \circ g_1(A(f_K^L \times \mathrm{id}) \circ aF)^{-1} : g_0 f_K^L \simeq g_1 f_K^L$ has the required property. ∎

2. Strong homology. Let $\mathbf{C}_*$ be a filtered chain complex of R-modules, which means we are given a chain complex C_* over a commutative, unitarian ring R with a decreasing filtration of subcomplexes $C_*(\lambda)$, where λ ranges over some nonempty directed set Λ. This ordered set determines a simplicial complex whose simplices are the finite increasing sequences of indices; we denote by Λ_* the corresponding chain complex based on *ordered* simplices with coefficients in R. Λ_* is filtered by the subcomplexes $\Lambda_*(\lambda)$ of chains of simplices $\langle \mu_0 \leq \ldots \leq \mu_p \rangle$ with $\mu_0 \geq \lambda$.

DEFINITION 2. The *total complex* Γ_* of $\mathbf{C}_*$ has as n-simplices the graded homomorphisms of degree n, $\varphi \in \mathrm{Hom}(\Lambda_*, C_{*+n})$, with $\varphi(\Lambda_*(\lambda)) \subseteq C_{*+n}(\lambda)$ for each index λ. The differential of Γ_* is defined by $(\partial\varphi)(x) := \partial(\varphi(x)) - (-1)^{\deg \varphi}\varphi(\partial x)$.

In plain terms an n-chain of Γ_* is a function assigning to each sequence of indices $\lambda_0 \leq \ldots \leq \lambda_p$ a $p + n$-chain of $C_*(\lambda_0)$, so Definition 2 coincides with the one given in [17, Section 3] for arbitrary inverse systems of chain complexes. We restrict our attention to filtered complexes, because it simplifies notation and is sufficient for the applications we have in mind. The differential defined in [17] coincides with ours apart from a change of sign, which does not affect the homology. Our sign convention is taken from the function complex definition familiar in homological algebra. Interpreting total complexes as function complexes will turn out to be advantageous when it comes to induced mappings.

LEMMA 3. *For every directed set Λ the chain complex Λ_* has the homology of a point.*

P r o o f. We consider an n-chain $x = \sum_i r_i \sigma_i$ of Λ; for $n = 0$ we suppose that x is reduced, $\sum_i r_i = 0$, and for $n > 0$ we assume $\partial x = 0$. If λ is an element of Λ dominating all vertices of the simplices σ_i, then we can form $n+1$-simplices $\widehat{\sigma}_i$ by adding the vertex λ to σ_i and set $y := (-1)^{n+1} \sum_i r_i \widehat{\sigma}_i$. This implies $\partial y = x$. ∎

Suppose we are given two filtered complexes $\mathbf{C}'_*$ and $\mathbf{C}_*$ and a chain map $f : C'_* \to C_*$ such that for each $\lambda \in \Lambda$ there exists $\alpha(\lambda) \in \Lambda'$ with

$$(1) \qquad\qquad f(C'_*(\alpha(\lambda))) \subseteq C_*(\lambda).$$

The map α is not assumed to be monotone. Lemma 3 tells us that for each $\lambda \in \Lambda$ the chain complex $\sum_{\mu \geq \lambda} \Lambda'_*(\alpha(\mu))$ has the homology of a point, because it is the complex of the directed set $\{\lambda' \in \Lambda \mid \exists \mu \geq \lambda : \lambda' \geq \alpha(\mu)\}$. Hence we can construct a chain map $\beta : \Lambda_* \to \Lambda'_*$ with

$$(2) \qquad\qquad \beta(\Lambda_*(\lambda)) \subseteq \sum_{\mu \geq \lambda} \Lambda'_*(\alpha(\mu))$$

for each λ, and any two of them are chain homotopic. Then we can define a chain map $F : \Gamma'_* \to \Gamma_*$ by

$$(3) \qquad\qquad F(\varphi) := f\varphi\beta.$$

LEMMA 4. *The chain homotopy class of F does not depend on the particular choice of the maps α and β, as long as conditions (1) and (2) are satisfied. The assignment $f \mapsto F$ is functorial up to chain homotopy.*

This follows easily from Lemma 3.

THEOREM 2. *If for any two indices $\lambda_0 \in \Lambda$ and $\lambda'_0 \in \Lambda'$ there are $\lambda_1 \geq \lambda_0$ and $\lambda'_1 \geq \lambda'_0$ with $f(C'_*(\lambda'_1)) \subseteq C_*(\lambda_1)$ such that $f : C'_*(\lambda'_1) \to C_*(\lambda_1)$ induces isomorphisms of all homology groups, then $F : \Gamma'_* \to \Gamma_*$ shares this property.*

P r o o f. In the special case of $\Lambda = \Lambda'$, for all $\lambda \in \Lambda$ we have $f(C'_*(\lambda)) \subseteq C_*(\lambda)$ and $f : C'_*(\lambda) \to C_*(\lambda)$ induces isomorphisms of all homology groups. Using a mapping cone argument (see Spanier [26, Lemma 4.6.9]) the proof that F induces an isomorphism of homology reduces to showing that Γ_* is acyclic whenever each $C_*(\lambda)$ is acyclic. But this follows readily from the definition of the differential.

To deal with the general case we consider the set $M \subseteq \Lambda' \times \Lambda$ of all pairs (λ', λ) with $f(C'_*(\lambda')) \subseteq C_*(\lambda)$ such that $f : C'_*(\lambda') \to C_*(\lambda)$ induces an isomorphism of homology, and let $\beta : M \to \Lambda'$, $\gamma : M \to \Lambda$ denote the projection maps. The assumption ensures that the set M is directed by the product order, that $\beta(M)$ is cofinal in Λ' and that $\gamma(M)$ is cofinal in Λ. We introduce new filtrations on our two chain complexes indexed with M by setting $\overline{C}'_*(\lambda', \lambda) := C'_*(\lambda')$ and $\overline{C}_*(\lambda', \lambda) := C_*(\lambda)$. From the corresponding total complexes we can build the following, chain homotopy commutative diagram:

$$
\begin{array}{ccc}
\Gamma'_* & \xrightarrow{F} & \Gamma_* \\
\downarrow & & \downarrow \\
\overline{\Gamma}'_* & \longrightarrow & \overline{\Gamma}_*
\end{array}
$$

