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The covering property for σ-ideals of compact sets

by

Carlos E. Uzcátegui A. (Mérida)

Abstract. The covering property for σ-ideals of compact sets is an abstract version of the classical perfect set theorem for analytic sets. We will study its consequences using as a paradigm the σ-ideal of countable closed subsets of 2^ω.

1. Introduction. The study of σ-ideals of compact sets has been motivated by problems in analysis and quite recently it has received considerable attention because of its connections with harmonic analysis (see [7]). The descriptive set theoretic approach was initiated by Kechris, Louveau and Woodin in [8] (see also [6]).

Throughout this article X will be a compact metric space. By $\mathcal{K}(X)$ we denote the collection of closed subsets of X. A subset $I \subseteq \mathcal{K}(X)$ is called *hereditary* if

$$\text{if } K, L \in \mathcal{K}(X),\ K \in I,\ L \subseteq K,\ \text{then } L \in I\,.$$

I is called an *ideal* if moreover

$$\text{if } K, L \in I,\ \text{then } K \cup L \in I\,,$$

and I is called a σ-*ideal* if in addition

$$\text{if } K, K_1, K_2, \ldots \in \mathcal{K}(X),\ K_i \in I \text{ for all } i \text{ and } K = \bigcup K_i, \text{ then } K \in I.$$

Let us give some examples:

(1) For each $A \subseteq X$, let $\mathcal{K}(A) = \{K \in \mathcal{K}(X) : K \subseteq A\}$.
(2) $K_\omega(X) = \{K \in \mathcal{K}(X) : K \text{ is countable}\}$.
(3) $I_{\text{meager}} = \{K \in \mathcal{K}(X) :\ K \text{ is meager}\}$.

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(4) Given a Borel measure μ over X, let

$$I_\mu = \{K \in \mathcal{K}(X) : \mu(K) = 0\}.$$

(5) Let R = Rajchman probability measures on the unit circle, i.e. those measures for which $\widehat{\mu}(n) \to 0$ as $|n| \to \infty$. Let

$$U_0 = \{K \in \mathcal{K}(X) : \mu(K) = 0 \text{ for all } \mu \in R\}.$$

U_0 are the closed sets of extended uniqueness (see [7]).

(6) Let $X = 2^\omega$, and I_c = the σ-ideal of closed subsets of 2^ω that avoid a cone of Turing degrees.

Given a σ-ideal I of closed subsets of X, the most natural way to extend I to a σ-ideal of arbitrary subsets of X is as follows: Let

$$I^{\text{ext}} = \Big\{A \subseteq X : \exists (K_n)_{n\in\omega} \text{ in } I,\ A \subseteq \bigcup_n K_n\Big\}.$$

I^{ext} is the smallest σ-ideal of subsets of X extending I. A typical example is when $I = I_{\text{meager}}$; the exterior extension of I is the σ-ideal of meager sets. Analogously the exterior extension of $K_\omega(X)$ is the σ-ideal of countable sets.

In some cases, however, the exterior extension is not the natural one. For example: if λ is the product measure on 2^ω and $I = I_\lambda$ then I^{ext} is not the σ-ideal of λ-measure zero sets. But this example suggests another way of extending I: Let

$$I^{\text{int}} = \{A \subseteq X : \mathcal{K}(A) \subseteq I\}$$

Clearly I^{int} is hereditary, $I^{\text{ext}} \subseteq I^{\text{int}}$ and $I^{\text{int}} \cap \mathcal{K}(X) = I$. But in general I^{int} is not even an ideal.

We say that a σ-ideal I on X has the *covering property* if $I^{\text{ext}} = I^{\text{int}}$ for $\boldsymbol{\Sigma}^1_1$ sets, i.e. a $\boldsymbol{\Sigma}^1_1$ set A is in I^{int} iff A is in I^{ext} (see e.g. [6] and also the notion of I-regularity of [9]). This is a quite strong property, in fact the only known σ-ideals of compact sets that have the covering property are $K_\omega(X)$ and U_0. For $K_\omega(X)$, the classical perfect set theorem for $\boldsymbol{\Sigma}^1_1$ sets is the assertion that $K_\omega(X)$ has the covering property. For U_0, it is a theorem of Debs and Saint Raymond (see [2]).

In the space ω^ω there have been studied some notions of σ-ideals which have a property similar to the covering property. For instance: the σ-ideal of σ-bounded sets of [4] and the well-founded and parametrized σ-ideal of [9].

In this article we undertake a study of the covering property from the descriptive set theoretic point of view. We will use as a paradigm the σ-ideal of countable closed sets, specifically the following five properties:

(1) The classical perfect set theorem.

(2) The collection of $\boldsymbol{\Sigma}^1_1$ countable sets is Π^1_1 on the codes.

(3) The effective version of the perfect set theorem says that a Σ^1_1 countable set contains only hyperarithmetic points.

(4) There is a largest Π^1_1 set without perfect subsets.

(5) The perfect set theorem can be extended to Σ^1_2 sets from large cardinals axioms, and it is false for Π^1_1 sets in the constructible universe.

This article is divided into five sections respectively dealing with the five properties mentioned above. In fact, we will show that similar results hold for σ-ideals of compact sets with the covering property.

2. The covering property and some related notions. We will work with the effective methods of descriptive set theory, so we assume that X is recursively presented (see [11]). We will use standard notions of descriptive set theory as in Moschovakis' book [11] and the notations from [8]. For instance, $\boldsymbol{\Sigma}^1_1$ denotes the *analytic sets*, i.e. the continuous images of Borel sets, and $\boldsymbol{\Pi}^1_1$ denotes the collection of *coanalytic sets*, i.e. sets whose complements are analytic. The corresponding effective pointclasses are denoted respectively by Σ^1_1 and Π^1_1.