The lower horizontal arrow is of the special form treated before, and the vertical arrows are chain homotopy equivalences by Lemma 4, and consequently F induces isomorphisms of all homology groups. ∎

Supplied with these tools we can now start to define the strong homology groups of a topological pair (X, A) with $X \neq \emptyset$. In the set M_X of all metrics on X we consider the subset $M_{(X,A)}$ of all metrics with trivial trace on A. We denote by $\widetilde{\mathbf{C}}_*(X, A; G)$ the *reduced* oriented simplicial chain complex of the *complex* N_X with coefficients in an R-module G, filtered by the subcomplexes determined by $N(d)$ for $d \in M_{(X,A)}$. We emphasize that $\widetilde{\mathbf{C}}_*(X, A; G)$ is based upon *oriented* simplices of the form $\langle x_0, \ldots, x_n \rangle$ with $x_i \in X$ (see [26, p. 158]), and that we do not use the more general singular simplices of the geometric realization of N_X (cf. [26, p. 173]). Thus an n-chain of $\widetilde{C}_*(X, A; G)$ is a finite linear combination of the form

$$
\sum_i g_i \langle x_0^i, \ldots, x_n^i \rangle
$$

with $g_i \in G$ and $x_j^i \in X$, for $n = 0$ we have $\sum_i g_i = 0$, and the chain belongs to $\widetilde{C}_*(d)$ if the d-diameter of the sets $\{x_0^i, \ldots, x_n^i\}$ does not exceed 1.

DEFINITION 3. The *reduced strong homology* $H_*^{\mathrm{s}}(X, A; G)$ of a topological pair (X, A), $X \neq \emptyset$, with coefficients in an R-module G is the homology

of the total complex of $\widetilde{\mathbf{C}}_*(X, A; G)$. Induced mappings are defined as in Section 1.

R e m a r k. a) Strong homology for arbitrary spaces has been introduced independently by Miminoshvili in [22] and by Lisica and Mardešić in [11, 12, 13] with further investigations in [19, 17, 16]. In [12] it is proved that strong homology of inverse systems of spaces is a functor on the coherent pro-homotopy category, and hence, by [8], on the Steenrod homotopy category. The strong homology of a space X is defined as the strong homology of a resolution $\widehat{\mathbf{X}}$ of X. By [14], $\widehat{\mathbf{X}}$ is a strong expansion of X, and since any two strong expansions are isomorphic in the Steenrod homotopy category it is isomorphic to the Vietoris system $\mathbf{V}(X)$. The strong homology of this inverse system is our $H_*^{\mathrm{s}}(X; G)$ with oriented simplices replaced by singular simplices, but Theorem 2 permits this replacement without affecting the homology.

b) For compact Hausdorff spaces X our $H_*^{\mathrm{s}}(X; G)$ is isomorphic to Massey's homology [20]. The proof will be given in Section 3.

The Eilenberg–Steenrod axioms are ready at our hand. The proof of the dimension axiom is left to the reader, but he should observe that we do not claim $H_n^{\mathrm{s}} = 0$ for $n < 0$ (for the reason we refer to [17, Theorem 3]), and that our strong homology functor is reduced.

THEOREM 3.
$$H_n^{\mathrm{s}}(S^0; G) = \begin{cases} G, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Here comes the excision axiom in the form proved by Mardešić and Miminoshvili in [16, Theorem 4]:

THEOREM 4. *For any topological pair (X, A) with $A \neq \emptyset$ the quotient map $p : (X, A) \to (X/A, *)$ induces isomorphisms of the strong homology groups $H_*^{\mathrm{s}}(p) : H_*^{\mathrm{s}}(X, A; G) \to H_*^{\mathrm{s}}(X/A, *; G)$.*

In the proof we will make use of the following observation:

LEMMA 5. *Consider two contiguous simplicial mappings $f, g : K \to L$ between simplicial complexes (see [26, p. 130]) and a subcomplex $K' \subseteq K$ on which f and g coincide. Then the induced chain maps $\widetilde{C}_*(f), \widetilde{C}_*(g) : \widetilde{C}_*(K; G) \to \widetilde{C}_*(L; G)$ can be connected by a chain homotopy ϱ which vanishes on $\widetilde{C}_*(K'; G)$.*

P r o o f. We introduce a linear order on the vertex set of K and define

$$\varrho(k_0 < k_1 < \ldots < k_n) := \sum_{i=0}^{n} (-1)^i \langle f(k_0), \ldots, f(k_i), g(k_i), \ldots, g(k_n) \rangle.$$

For $k_i \in K'$ the simplex on the right hand side is degenerate and therefore determines the zero element of $\widetilde{C}_*(L; G)$. ∎

Now the proof of Theorem 4 is a straightforward application of Theorem 2. In fact, the chain map $\widetilde{\mathbf{C}}_*(p) : \widetilde{\mathbf{C}}_*(X, A; G) \to \widetilde{\mathbf{C}}_*(X/A, *; G)$ is of the special form mentioned at the beginning of the proof of Theorem 2, because $p^* : M_{(X/A,*)} \to M_{(X,A)}$ is one-to-one. Furthermore, the vertex map $p_* : N(p^*d) \to N(d)$ is surjective, and any sequence of points $x_0, \ldots, x_n \in X$ determines a simplex of $N(p^*d)$ if and only if $p(x_0), \ldots, p(x_n)$ determines a simplex of $N(d)$. Hence $p_* : N(p^*d) \to N(d)$ is invertible up to contiguity, and by Lemma 5, $\widetilde{C}_*(p) : \widetilde{C}_*(p^*d) \to \widetilde{C}_*(d)$ is a chain homotopy equivalence. ∎

Our next topic is the exact homology sequence of a topological pair (X, A) with normally embedded subspace $A \neq \emptyset$. We recall that a subspace A of a topological space X is *normally embedded* if every metric on A can be extended over X; the equivalence of this characterization with the familiar covering property [18, p. 89] is proved in [1, Theorem 14.7]. Clearly for any increasing sequence of metrics on A we can inductively construct an increasing sequence of extensions over X, so that the restriction map induces a surjection of the complex of the directed set M_X onto that of M_A. Therefore the inclusion map $i : A \hookrightarrow X$ induces a monomorphism $\Gamma_*(i) : \Gamma_*(A) \to \Gamma_*(X)$ of the total complexes, by means of which we consider $\Gamma_*(A)$ as a subcomplex of $\Gamma_*(X)$. Now we remind ourselves of the quotient map $p : (X, A) \to (X/A, *)$; we have already learned that $\Gamma_*(p) : \Gamma_*(X, A) \to \Gamma_*(X/A)$ induces an isomorphism of homology. $\Gamma_*(p) : \Gamma_*(X) \to \Gamma_*(X/A)$ maps $\Gamma_*(A)$ to 0 and hence determines a map $q : \Gamma_*(X)/\Gamma_*(A) \to \Gamma_*(X/A)$.