The collection of compact subsets of X becomes itself a compact, metric space under the usual metric:

$$\varrho(K,L)=\begin{cases}\sup\{\max\{d(x,K),d(y,L)\}:x\in K,\ y\in L\} & \text{if } K,L\neq\emptyset,\\ \operatorname{diam}(X) & \text{if } K \text{ or } L=\emptyset,\\ 0 & \text{if } K=L=\emptyset.\end{cases}$$

All topological and descriptive set theoretic notions concerning $\mathcal{K}(X)$ refer to this space (for more details about the topology of $\mathcal{K}(X)$ see [8] and the references given there). For instance, for the most part we will impose a definability condition on I, namely, it has to be a Π^1_1 subset of $\mathcal{K}(X)$.

As noted in the introduction, for each σ-ideal I of closed subsets of X, there are two classes of (arbitrary) subsets of X associated with I: I^{int} and I^{ext}.

DEFINITION 2.1. We say that I has the *covering property* if for every $\boldsymbol{\Sigma}^1_1$ set $A\in I^{\text{int}}$, there is a countable collection $\{F_n\}$ of closed sets in I such that $A\subseteq\bigcup_n F_n$. In general for a pointclass Γ we say that I has the *covering property for Γ-sets* if for every $A\in\Gamma$ with $A\in I^{\text{int}}$ there is a countable collection $\{F_n\}$ of closed sets in I such that $A\subseteq\bigcup_n F_n$.

Observe that for a σ-ideal I consisting of meager sets, the covering property implies that $\boldsymbol{\Sigma}^1_1$ sets in I^{int} are of first category, i.e., they are also small in the sense of category.

As mentioned before, the classical perfect set theorem for $\boldsymbol{\Sigma}^1_1$ sets says that $K_\omega(X)$ has the covering property. So, we can regard this property as an abstraction of the content of the perfect set theorem. Since in ZFC this theorem cannot be extended to Π^1_1 sets, we do not expect to have (in ZFC) the covering property for Π^1_1 sets (we will look at this problem in Section 6).

In this section we will introduce some notions related to the covering property and show some structural and definability consequences of the covering property. As a corollary we will obtain a result of Kaufman about sets of extended uniqueness and also a partial answer to a question raised in [8].

Definition 2.2. A σ-ideal I is *calibrated* if for every closed set F the following holds: If for some collection $\{F_n\}$ of closed sets in I, $F - \bigcup_n F_n \in I^{\mathrm{int}}$, then $F \in I$.

A typical calibrated σ-ideal is the collection of closed null sets with respect to some Borel measure. On the other hand, the σ-ideal of closed meager sets is not calibrated. Notice also that the covering property clearly implies calibration.

Let B be a hereditary subset of $\mathcal{K}(X)$. Then B_σ denotes the smallest σ-ideal (of closed sets) containing B, i.e., $K \in B_\sigma$ if there is a sequence $\{K_n\}$ of elements of B such that $K = \bigcup_n K_n$. We say that I has a *Borel basis* if there is a Borel hereditary set $B \subseteq I$ such that $I = B_\sigma$. I is called *locally non-Borel* if for every closed set $F \notin I$, $I \cap \mathcal{K}(F)$ is not Borel.

The only criterion known to show that a σ-ideal has the covering property is the following theorem, which was originally used to show that the σ-ideal of closed sets of uniqueness does not have a Borel basis (see [7] for a proof of both results).

Theorem 2.3 (Debs–Saint Raymond [2]). *Let I be a calibrated, locally non-Borel, Π^1_1 σ-ideal. If I has a Borel basis, then I has the covering property.* ■

Kechris [6] has asked to characterize the σ-ideals which have the covering property. As already noted, it implies calibration, but it is not known if the other hypotheses of the previous theorem are necessary. Recall here that a Π^1_1 σ-ideal I satisfies the so-called *dichotomy theorem*: It is either a true Π^1_1 set or a G_δ set (see [8]). So, the problem is to show that no G_δ σ-ideal has the covering property.

The usual way to show that the covering property fails for a σ-ideal I consisting of meager sets is to find a dense G_δ set G with $G \in I^{\mathrm{int}}$. In fact, by the Baire category theorem such a G cannot be covered by countably many closed meager sets. In other words, the covering property fails for a G_δ set. This is the case, for instance, when I consists of the null sets with respect to a Borel measure.

The following notion is quite useful: A non-empty set A is said to be *locally not in I* (or *I-perfect*) if for every open set V with $V \cap A \neq \emptyset$, we have $\overline{V \cap A} \notin I$. Notice that A is I-perfect iff $\overline{A}$ is I-perfect. Given a closed set $F \notin I$, there is a closed $F' \subseteq F$ such that F' is locally not in I. In fact, let

$O = \bigcup\{V \subseteq X : V$ is open and $F \cap V \in I^{\text{ext}}\}$. Put $F' = F - O$. It is easy to check that F' is locally not in I. F' is called the *I-perfect kernel* of F.

We will see later on that it is convenient to restrict attention to the covering property for $\boldsymbol{\Pi}_2^0$ sets. We have the following useful characterization of this notion:

LEMMA 2.4. *Let I be a σ-ideal of compact sets. The following are equivalent*:

(i) *I has the covering property for $\boldsymbol{\Pi}_2^0$ sets.*
(ii) *For each $\boldsymbol{\Pi}_2^0$ set G such that $\overline{G}$ is locally not in I, we have $G \notin I^{\text{int}}$.*

Proof. (i)⇒(ii). Let G be a G_δ set such that $M = \overline{G}$ is locally not in I. Suppose, towards a contradiction, that $G \in I^{\text{int}}$. By (i) there is a sequence $\{F_n\}$ of sets in I such that $G \subseteq \bigcup_n F_n$. By the Baire category theorem there is an n and an open set V such that $\emptyset \neq G \cap V \subseteq F_n$. Hence $\overline{V \cap M} = \overline{V \cap G} \subseteq F_n$. So, $\overline{V \cap M} \in I$, which contradicts M being locally not in I.