THEOREM 5. *For every topological pair* (X, A) *with nonempty, normally embedded subspace the quotient chain map* $q : \Gamma_*(X)/\Gamma_*(A) \to \Gamma_*(X/A)$ *induces isomorphisms of the homology groups*; *therefore the short exact sequence of chain complexes*

$$0 \to \Gamma_*(A) \to \Gamma_*(X) \to \Gamma_*(X)/\Gamma_*(A) \to 0$$

provides us with a natural exact homology sequence

$$\ldots \to H_n^{\mathrm{s}}(A; G) \to H_n^{\mathrm{s}}(X; G) \to H_n^{\mathrm{s}}(X, A; G) \to H_{n-1}^{\mathrm{s}}(A; G) \to \ldots$$

For the proof we need the following variant of Morita's lemma (cf. [18, p. 112]):

LEMMA 6. *Suppose we are given a filtered chain complex* $\mathbf{C}_*$ *and a subcomplex* $C'_* \subseteq C_*$; *on* C'_* *we consider the filtration* $C'_*(\lambda) = C'_* \cap C_*(\lambda)$. *If for every index* λ *there are* $\mu \geq \lambda$, *a chain map* $\tau : C_*(\mu) \to C'_*(\lambda)$ *extending the inclusion map* $C'_*(\mu) \hookrightarrow C'_*(\lambda)$ *and a chain homotopy* ϱ *from* τ *to the*

inclusion homomorphism $C_*(\mu) \hookrightarrow C_*(\lambda)$, with $\varrho(C'_*(\mu)) \subseteq C'_*(\lambda)$, then the inclusion chain map of the total complexes $\Gamma'_* \hookrightarrow \Gamma_*$ induces isomorphisms of the homology groups.

$$
\begin{array}{ccc}
C'_*(\mu) & \xhookrightarrow{\hspace{2cm}} & C_*(\mu) \\[6pt]
\downarrow & \quad\tau\;\nearrow\quad{}^{\textstyle @} & \downarrow \\[6pt]
C'_*(\lambda) & \xhookrightarrow{\hspace{2cm}} & C_*(\lambda)
\end{array}
$$

P r o o f. We consider the quotient complex $C''_* := C_*/C'_*$ filtered by the subcomplexes $C''_*(\lambda) := C_*(\lambda)/C'_*(\lambda)$. If λ and μ are as in the assumption, then the inclusion chain map $C''_*(\mu) \hookrightarrow C''_*(\lambda)$ is nullhomotopic. Now it suffices to show that the total complex Γ''_* is acyclic. By Theorem 2 we may assume that the index set Λ is cofinite, because else we could reindex $\mathbf{C}''_*$ with the directed set of all finite subsets of Λ containing a greatest element. If Λ is cofinite, then we can construct a nondecreasing index map $\alpha : \Lambda \to \Lambda$ such that $C''_*(\alpha(\lambda)) \hookrightarrow C''_*(\lambda)$ is nullhomotopic. Suppose φ is a cycle of Γ''_*; by induction on $\deg x$, $x \in \Lambda_*$, we can construct elements $\psi(x) \in C''_*$ with

(4) $$\partial(\psi(x)) = \varphi(\alpha_* x) - (-1)^{\deg \varphi} \psi(\partial x),$$

(5) $$\psi(x) \in C''_*(\alpha(\lambda)) \quad \text{for } x \in \Lambda_*(\lambda),$$

because the right hand side of equation (4) is a cycle of $C''_*(\alpha(\lambda))$ and hence a boundary in $C''_*(\lambda)$; $\alpha_* : \Lambda_* \to \Lambda_*$ is the chain map induced by our nondecreasing index map. Then $\partial\psi = \alpha^*\varphi$, but by Lemma 4, $\alpha^*\varphi$ and φ are homologous. ∎

LEMMA 7. *Let $A \subseteq X$ be a nonempty, normally embedded subspace. Denote by $\mathbf{C}^0_*(X, A; G) \subseteq \widetilde{\mathbf{C}}_*(X, A; G)$ the filtered subcomplex such that for each metric $d \in M_X$ the subcomplex $C^0_*(d) \subseteq \widetilde{C}_*(d)$ is determined by the simplices $\langle x_0, \ldots, x_n \rangle$ with $d(x_i, A) \le 2$ for all i, and denote by $\Gamma^0_*(X, A) \subseteq \Gamma_*(X, A)$ the derived total complex. Then the natural map $\Gamma_*(A) \to \Gamma^0_*(X, A)$ induces isomorphisms of all homology groups.*

P r o o f. Since A is normally embedded in X we can reindex $\widetilde{\mathbf{C}}_*(A; G)$ using M_X as index set, and then we are in a position to apply Lemma 6. Let $d \in M_X$ be given. For any $x \in X$ with $7d(x, A) \le 2$ we choose $\alpha(x) \in A$ with $d(x, \alpha(x)) \le 3/7$, assuming $\alpha(x) = x$ for $x \in A$. Then α may be considered as simplicial map $\alpha : N^0(7d) \to N(d_{|A})$, where $N^0(7d) \subseteq N(7d)$ is the full subcomplex determined by all vertices x with $7d(x, A) \le 2$, which operates as identity on $N(7d_{|A})$ and which is contiguous in $N^0(d)$ to the inclusion map $N^0(7d) \hookrightarrow N^0(d)$. Now α induces our chain map $\tau : \widetilde{C}^0_*(7d) \to \widetilde{C}_*(d_{|A})$, and the existence of ϱ follows from Lemma 5. ∎

P r o o f   o f   T h e o r e m   5. We consider the following commutative diagram:

$$
\begin{array}{ccc}
\Gamma_*(X)/\Gamma_*(A) & \longrightarrow & \Gamma_*(X)/\Gamma_*^0(X,A) \\
\Big\downarrow{\scriptstyle q} & {\scriptstyle g}\nearrow & \Big\downarrow \\
\Gamma_*(X/A) & \longrightarrow & \Gamma_*(X/A)/\Gamma_*^0(X/A,*)
\end{array}
$$
(6)

The vertical maps are induced by $p : X \to X/A$ and the horizontal maps are algebraic quotient maps. By Lemma 7 the horizontal maps induce an isomorphism of homology, because the one-point subspace $*$ is certainly normally embedded in X/A. If we can construct a filler g to the diagram above, then q induces an isomorphism of homology. We start by constructing a nondecreasing index map $M_X \to M_{X/A}$, $d \mapsto d'$, with

$$d'(p(x), p(y)) := \min(d(x,y), d(x,A) + d(y,A)) \,. \tag{7}$$

Furthermore, we assign to each metric $d \in M_X$ a graded homomorphism $\vartheta_d : \widetilde{C}_*(X/A; G) \to \widetilde{C}_*(X; G)$, which is defined on the generating simplices $\sigma = \langle p(x_0), \ldots, p(x_n) \rangle$ by $\vartheta_d(\sigma) := \langle x_0, \ldots, x_n \rangle$ if $d(x_i, A) \geq 1$ for all i, and $\vartheta_d(\sigma) = 0$ else. For $d'(p(x_i), p(x_j)) \leq 1$ and $d(x_i, A) \geq 1$, $d(x_j, A) \geq 1$ we necessarily have $d(x_i, x_j) = d'(p(x_i), p(x_j)) \leq 1$ and hence

$$\vartheta_d(\widetilde{C}_*(d')) \subseteq \widetilde{C}_*(d) \,. \tag{8}$$

ϑ_d is not a chain map, but at least we have

$$(\partial\vartheta_d - \vartheta_d\partial)(\widetilde{C}_*(d')) \subseteq C_*^0(d) \,, \tag{9}$$

because the only possibility for $\partial\vartheta_d(\sigma)$ and $\vartheta_d\partial(\sigma)$ to be distinct from one another is the existence of some i with $d(x_i, A) < 1$, and then all x_j necessarily satisfy $d(x_j, A) \leq 2$. Similarly we see that

$$(\vartheta_d p_* - \mathrm{id})(\widetilde{C}_*(d)) \subseteq C_*^0(d) \,, \tag{10}$$

$$(p_*\vartheta_d - \mathrm{id})(\widetilde{C}_*(d')) \subseteq C_*^0(d') \,, \tag{11}$$

$$(\vartheta_{d_0} - \vartheta_{d_1})(\widetilde{C}_*(d_1')) \subseteq C_*^0(d_0) \quad \text{for } d_0 \leq d_1 \,. \tag{12}$$

Equations (8), (9) and (12) allow us to define a chain map $g : \Gamma_*(X/A) \to \Gamma(X)/\Gamma_*^0(X,A)$ by

$$g(\psi)\langle d_0 \leq \ldots \leq d_p \rangle := \vartheta_{d_0}(\psi\langle d_0' \leq \ldots \leq d_p' \rangle) \,,$$

and by equations (10) and (11), g fits commutatively into diagram (6). ∎

Now we prove at last that our homology functor H_*^{s} is homotopy invariant:

THEOREM 6. *For every topological pair (X, A) the projection map p : $(X \times I, A \times I) \to (X, A)$ induces isomorphisms of the homology groups $H_*^s(p) : H_*^s(X \times I, A \times I; G) \to H_*^s(X, A; G)$.*

P r o o f. We want to apply Theorem 2 and therefore we need some special metrics on $X \times I$ (which will serve us as substitutes for stacked coverings). To any metric d on X and any map $\alpha : X \to \mathbb{R}_+$ into the positive reals we assign a metric d^α on $X \times I$ by setting

$$d^\alpha(x, s; y, t) := \max(d(x, y), |\alpha(x)s - \alpha(y)t|).$$

If d and α vanish on A, then d^α also has trivial trace on A. We claim:

(a) For every metric $d' \in M_{(X \times I, A \times I)}$ there exist a metric $d \in M_{(X,A)}$ and a map $\alpha : X \to \mathbb{R}_+$ with $\alpha_{|A} = 0$ and $d^\alpha(x, s; y, t) \leq 1 \Rightarrow d'(x, s; y, t) \leq 1$.

(b) $p_* : N(d^\alpha) \to N(d)$ induces an isomorphism of homology.

(a), (b) and Theorem 2 imply that $H_*^s(p)$ is an isomorphism, because for $d_0 \in M_{(X,A)}$ and $d_0' \in M_{(X \times I, A \times I)}$ we can choose $d_1 \in M_{(X,A)}$, $d_1 \geq d_0$, and $\alpha : X \to \mathbb{R}_+$ with property (a). Then we set $d_1' := \max(d_0', d_1^\alpha)$ to get a metric $d_1' \geq d_0'$ whose Vietoris nerve coincides with that of d_1^α, so that by (b), $p_* : N(d_1') \to N(d_1)$ induces isomorphisms of the homology groups.