(ii)⇒(i). Let G be a $\boldsymbol{\Pi}_2^0$ set in I^{int}. Assume towards a contradiction that $G \notin I^{\text{ext}}$. Let $O = \bigcup\{V \subseteq X : V$ is an open set and $V \cap G \in I^{\text{ext}}\}$. Let $G' = G - O$. As $G \notin I^{\text{ext}}$, we have $G' \neq \emptyset$. It is clear that for all V open, if $V \cap G' \neq \emptyset$ then $V \cap G' \notin I^{\text{ext}}$. Clearly G' is a $\boldsymbol{\Pi}_2^0$ set in I^{int} and for every open set V, if $V \cap G' \neq \emptyset$ then $\overline{V \cap G'} \notin I$. Therefore $M = \overline{G}$ is locally not in I, which contradicts (ii). ∎

The next type of σ-ideals that we are going to consider are the thin σ-ideals. This notion was introduced in [8] and it corresponds dually to the countable chain condition. We say that I is *thin* if every collection of pairwise disjoint closed sets not in I is at most countable. The typical example of a thin σ-ideal is the collection of null sets for some Borel measure. The next theorem relates thinness to the covering property.

THEOREM 2.5. *Let I be a σ-ideal of closed sets which satisfies one of the following non-triviality conditions*:

(i) *$I \neq \mathcal{K}(X)$ and for every $x \in X$, $\{x\} \in I$.*
(ii) *Every $K \in I$ is a meager set.*

If I is thin, then I does not have the covering property for $\boldsymbol{\Pi}_2^0$ sets. Actually, if I is thin and (ii) *holds, then there is a dense G_δ set in I^{int}.*

Proof. Assume first that (i) holds. Let $O = \bigcup\{V \subseteq X : V$ is open and $V \in I^{\text{ext}}\}$. Then O is the largest open set in I^{ext}. Put $K = X - O$; then K is locally not in I (if $V \cap K \neq \emptyset$, then $\overline{V \cap K} \notin I$, otherwise $V \subseteq O$). As $I \neq \mathcal{K}(X)$ and every singleton is in I, K is a (non-empty) perfect set. Let G be a dense G_δ subset of K with empty interior with respect to the relative topology of K. Let $\{K_n\}$ be a maximal collection of pairwise disjoint closed subsets of G with each $K_n \notin I$. Each K_n is meager in K. Put $F = \bigcup_n K_n$ and $H = G - F$. Then H is a dense (in K) G_δ subset of K. Clearly $H \in I^{\text{int}}$, hence by 2.4, I does not have the covering property for $\boldsymbol{\Pi}^0_2$ sets.

Now if (ii) holds, then X is locally not in I, hence the same proof applies. Finally, observe that in this case we get a dense G_δ set in I^{int}. ■

Remark. (i) Beside $I \neq \mathcal{K}(X)$, some other non-triviality condition has to be imposed on I in order to get the conclusion of 2.5, as the following example shows: let $F \subseteq X$ be a countable closed set and $V = X - F$. Put $I = \mathcal{K}(V)$. I is thin, because $K \notin I$ iff $K \cap F \neq \emptyset$. Thus there are only countably many disjoint sets not in I. However, I trivially satisfies the covering property (because $V \in I^{\text{ext}}$ and if $H \in I^{\text{int}}$ then $H \subseteq V$).

(ii) Normally, we will use 2.5 as follows. Suppose that every Borel set in I^{int} is of the first category ($\boldsymbol{\Pi}^0_2$ sets suffice). Then I is not thin. Just notice that in this case every set in I is meager.

The following notion was introduced in [8]. A set $A \subseteq X$ is called *I-thin* if there is no uncountable family of pairwise disjoint closed subsets of A which are not in I. In other words, A is I-thin if the restriction of I to $\mathcal{K}(A)$ is a thin σ-ideal. Given a σ-ideal I define another σ-ideal J_I as follows:

$$K \in J_I \quad \text{iff} \quad K \text{ is } I\text{-thin}.$$

It was proved in [8] that if I is a Π^1_1 calibrated σ-ideal then so is J_I. It was asked there to find out for a given I whether $J_I = I$. In this connection we have the following

COROLLARY 2.6. *Let I be a σ-ideal of closed subsets of X containing all singletons. If I has the covering property for $\boldsymbol{\Pi}^0_2$ sets, then $I = J_I$.*

Proof. It is clear that $I \subseteq J_I$. Now, let F be a closed set not in I. We want to show that $F \notin J_I$. We can assume without loss of generality that F is locally not in I. Hence as I contains all singletons, F is perfect. Put $\widetilde{I} = \mathcal{K}(F) \cap I$. Then $\widetilde{I}$ is non-trivial in the sense of 2.5(i) and it has the covering property for $\boldsymbol{\Pi}^0_2$ sets: if $H \subseteq F$ is a $\boldsymbol{\Pi}^0_2$ set in $\widetilde{I}^{\text{int}}$ then $H \in I^{\text{int}}$. Hence, by the covering property for I, $H \in I^{\text{ext}}$. This clearly implies that $H \in \widetilde{I}^{\text{ext}}$. Therefore, by 2.5, $\widetilde{I}$ is not thin, i.e., $F \notin J_I$. ■

COROLLARY 2.7 (Kaufman). *Let U_0 denote the σ-ideal of closed sets of extended uniqueness in the unit circle. Then $U_0 = J_{U_0}$.*

Proof. Debs and Saint Raymond [2] have shown that U_0 has the covering property. ■

Theorem 2.5 says that a non-trivial Π^1_1 thin σ-ideal I does not have the covering property. In [8] (p. 287, question 1) it was asked whether every calibrated thin Π^1_1 σ-ideal is $\boldsymbol{\Pi}^0_2$; this question and 2.5 seem to be related to the conjecture that no $\boldsymbol{\Pi}^0_2$ σ-ideal has the covering property. The next theorem is a partial answer to this question.