To prove (a) we set $d_0(x, y) := \sup_t d'(x, t; y, t)$ and consider the following subsets of X:

$$U_n := \begin{cases} \{x \in X \mid d_0(x, A) \leq 1/4\}, & n = 0, \\ \{x \in X \setminus A \mid |s - t| \leq 2/n \Rightarrow d'(x, s; x, t) \leq 1/2\}, & n \geq 1, \end{cases}$$

$$V_n := \begin{cases} \{x \in X \mid d_0(x, A) \leq 1/8\}, & n = 0, \\ \{x \in X \setminus A \mid |s - t| \leq 2/n \Rightarrow d'(x, s; x, t) \leq 1/4\}, & n \geq 1. \end{cases}$$

By Lebesgue's Theorem the sets V_n cover X, and each V_n is contained in the interior of U_n with respect to the metric d_0. Therefore there exists a continuous, locally finite partition of unity $\{\varphi_n\}$ on the pseudometric space (X, d_0) with $\varphi_n^{-1}]0, 1] \subseteq U_n$. All requirements of (a) are satisfied if we define

$$\alpha(x) := \sum_{n=0}^{\infty} n\varphi_n(x), \quad d(x, y) := \max(2d_0(x, y), |\alpha(x) - \alpha(y)|).$$

To prove (b) we consider the map $\chi : \widetilde{C}_*(N(d); G) \to \widetilde{C}_*(N(d^\alpha); G)$ induced by the vertex map $x \mapsto (x, 0)$. Clearly χ is right inverse to $p : \widetilde{C}_*(N(d^\alpha); G) \to \widetilde{C}_*(N(d); G)$, and we are going to construct a chain homotopy ϱ connecting χp with the identity map. We define functions $f_x^n : I \to I$ for $x \in X$ and $0 \leq n < \infty$ by

$$f_x^n(t) := \begin{cases} \alpha(x)^{-1}n, & n < \alpha(x)t, \\ t, & n \geq \alpha(x)t, \ n > 0, \\ 0, & n = 0, \end{cases}$$

and observe that $|\alpha(x)s - \alpha(y)t| \leq 1$ implies $|\alpha(x)f_x^n(s) - \alpha(y)f_y^n(t)| \leq 1$ and $|\alpha(x)f_x^n(s) - \alpha(y)f_y^{n+1}(t)| \leq 1$; furthermore, $f_x^0(t) = 0$ and $f_x^n(t) = t$ for $n > \alpha(x)$. Let $F_n : N(d^\alpha) \to N(d^\alpha)$ be the simplicial map $F_n(x, t) := (x, f_x^n(t))$; then F_n and F_{n+1} are contiguous and coincide for $n > \alpha(x)$. By Lemma 5 the induced chain mappings $\widetilde{C}_*(F_n), \widetilde{C}_*(F_{n+1}) : \widetilde{C}_*(d^\alpha) \to \widetilde{C}_*(d^\alpha)$ can be connected by a chain homotopy ϱ_n, $\partial \varrho_n + \varrho_n \partial = \widetilde{C}_*(F_{n+1}) - \widetilde{C}_*(F_n)$, with $\varrho_n \langle (x_0, t_0), \ldots, (x_m, t_m) \rangle = 0$ for $n > \max_i \alpha(x_i)$. Therefore we can define $\varrho := \sum_{n=0}^\infty \varrho_n$, because the apparently infinite sum terminates for every chain of the domain. Then ϱ is a chain homotopy: $\partial \varrho + \varrho \partial = \mathrm{id} - \chi p$. ∎

3. Strong homology with compact supports. There is a natural notion of support of a chain $\varphi \in \Gamma_*(X, A; G)$: If for each increasing sequence $\mu = \langle d_0 \leq \ldots \leq d_p \rangle$ of metrics in $M_{(X,A)}$ we have $\varphi(\mu) = \sum_i g_i^\mu \langle x_{\mu 0}^i, \ldots, x_{\mu, n+p}^i \rangle$, then $\mathrm{Supp}(\varphi)$ is the smallest closed subset of X containing all the points $x_{\mu k}^i$ with $g_i^\mu \neq 0$. The chains with compact support form a subcomplex $\Gamma_*^c(X, A; G) \subseteq \Gamma_*(X, A; G)$, whose homology is denoted by $H_*^c(X, A; G)$ and is called *strong homology with compact supports*. We are going to show that under some restrictions on (X, A) its homology group can be obtained by taking direct limits.

LEMMA 8. *Every compact subspace A of a completely regular Hausdorff space X is normally embedded.*

P r o o f. Let d be a metric on A, $\widetilde{A}$ the metric space obtained by identifying points of distance 0 and $q : A \to \widetilde{A}$ the quotient map. Since $\widetilde{A}$ is compact metric there is an embedding $\varphi : \widetilde{A} \hookrightarrow Q$ into the Hilbert cube. $\varphi(\widetilde{A})$ as a closed subset of a compact space is normally embedded, and hence there is a metric ϱ on Q with $\varrho(\varphi q(a), \varphi q(b)) = d(a, b)$ for $a, b \in A$. Let $X \hookrightarrow \beta X$ be the embedding of X into its Stone–Čech compactification and consider A as a closed subspace of βX. Since Q is an ANE for compact spaces there is a map $\psi : \beta X \to Q$ extending φq, and then $d'(x, y) := \varrho(\psi(x), \psi(y))$ is a metric on X extending d. ∎

PROPOSITION 4. *For each normal Hausdorff space X and each closed subspace A the natural homomorphism $\varinjlim_C H_*^s(C, C \cap A; G) \to H_*^c(X, A; G)$ is an isomorphism, where the limit is taken over all compact subsets C of X.*

P r o o f. We want to show that for each compact subset $B \subseteq X$ the restriction map $M_{(X,A)} \to M_{(B, B \cap A)}$ is surjective. Consider a metric d on B with trivial trace on $B \cap A$. Then d may be considered as a metric on $B/B \cap A$, and this space is homeomorphic to the canonical image of B in X/A. The assumptions assure that X/A is normal and in particular completely regular, thus by Lemma 8, $B/B \cap A$ is normally embedded in

X/A. Therefore d has an extension over X/A, i.e. an extension over X with trivial trace on A.

Now Theorem 2 permits us to use the directed set $M_{(X,A)}$ as index set of the filtered complexes $\widetilde{\mathbf{C}}_*(B, B \cap A; G)$, and with this understanding we have $\Gamma_*^{\mathrm{c}}(X, A; G) = \bigcup_B \Gamma_*(B, B \cap A; G)$. ∎

By means of Proposition 4 the Eilenberg–Steenrod axioms from Section 2 carry over to strong homology with compact supports in the following form:

THEOREM 7. (a)

$$H_n^{\mathrm{c}}(S^0; G) = \begin{cases} G, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

In (b)–(d), (X, A) is a topological pair consisting of a normal Hausdorff space X and a closed subspace A:

(b) There is a natural exact homology sequence

$$\ldots \to H_n^{\mathrm{c}}(A; G) \to H_n^{\mathrm{c}}(X; G) \to H_n^{\mathrm{c}}(X, A; G) \to H_{n-1}^{\mathrm{c}}(A; G) \to \ldots$$

(c) The projection map $p : (X \times I, A \times I) \to (X, A)$ induces isomorphisms of the homology groups $H_*^{\mathrm{c}}(p) : H_*^{\mathrm{c}}(X \times I, A \times I; G) \to H_*^{\mathrm{c}}(X, A; G)$.

(d) If B is a closed subspace of X with $A \cap B \neq \emptyset$ and $A \cup B = X$, then the inclusion map $i : (B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms of homology: $H_*^{\mathrm{c}}(i) : H_*^{\mathrm{c}}(B, A \cap B; G) \to H_*^{\mathrm{c}}(X, A; G)$.

In comparison to the homology functor of Section 2 the exactness property of H_*^{c} is improved, because the subspace is no longer required to be normally embedded, but the excision property does not carry over in full strength. The reason is that a compact subset of X/A need not be the image of a compact subset of X. However:

PROPOSITION 5. (a) Let X be a paracompact Hausdorff space and A a closed subspace such that X/A is compact. Then $X \setminus A$ is relatively compact in X.

(b) For every paracompact space X and every closed, nonempty subspace A the quotient map $p : (X, A) \to (X/A, *)$ induces isomorphisms $H_*^{\mathrm{c}}(p) : H_*^{\mathrm{c}}(X, A; G) \to H_*^{\mathrm{c}}(X/A, *; G)$.

P r o o f. (a) We set $B := \overline{X \setminus A}$; then by [7, p. 206, Theorem 3.10.21, and p. 306, Theorem 5.1.20] it suffices to show that B is pseudocompact. Now suppose this is false, and let $f : B \to \mathbb{R}$ be a map and b_n a sequence of points with $f(b_n) > n$. By continuity we can find $x_n \in B \setminus A$ in some neighborhood of b_n with $f(x_n) > n$. The set C formed by all of these points x_n cannot

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(^{1}) Actually this follows from [21, Corollary 1.2]; we include our proof merely for the sake of completeness.

have an accumulation point, hence it is a closed discrete set disjoint from A. The quotient map $p : X \to X/A$ maps C homeomorphically onto a closed, discrete subset of X/A, which is absurd if X/A is compact.

(b) follows from (a) and Theorem 4. ■

EXAMPLE. To show that neither part of the preceding proposition holds for arbitrary spaces X we use a counterexample due to E. Michael [21, Example 3.3]. Let X be the space of all countable ordinals and $A \subseteq X$ the subset of all limit ordinals [7, p. 130, Example 3.1.27]; X is a noncompact, normal Hausdorff space and A is a closed subspace of empty interior. The quotient space X/A is compact, because every neighborhood of A in X has finite complement (hence X/A is homeomorphic to the space $A(\aleph_1)$ in the notation of [7, p. 129]).

To show that the quotient map $p : (X, A) \to (X/A, *)$ need not induce isomorphisms $H_*^c(p)$ we use coefficients in a field, say $\mathbb{Z}_2$. Strong homology of compact pairs with coefficients in a field coincides with Čech homology, because the inverse limit functor operates exactly on finite-dimensional vector spaces, hence $H_0^c(X/A, *; \mathbb{Z}_2) = H_0^s(X/A, *; \mathbb{Z}_2) = \prod_{\lambda \in X \setminus A} \mathbb{Z}_2^{(\lambda)}$, one factor $\mathbb{Z}_2$ for each countable nonlimit ordinal. We observe that the number of such ordinals is uncountable. The sets of the form $[0, \lambda]$, where λ ranges over the countable ordinals, form a cofinal subset of the set of compact subsets of X; this allows us to compute $H_0^c(X, A; \mathbb{Z}_2)$: It equals the subgroup of $\prod_{\lambda \in X \setminus A} \mathbb{Z}_2^{(\lambda)}$ formed by the elements with at most countably many coordinates different from 0, with $H_0^c(p)$ being the inclusion map. Therefore $H_0^c(p)$ is not an isomorphism.

Now finally we show that A is normally embedded in X, and hence, by [9, Corollary 4], H_*^c does not factor over the strong shape category. Let d be a metric on A. Then as in [7, Example 3.1.21] one shows that there is an ordinal $\lambda \in A$ such that $d(\mu, \nu) = 0$ for all $\mu, \nu \geq \lambda$, $\mu, \nu \in A$. Since $[0, \lambda]$ is compact and A is closed, the intersection $A \cap [0, \lambda]$ is normally embedded in $[0, \lambda]$, and there is a metric d' on $[0, \lambda]$ extending the restriction of d. The retraction map $f : X \to [0, \lambda]$ with $f(\mu) = \mu$ for $\mu \leq \lambda$ and $f(\mu) = \lambda$ for $\mu > \lambda$ is continuous, and $\varrho(\mu, \nu) := d'(f(\mu), f(\nu))$ is a metric on X extending d. Thus we have proved:

THEOREM 8. *Strong homology with compact supports H_*^c does not factor over the strong shape category.*

THEOREM 9. *In ZFC strong homology H_*^s does not have compact supports.*

This follows from our counterexample simply because $H_*^c \neq H_*^s$. In [17] Mardešić and Prasolov constructed a different counterexample, but it depended on the continuum hypothesis.