THEOREM 2.8. *If I is a calibrated, thin, Π^1_1 σ-ideal of closed sets with a Borel basis, then I is $\boldsymbol{\Pi}^0_2$.*

Proof. Let $\{F_n\}$ be a maximal pairwise disjoint countable collection of closed sets such that for each n, $F_n \notin I$ and $I \cap \mathcal{K}(F_n)$ is $\boldsymbol{\Pi}^0_2$. Put $F = \bigcup_n F_n$ and $H = X - F$. We claim that $H \in I^{\mathrm{int}}$. Granting this claim we have:

$$(*) \qquad K \in I \quad \text{iff} \quad (\forall n)(K \cap F_n \in I)\,.$$

The direction $\Rightarrow$ is trivial. On the other hand, let $K \subseteq X$ be a closed set. Then $K = (K \cap H) \cup \bigcup_n (K \cap F_n)$. Suppose that each $K \cap F_n \in I$. As I is calibrated and $K \cap H \in I^{\mathrm{int}}$, we obtain $K \in I$.

Now, the map $K \mapsto K \cap F_n$ is Borel, so $(*)$ says that I is Borel. Therefore by the dichotomy theorem (see [8], Theorem 1.7), I is $\boldsymbol{\Pi}^0_2$.

It remains to show that H is in I^{int}. Suppose not. Let $M \subseteq H$ be a closed set locally not in I. Since $\{F_n\}$ is maximal, $\{x\} \in I$ for every $x \in M$. Hence M is a perfect set. Consider the σ-ideal $I_0 = \mathcal{K}(M) \cap I$. It is clearly a calibrated, thin (non-trivial as in 2.5,) Π^1_1 σ-ideal with a Borel basis. As $\{F_n\}$ is maximal, $\mathcal{K}(F) \cap I_0 = \mathcal{K}(F) \cap I$ is not $\boldsymbol{\Pi}^0_2$ for every $F \subseteq M$ with $F \notin I_0$. Hence I_0 is locally non-Borel and thus all the hypotheses of the Debs–Saint Raymond theorem (2.3) are satisfied. Therefore I_0 has the covering property, but also it is non-trivial and thin, which contradicts 2.5. ∎

This raises the following question: Does every calibrated, thin Π^1_1 σ-ideal have a Borel basis?

3. Complexity of the codes. As noted in the introduction, another feature of the σ-ideal of countable sets is that it is Π^1_1 on the codes of $\boldsymbol{\Sigma}^1_1$ sets. We will present an abstract version of this result as a consequence of the covering property. The key notion involved is the following:

DEFINITION 3.1. A σ-ideal I is *strongly calibrated* if for every closed set $F \subseteq X$ with $F \notin I$ and every $\boldsymbol{\Pi}^0_2$ set $H \subseteq X \times 2^\omega$ such that $\mathrm{proj}(H) = F$, there is a closed set $K \subseteq H$ such that $\mathrm{proj}(K) \notin I$.

This notion was introduced in [8] and proved to imply calibration (see [8], p. 283). Also, it is easy to check that one can take projections of $\boldsymbol{\Sigma}^1_1$ subsets of any compact Polish space in the definition of strong calibration as follows: Let Y be a compact Polish space. If $F \subseteq X$ is a closed set not in I and $Q \subseteq X \times Y$ is a $\boldsymbol{\Sigma}^1_1$ set such that $\mathrm{proj}(Q) = F$, then there is a closed set $K \subseteq Q$ such that $\mathrm{proj}(K) \notin I$.

Strong calibration resembles the conclusion of Choquet's capacitability theorem and in fact this theorem implies that the σ-ideal of closed measure zero sets for a collection of Borel measures is strongly calibrated: Let $\mathcal{M}$ be a collection of Borel measures on X and let $I = \mathrm{Null}(\mathcal{M})$. Let $Q \subseteq X \times 2^\omega$ be a $\boldsymbol{\Pi}^0_2$ set such that $\mathrm{proj}(Q) = F \notin I$, and say $\mu(F) > 0$ for some $\mu \in \mathcal{M}$. Define a capacity γ on $X \times 2^\omega$ as follows:

$$\gamma(A) = \mu^*(\mathrm{proj}(A)), \quad \text{for } A \subseteq X \times 2^\omega .$$

As Q is $\boldsymbol{\Pi}^0_2$ and $\gamma(Q) > 0$, by Choquet's capacitability theorem there is a compact set $K \subseteq Q$ such that $\gamma(K) > 0$. Hence $\mathrm{proj}(K) \notin I$.

This type of σ-ideals have the property that the collection of $\boldsymbol{\Sigma}^1_1$ sets in I^{int} is Π^1_1 on the codes of $\boldsymbol{\Sigma}^1_1$ sets (assuming that I is Π^1_1). The usual argument to show this uses the capacitability theorem. We show next that strongly calibrated σ-ideals also have this property.