The last part of this section is devoted to proving isomorphism between strong homology with compact supports and Massey's homology with compact supports. This requires some additional tools, for instance we have to pass from filtered chain complexes to general inverse systems of chain complexes $C_*(\lambda)$ and chain maps $h_\mu^\lambda : C_*(\lambda) \to C_*(\mu)$ over some *cofinite* directed set Λ. The total complex Γ_* is defined as in [17, Section 3] using our sign convention of Definition 2. There is an inclusion chain map $i : \varprojlim_\lambda C_*(\lambda) \hookrightarrow \Gamma_*$ mapping a thread of n-chains $x_\lambda \in C_n(\lambda)$ to $\varphi \in \Gamma_n$ with

$$\varphi\langle\lambda_0 \leq \ldots \leq \lambda_p\rangle := \begin{cases} x_{\lambda_0} & \text{for } p = 0, \\ 0 & \text{for } p > 0. \end{cases}$$

LEMMA 9. *If for each index λ the chain map $\varprojlim_{\mu<\lambda} h_\mu^\lambda : C_*(\lambda) \to \varprojlim_{\mu<\lambda} C_*(\mu)$ is surjective and splits as an epimorphism of graded groups, then $i : \varprojlim_\lambda C_*(\lambda) \hookrightarrow \Gamma_*$ is a chain homotopy equivalence.*

P r o o f. (a) First we deal with the special case where the index set Λ contains a greatest element α. Let $p : \Gamma_* \to C_*(\alpha)$ be the evaluation map $p(\varphi) := \varphi(\alpha)$, and $\varrho : \Gamma_* \to \Gamma_{*+1}$ the chain homotopy $\varrho(\varphi)\langle\lambda_0 \leq \ldots \leq \lambda_p\rangle := (-1)^{p+\deg\varphi}\varphi\langle\lambda_0 \leq \ldots \leq \lambda_p \leq \alpha\rangle$; one checks $pi = \mathrm{id}$, $\partial\varrho + \varrho\partial = p - \mathrm{id}$ and furthermore $\varrho i = 0$.

(b) In the proof of the general case we will encounter the following situation: We consider a commutative square of chain complexes as below, and we suppose there exist a chain map $\varphi : X_* \to A_*$ with $\varphi i = \mathrm{id}$ and a chain homotopy $\varrho : X_* \to X_{*+1}$ with $\partial\varrho + \varrho\partial = i\varphi - \mathrm{id}$ and $\varrho i = 0$; in addition we assume that $p : E_* \to B_*$ is a split epimorphism of graded groups.

(13)
$$\begin{array}{ccc} A_* & \xrightarrow{f} & E_* \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X_* & \xrightarrow{g} & B_* \end{array}$$

Let $s : B_* \to E_*$ be a homomorphism of graded groups with $ps = \mathrm{id}$ and set $h := f\varphi - \partial sg\varrho - sg\varrho\partial : X_* \to E_*$. Then h is a chain map fitting commutatively into our diagram.

If *two* chain maps $h_0, h_1 : X_* \to E_*$ with $h_0 i = h_1 i$ and a chain homotopy $\tau : X_* \to B_{*+1}$, $\partial\tau + \tau\partial = ph_0 - ph_1$, with $\tau i = 0$ are given, then $\eta := h_1\varrho - h_0\varrho + \partial s\tau\varrho - s\tau\varrho\partial : X_* \to E_{*+1}$ is a chain homotopy with $\partial\eta + \eta\partial = h_0 - h_1$, $\eta i = 0$ and $p\eta = \tau$.

(c) The general case of the lemma. We consider the subsystems with index set $\{\mu \in \Lambda \mid \mu \leq \lambda\}$ with λ fixed and denote the corresponding total complexes by $\Gamma_*(\lambda)$; they form an inverse system with limit Γ_*. The

natural inclusion maps $i_\lambda : C_*(\lambda) \hookrightarrow \Gamma_*(\lambda)$ form a level system of chain maps with limit $i : \varprojlim_\lambda C_*(\lambda) \hookrightarrow \Gamma_*$. Since Λ is cofinite we can apply (b) to the diagrams

(14)
$$
\begin{array}{ccc}
C_*(\lambda) & \xrightarrow{\mathrm{id}} & C_*(\lambda) \\
{\scriptstyle i_\lambda}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\mu<\lambda} h_\mu^\lambda} \\
\Gamma_*(\lambda) & \longrightarrow & \varprojlim_{\mu<\lambda} C_*(\mu)
\end{array}
$$

and construct inductively a family of chain maps $f_\lambda : \Gamma_*(\lambda) \to C_*(\lambda)$ with $f_\lambda i_\lambda = \mathrm{id}$ and $h_\mu^\lambda f_\lambda = f_\mu \gamma_\mu^\lambda$ for $\mu \leq \lambda$, where $\gamma_\mu^\lambda : \Gamma_*(\lambda) \to \Gamma_*(\mu)$ is the natural projection. These maps determine a limit chain map $f := \varprojlim_\lambda f_\lambda : \Gamma_* \to \varprojlim_\lambda C_*(\lambda)$ with $fi = \mathrm{id}$.

It follows readily from the definitions that $\Gamma_n(\lambda)$ equals

$$
(\varprojlim_{\mu<\lambda} \Gamma_n(\mu)) \times \prod_{p\geq 0} \prod_{\lambda_0 \leq \ldots \leq \lambda_{p-1} \leq \lambda} C_{n+p}(\lambda_0),
$$

so that $\varprojlim_{\mu<\lambda} \gamma_\mu^\lambda : \Gamma_*(\lambda) \to \varprojlim_{\mu<\lambda} \Gamma_*(\mu)$ is a split epimorphism of graded groups. Applying (b) to the diagrams

(15)
$$
\begin{array}{ccc}
C_*(\lambda) & \xrightarrow{i_\lambda} & \Gamma_*(\lambda) \\
{\scriptstyle i_\lambda}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\mu<\lambda} \gamma_\mu^\lambda} \\
\Gamma_*(\lambda) & \xrightarrow{\varprojlim_{\mu<\lambda} \gamma_\mu^\lambda} & \varprojlim_{\mu<\lambda} \Gamma_*(\mu)
\end{array}
$$

and the maps $i_\lambda f_\lambda, \mathrm{id} : \Gamma_*(\lambda) \to \Gamma(\lambda)$ we get a family of chain homotopies $\varrho_\lambda : \Gamma_*(\lambda) \to \Gamma_{*+1}(\lambda)$ with $\partial\varrho_\lambda + \varrho_\lambda\partial = i_\lambda f_\lambda - \mathrm{id}$, $\varrho_\lambda i_\lambda = 0$ and $\gamma_\mu^\lambda \varrho_\lambda = \varrho_\mu \gamma_\mu^\lambda$ for $\mu \leq \lambda$. We obtain a limit chain homotopy $\varrho := \varprojlim_\lambda \varrho_\lambda : \Gamma_* \to \Gamma_{*+1}$ with $\partial\varrho + \varrho\partial = if - \mathrm{id}$. ∎

THEOREM 10. *On the category of compact Hausdorff spaces and closed subspaces strong homology H_*^{s} is isomorphic to Massey's homology. On the category of normal Hausdorff spaces and closed subspaces H_*^{c} is isomorphic to Massey's homology with compact supports.*

COROLLARY 1. $H_n^{\mathrm{c}} = 0$ *for* $n < 0$.