THEOREM 3.2. *Let I be a Π^1_1 strongly calibrated σ-ideal of closed subsets of X. Then the collection of $\boldsymbol{\Sigma}^1_1$ sets in I^{int} is Π^1_1 on the codes of $\boldsymbol{\Sigma}^1_1$ sets.*

Proof. Let $\mathcal{U} \subseteq 2^\omega \times X$ be a Σ^1_1 universal set for $\boldsymbol{\Sigma}^1_1$ subsets of X. Let $Q \subseteq (2^\omega \times X) \times 2^\omega$ be a Π^0_2 set such that $\mathcal{U} = \mathrm{proj}(Q)$. Consider the following relation:

$$R(F,\alpha) \quad \text{iff} \quad F \subseteq \mathcal{U}_\alpha \ \&\ F \notin I .$$

Then we have

$$\mathcal{U}_\alpha \notin I^{\mathrm{int}} \quad \text{iff} \quad (\exists F) R(F,\alpha) .$$

Hence it suffices to show that R is Σ^1_1. We claim that

$$(*) \qquad R(F,\alpha) \quad \text{iff} \quad (\exists K \in \mathcal{K}(2^\omega \times X))(K \subseteq Q^\alpha \ \&\ \mathrm{proj}(K) \notin I) .$$

The direction $\Leftarrow$ clearly holds. For the other, suppose that $R(F,\alpha)$ holds and put $H = Q^\alpha \cap (2^\omega \times F)$. Then $\mathrm{proj}(H) = F$. As H is $\boldsymbol{\Pi}^0_2$, by strong calibration there is a closed $K \subseteq H$ such that $\mathrm{proj}(K) \notin I$; this K clearly works.

To see that $(*)$ is a Σ^1_1 relation we use the uniformization theorem for relations with K_σ sections (see Theorem 4F.16 in [11]) to conclude that the relation $K \subseteq Q^\alpha$ is Δ^1_1. (In fact, since we are working with compact spaces the projection of a $\boldsymbol{\Sigma}^0_2$ is also $\boldsymbol{\Sigma}^0_2$; with this in mind it is easy to check that $K \subseteq Q^\alpha$ is a $\boldsymbol{\Pi}^0_2$ relation between K and α.) On the other hand, by a similar argument it is easy to see that the function $K \mapsto \mathrm{proj}(K)$ is Δ^1_1-recursive (it is clearly continuous). ■

THEOREM 3.3. *Let I be a σ-ideal of closed subsets of X. If I has the covering property for $\boldsymbol{\Pi}^0_2$ sets, then I is strongly calibrated.*

Proof. Let F be a closed set not in I and let $Q \subseteq X \times 2^\omega$ be a $\boldsymbol{\Pi}^0_2$ set such that $F = \mathrm{proj}(Q)$. Without loss of generality we can assume that F is locally not in I. By the von Neumann selection theorem (see 4E.9 in [11]) there is a Baire measurable function f such that for all $x \in F$, $(x, f(x)) \in Q$.

By the analog of the Lusin theorem for category (see [12]), there is a G_δ set $G \subseteq F$ dense in F such that f is continuous on G. Since I has the covering property for $\boldsymbol{\Pi}^0_2$ sets, 2.4 shows that $G \notin I^{\mathrm{int}}$. Thus, there is a closed set $K \subseteq F$ with $K \notin I$. Let $K^* =$ graph of f restricted to K. As f is continuous on K, K^* is a closed set and clearly $\mathrm{proj}(K^*) = K$. ■

COROLLARY 3.4. *Let I be a Π^1_1 locally non-Borel σ-ideal with a Borel basis. Then I is calibrated iff I is strongly calibrated.*

P r o o f. It was proved in [8] (p. 283) that strong calibration implies calibration. On the other hand, by the Debs–Saint Raymond theorem (2.3) every σ-ideal as in the hypothesis above has the covering property. Hence, by the previous theorem it is strongly calibrated. ■

From the proof of 3.3 one gets the following: Let us say that a σ-ideal I has the *continuity property* if for every Baire measurable function f with $\mathrm{dom}(f) = F \notin I$ (F a closed set), there is a closed set $K \subseteq F$ such that $K \notin I$ and f is continuous on K.

COROLLARY 3.5 (of the proof of 3.3). *Let I be a σ-ideal of closed subsets of X.*

(i) *If I has the covering property for $\boldsymbol{\Pi}^0_2$ sets, then I has the continuity property.*

(ii) *If I has the continuity property, then I is strongly calibrated.* ■

R e m a r k. Observe that if I is strongly calibrated, then I has the continuity property for Borel functions: Just apply the definition of strong calibration to the graph of f.

Strong calibration is not equivalent to the covering property for $\boldsymbol{\Pi}^0_2$ sets, because as already mentioned $\mathrm{Null}(\mu)$ is strongly calibrated but it does not have the covering property.

Calibration is equivalent to being $I^{\mathrm{int}} \cap \boldsymbol{\Pi}^0_2(X)$ being a σ-ideal (see Proposition 1 of §3 in [8]). The next lemma shows that for strong calibration we get a similar result for $\boldsymbol{\Sigma}^1_1$ sets.

LEMMA 3.6. *Let I be a strongly calibrated σ-ideal.*

(i) *If F is a closed set such that $F = P \cup \bigcup_n F_n$, for some $\boldsymbol{\Sigma}^1_1$ set P in I^{int} and each F_n in I, then $F \in I$. In particular, I is calibrated.*

(ii) *$\{P \subseteq X : P$ is a $\boldsymbol{\Sigma}^1_1$ set in $I^{\mathrm{int}}\}$ is a σ-ideal.*

(iii) *Define a collection $J \subseteq \mathcal{K}(X \times 2^\omega)$ as follows:*

$$K \in J \quad \textit{iff} \quad \mathrm{proj}(K) \in I.$$

Then J is a calibrated σ-ideal.

P r o o f. (i) Let $F = P \cup \bigcup_n F_n$ be a closed set not in I with P a $\boldsymbol{\Sigma}^1_1$ set and each F_n in I. We will show that $P \notin I^{\mathrm{int}}$. Let $G \subseteq X \times 2^\omega$ be a $\boldsymbol{\Pi}^0_2$

set such that $\mathrm{proj}(G) = P$. Put

$$Q = (G \times \{0\}) \cup \bigcup_n (F_n \times 2^\omega \times \{1\}) .$$

Then $Q \subseteq X \times (2^\omega \times (\omega + 1))$ and $\mathrm{proj}(Q) = F$. By strong calibration there is a $K \subseteq Q$ closed such that $\mathrm{proj}(K) \notin I$. Now, we have

$$K = K \cap (G \times \{0\}) \cup \bigcup_n K \cap (F_n \times 2^\omega \times \{1\}) .$$

Hence

$$\mathrm{proj}(K) = \mathrm{proj}(K \cap (G \times \{0\})) \cup \bigcup_n \mathrm{proj}(K \cap (F_n \times 2^\omega \times \{1\})) .$$

Since $K \cap (G \times \{0\})$ is closed in $X \times (2^\omega \times (\omega + 1))$ and $\mathrm{proj}(K \cap (F_n \times 2^\omega \times \{1\})) \subseteq F_n \in I$, we have $\mathrm{proj}(K \cap (G \times \{0\})) \notin I$. Thus $\mathrm{proj}(G) = P \notin I^{\mathrm{int}}$.

We show (iii) first. It is clear that J is a σ-ideal. Let $K = G \cup \bigcup_n K_n$, where $K \subseteq X \times 2^\omega$ is closed, G is a $\boldsymbol{\Pi}^0_2$ set in J^{int} and each K_n is in J. Now, $\mathrm{proj}(K) = \mathrm{proj}(G) \cup \bigcup_n \mathrm{proj}(K_n)$. As $\mathrm{proj}(K_n)$ is a closed set in I, it suffices to show that $\mathrm{proj}(G) \in I^{\mathrm{int}}$ and then apply (i). Let $F \subseteq \mathrm{proj}(G)$ and suppose toward a contradiction that $F \notin I$. By strong calibration there is a $K \subseteq (F \times 2^\omega) \cap G$ closed such that $\mathrm{proj}(K) \notin I$. This contradicts G begin in J^{int}.

(ii) It is easy to check (as in (iii)) that strong calibration implies that

$$\{P \subseteq X : P \in \boldsymbol{\Sigma}^1_1(X) \cap I^{\mathrm{int}}\} = \{\mathrm{proj}(G) : G \in \boldsymbol{\Pi}^0_2(X \times 2^\omega) \cap J^{\mathrm{int}}\} .$$

Since J is calibrated the collection of $\boldsymbol{\Pi}^0_2$ sets in J^{int} is a σ-ideal (in fact $\boldsymbol{\Sigma}^0_3$ sets, see Proposition 1, §3 in [8]), from which the claim follows. ∎

The next lemma relates the covering property of I and J; it will be used in Section 6.

Lemma 3.7. *Let I be a σ-ideal and J be the σ-ideal defined in* 3.6(iii). *Then the following are equivalent*:

(i) *J has the covering property.*
(ii) *J has the covering property for $\boldsymbol{\Pi}^0_2$ sets.*
(iii) *I has the covering property.*

Proof. Clearly (i)⇒(ii).

(ii)⇒(iii). Let P be a $\boldsymbol{\Sigma}^1_1$ set in I^{int} and let $G \subseteq X \times 2^\omega$ be a $\boldsymbol{\Pi}^0_2$ set such that $\mathrm{proj}(G) = P$. Clearly $G \in J^{\mathrm{int}}$. Hence there are closed sets $K_n \in J$ such that $G \subseteq \bigcup_n K_n$. Each $\mathrm{proj}(K_n) \in I$ and $\mathrm{proj}(G) \subseteq \bigcup_n \mathrm{proj}(K_n)$.

(iii)⇒(i). Let $G \subseteq X \times 2^\omega$ be a $\boldsymbol{\Sigma}^1_1$ set with $G \in J^{\mathrm{int}}$. By 3.3, I is strongly calibrated, hence (as in the proof of 3.6(ii)) $\mathrm{proj}(G) \in I^{\mathrm{int}}$. So, there are closed sets F_n in I such that $\mathrm{proj}(G) \subseteq \bigcup_n F_n$. Thus $G \subseteq \bigcup_n F_n \times 2^\omega$ and clearly for all n, $F_n \times 2^\omega \in J$. ∎

If I has the covering property then for every $\boldsymbol{\Sigma}^1_1$ set $A \in I^{\text{int}}$ there is a Borel (actually an F_σ) set $B \in I^{\text{int}}$ with $A \subseteq B$. The next result shows that this is also a consequence of strong calibration, which in particular says that the covering property for Borel sets implies the covering property (for $\boldsymbol{\Sigma}^1_1$ sets).