Proof of Theorem 10. It is sufficient to consider the compact case and to deal with nonempty spaces X rather than pairs; if we can show that our complex $\widetilde{\Gamma}_*(X; G)$ is naturally equivalent to Massey's chain complex via a sequence of natural chain maps which induce isomorphisms of homology, then the two homology functors will also coincide for pairs and the isomorphism will commute with the boundary operator.

First we alter the definition of the filtered complex $\widetilde{\mathbf{C}}_*(X; G)$ by replacing oriented simplices $\langle x_0, \ldots, x_n \rangle$ of N_X by ordered simplices $(x_0, \ldots, x_n)$; Theorem 2 ensures that the homology of the total complex remains unaffected. During this proof we will use the old symbol $\widetilde{\mathbf{C}}_*(X; G)$ to denote the new filtered complex.

We set

$$\Phi^n(X) := \{\alpha : X^{n+1} \to \mathbb{Z} \mid \alpha(X^{n+1}) \text{ finite}\}$$

and for $d \in M_X$ we denote by $\Phi_d^n(X) \subseteq \Phi^n(X)$ the subgroup of all functions $\alpha : X^{n+1} \to \mathbb{Z}$ such that $\alpha(x_0, \ldots, x_n) = 0$ if the set $\{x_0, \ldots, x_n\}$ has d-diameter at most 1 [20, Ch. 1, §1.1]. The groups

$$\overline{C}_n(d) := \operatorname{Hom}(\Phi^n(X)/\Phi_d^n(X), G)$$

constitute a filtered complex $\overline{\mathbf{C}}_*(X; G)$, and the intersection $\bigcap_d \overline{C}_*(d)$ is precisely Massey's chain complex. For $n = 0$ by $\overline{C}_0(d)$ we will understand the corresponding reduced complex. There is an obvious chain map $f : \widetilde{\mathbf{C}}_*(X; G) \to \overline{\mathbf{C}}_*(X; G)$ given by evaluation: $f((x_0, \ldots, x_n) \otimes g) = \varphi$ with $\varphi(\alpha) := \alpha(x_0, \ldots, x_n)g$. f is injective and has the property $\widetilde{C}_*(d) = f^{-1}(\overline{C}_*(d))$, in particular it is filtration preserving. For each metric $d \in M_X$ we can find a finite subset $A \subseteq X$ and a not necessarily continuous function $g : X \to A$ with $d(x, g(x)) \leq 1/4$ for all $x \in X$. The induced vertex map $g : N(2d) \to N(d)$ is contiguous to the inclusion map, and applying Lemma 5 we can construct chain homotopies $\widetilde{\varrho} : \widetilde{C}_*(2d) \to \widetilde{C}_{*+1}(d)$ and $\overline{\varrho} : \overline{C}_*(2d) \to \overline{C}_{*+1}(d)$ connecting the inclusion map and the chain map induced by g such that the equation $f\widetilde{\varrho} = \overline{\varrho}f$ holds. But since g has finite image one readily sees $g(\overline{C}_*(2d)) \subseteq \widetilde{C}_*(d)$, and now Lemma 6 tells us that the chain map of the total complexes $F : \widetilde{\Gamma}_*(X; G) \to \overline{\Gamma}_*(X; G)$ determined by f induces isomorphisms of homology.

As explained in the remarks preceding Lemma 9 the inverse limit $\bigcap_d \overline{C}_*(d) = \varprojlim_d \overline{C}_*(d)$ may be naturally embedded in the total complex $\overline{\Gamma}_*(X; G)$; and if we can show that the inclusion map induces isomorphisms of homology we are finished. To do this we start by replacing M_X by a cofinite index set, taking into account Theorem 2. It follows from Nöbeling's theorem [20, p. 88] that the sequence

$$(16) \qquad 0 \to \overline{C}_n(d) \to \operatorname{Hom}(\Phi^n(X), G) \to \operatorname{Hom}(\Phi_d^n(X), G) \to 0$$

is split exact; and the collection of them all forms an inverse system of short exact sequences over our new, cofinite index set, giving rise to a short exact sequence of total complexes

$$(17) \qquad\qquad 0 \to \overline{\Gamma}_*(X; G) \to \Gamma'_* \to \Gamma''_* \to 0\,.$$

The inverse limit of the sequences (16) is

$$(18) \qquad 0 \to \bigcap_d \overline{C}_*(d) \to \mathrm{Hom}(\varPhi^*(X), G) \to \mathrm{Hom}(\varinjlim_d \varPhi_d^n(X), G) \to 0 \,,$$

and by Nöbeling's theorem this sequence is also exact. The natural inclusions of limit complexes into total complexes provide us with a transformation of the sequence (18) into the sequence (17), whence we derive a transformation of the long exact homology sequences. $\varPhi^*(X)$ is chain contractible, and hence $\mathrm{Hom}(\varPhi^*(X), G)$ and $\varGamma'_*$ are acyclic, so it remains to show that the natural inclusion

$$\mathrm{Hom}(\varinjlim_d \varPhi_d^n(X), G) = \varprojlim_d \mathrm{Hom}(\varPhi_d^n(X), G) \hookrightarrow \varGamma''_*$$

induces isomorphisms of homology, but since for $d \leq e$ the monomorphism $\varPhi_d^n(X) \hookrightarrow \varPhi_e^n(X)$ splits by Nöbeling's theorem, this follows from Lemma 9. ∎

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