THEOREM 3.8. *Let I be a strongly calibrated Π^1_1 σ-ideal. Let A be a Σ^1_1 set in I^{int}. Then there is a Δ^1_1 set $B \in I^{\text{int}}$ such that $A \subseteq B$. Therefore, if we let*

$$H(I) = \bigcup \{B \subseteq X : B \text{ is } \Delta^1_1 \text{ and } B \in I^{\text{int}}\},$$

then

(i) *$H(I)$ is a Π^1_1 set in I^{int}.*
(ii) *For every Σ^1_1 set A, $A \in I^{\text{int}}$ iff $A \subseteq H(I)$.*

Proof. The first claim follows from the reflection principle but we give a direct proof anyway. Let A be a Σ^1_1 set in I^{int} and put $P = X - A$. Let φ be a Π^1_1 norm on P and consider

$$M = \{x \in X : \{y : \neg(y <^*_\varphi x)\} \in I^{\text{int}}\}.$$

As in the proof of Proposition 3.2, M is Π^1_1. We claim that $A \subseteq M$. In fact, if $x \in A$ then by definition of $<^*_\varphi$ we have

$$\{y : \neg(y <^*_\varphi x)\} = A.$$

By separation, let $B \subseteq M$ be a Δ^1_1 set with $A \subseteq B \subseteq M$. If $A = B$ we are done. Else let ξ be the least ordinal in $\{\varphi(x) : x \in B\}$ and let $x \in B$ with $\varphi(x) = \xi$. Then

$$B \subseteq \{y : \neg(y <^*_\varphi x)\}.$$

Hence $B \in I^{\text{int}}$.

From Lemma 3.6 we know that the collection of $\boldsymbol{\Sigma}^1_1$ sets in I^{int} forms a σ-ideal, so $H(I) \in I^{\text{int}}$. As in the proof of 3.2 we can show that $H(I)$ is Π^1_1. This proves (i). And (ii) follows from (i) and the first claim. ■

The set $H(I)$ can be thought of as an abstract version of the hyperarithmetic reals. A better description of it will be given in the next section. By Theorem 3.3 the covering property for G_δ sets implies strong calibration, thus from the relativized version of the previous theorem we immediately get

THEOREM 3.9. *Let I be a Π^1_1 σ-ideal. If I has the covering property for Borel sets, then it has the covering property.* ■

4. Analog of the hyperarithmetic reals. One of the consequences of the effective perfect set theorem is that a countable Σ^1_1 set contains only hyperarithmetic points, i.e., Δ^1_1 points. We will present an abstract version of this result for Π^1_1 σ-ideals with the covering property. The main theorem is the following strengthening of 3.8:

THEOREM 4.1. *Let I be a Π^1_1 σ-ideal on 2^ω with the covering property. Let*

$$H(I)=\bigcup\{[T]:T \text{ is a } \Delta^1_1 \text{ binary tree and } [T]\in I\}\,.$$

Then

(i) *$H(I)$ is a Π^1_1 set in I^{ext}.*
(ii) *For every Σ^1_1 set A, $A\in I^{\mathrm{int}}$ iff $A\subseteq H(I)$.*

The key lemma used in the proof of this theorem is the following result due to Barua–Srivatsa ([1]); its proof is similar to the proof of Theorem 3.8 and slightly different from the one given in [1].

LEMMA 4.2 (see Barua–Srivatsa [1]). *Let I be a Π^1_1 σ-ideal on 2^ω and T a Σ^1_1 binary tree such that $[T]\in I$. Then there is a Δ^1_1 binary tree S such that $[T]\subseteq[S]$ and $[S]\in I$.*

Proof. Let Seq denote the collection of binary sequences. Fix a Π^1_1 norm φ on Seq $-T$ and set

$$A_s=\{t\in \mathrm{Seq}:\exists t' \text{ extending } t \ \&\ \neg(t'<^*_\varphi s)\}\,.$$

Notice that A_s is a tree. Let

$$M=\{s\in\mathrm{Seq}:[A_s]\in I\}\,.$$

As A_s is Σ^1_1 one can easily check (as in 3.2) that the property $[A_s]\in I$ is Π^1_1. Hence M is a Π^1_1 set. Observe that $T\subseteq M$: If $s\in T$, it is easy to see that $A_s=T$, hence $s\in M$. So, unless T is Δ^1_1 (in which case there is nothing to prove), there is an $s\in M-T$. Thus we let

$$\alpha=\text{least ordinal of } \{\varphi(s):s\in M\}$$

and let $s_0\in M$ such that $\varphi(s_0)=\alpha$.

It is clear that $M\subseteq A_{s_0}$: If $s\in M$ then $\neg(s<^*_\varphi s_0)$, hence $s\in A_{s_0}$. Let B be a Δ^1_1 set such that $T\subseteq B\subseteq M$ and let $S=\{t\in\mathrm{Seq}:\exists t'$ extending $t\ \&\ t'\in B\}$. Then S is a Δ^1_1 tree and thus $S\subseteq A_{s_0}$. Hence $[S]\in I$. ■

Proof of Theorem 4.1. (i) It is obvious that $H(I)$ is in I^{ext}, and by the theorem on restricted quantification (4D.3 in [11]) we conclude that $H(I)$ is Π^1_1.

To see (ii), let $H=H(I)$, and suppose $A\not\subseteq H$. Let $A^*=A-H$. Then A^* is a Σ^1_1 set in I^{int}. So, let $\{K_n\}$ be closed sets in I such that $A^*\subseteq\bigcup_n K_n$. By working with the Σ^1_1-topology (see [10]) and by the Baire category theorem we know that there is a Σ^1_1 set V such that $\emptyset\neq V\cap A^*\subseteq K_n$ for some n. Let T be the tree of $\overline{V\cap A^*}$. It is clearly Σ^1_1 and $[T]\in I$. Hence by Lemma 4.2

there is a Δ^1_1 tree S such that $T \subseteq S$ and $[S] \in I$. Thus $[S] \subseteq H$, contrary to $A^* \cap H = \emptyset$. ∎

As a corollary we find that the covering property holds effectively.

COROLLARY 4.3 (see Barua–Srivatsa [1]). *Let I be a Π^1_1 σ-ideal on 2^ω with the covering property. Let A be a Σ^1_1 set in I^{int}. Then there is a Δ^1_1 recursive function $f : \mathbb{N} \to \omega^\omega$ such that, for all n, $f(n)$ is a binary tree with $[f(n)] \in I$ and $A \subseteq \bigcup_n [f(n)]$.*

Proof. The proof is a standard application of the selection principle.

By separation there is a Δ^1_1 set A^* such that $A \subseteq A^* \subseteq H(I)$. Let $\mathbf{d}(i)$ be the canonical function that enumerates the Δ^1_1 points (see [11], 4D.2). Consider the following relation:

$$D(x,i) \quad \text{iff} \quad \mathbf{d}(i){\downarrow} \ \&\ \mathbf{d}(i) \text{ codes a } \Delta^1_1 \text{ binary tree } T \ \&\ x \in [T] \ \&\ [T] \in I\,.$$

It is easy to check that D is Π^1_1.

We see that for all $x \in A^*$, there is an i such that $D(x,i)$ holds. Hence by the △-selection principle (see [11], 4B.5), there is a Δ^1_1-recursive function $g : 2^\omega \to \omega$ such that for all $x \in A^*$, $D(x, g(x))$ holds. Let R be the range of g; then R is Σ^1_1. Put

$$S = \{i : \mathbf{d}(i){\downarrow} \ \&\ \mathbf{d}(i) \text{ codes a } \Delta^1_1 \text{ binary tree } T \text{ with } [T] \in I\}\,.$$

Then S is Π^1_1 and $R \subseteq S$. So, by separation there is a Δ^1_1 set R^* such that $R \subseteq R^* \subseteq S$. Define f as follows:

$$f(i) = \begin{cases} T & \text{if } i \in R^* \text{ and } \mathbf{d}(i) \text{ codes } T, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then f is clearly Δ^1_1-recursive. For all i, $[f(i)] \in I$ and $A \subseteq \bigcup_i f(i)$. Also, $f(i)$ is a Δ^1_1 binary tree. ∎

5. The largest Π^1_1 set in I^{int}. It is a well known fact that there is a largest Π^1_1 thin set, i.e., a set without a perfect subset. This set is denoted by C_1 and it is characterized by $\alpha \in C_1$ iff $\alpha \in L_{\omega_1^\alpha}$ (see [3], [4] and [9] for similar results on σ-ideals on ω^ω defined by games). Another consequence of the covering property for a σ-ideal I is that there is a largest Π^1_1 set in I^{int}. In this section we will present a proof of this fact. Moreover, for σ-ideals defined on 2^ω such a set can be characterized in a fashion similar to C_1.

There is a theorem due to Kechris (see [3], 1A-2) that gives sufficient conditions for the existence of such a largest Π^1_1 set for σ-ideals of subsets of X. One of these conditions is the so-called Π^1_1-additivity. We will show next that for every σ-ideal I of meager subsets of X, if I has the covering property, then I^{int} is Π^1_1-additive. The proof is based on a representation of I as the common meager closed sets for a collection of Polish topologies on X.

Definition 5.1. For every topology τ on X, let Meager(τ) be the collection of τ-meager sets. We say that a topology τ on X is *compatible with* I if τ extends the original topology on X, every τ-open set is Borel and $I \subseteq \text{Meager}(\tau)$.

Observe that in this case the Borel structure of X and (X, τ) are the same. In particular, every C-measurable subset $B \subseteq X$ has the Baire property with respect to τ (C is the least σ-algebra containing the open sets and closed under the Suslin operation).

Lemma 5.2. *Let I be a σ-ideal of meager closed subsets of a compact Polish space X. Then*

$$I = \bigcap \{\text{Meager}(\tau) \cap \mathcal{K}(X) : \tau \textit{ is a Polish topology on } X \textit{ compatible with } I\}.$$

Proof. One direction is obvious. Let $K \notin I$. We want to find a Polish topology τ on X compatible with I and such that K is not τ-meager. Without loss of generality we assume that K is locally not in I. Let τ_0 be the given topology on X and consider the topology τ generated by

$$\tau_0 \cup \{V \cap K : V \in \tau_0\}.$$

It is a standard fact that τ is the least Polish topology for which K is τ-clopen. It remains to show that $I \subseteq \text{Meager}(\tau)$. But this is clear, because as K is locally not in I, for every $V \in \tau_0$ such that $V \cap K \neq \emptyset$ we have $\overline{V \cap K} \notin I$. Hence for every $F \in I$, $V \cap K \not\subseteq F$. ∎

Definition 5.3. We say that a subset of X is *I-meager* if it is τ-meager for every topology τ-compatible with I.

Thus the previous lemma says that a closed set is in I iff it is I-meager. As mentioned before, the key fact in the proof of the existence of the largest Π^1_1 set in I^{int} is the following

Theorem 5.4. *Let I be a σ-ideal of meager subsets of X with the covering property and let B be a subset of X with the Baire property with respect to every Polish topology compatible with I. The following are equivalent*:

(i) $B \in I^{\text{int}}$.
(ii) *B is I-meager.*

Proof. (i)⇒(ii). Suppose that B is not τ-meager for some topology τ compatible with I. As B has the Baire property for τ, there is a τ-open set V such that B is τ-comeager in V. So, let G be a τ-G_δ set τ-dense in V and $G \subseteq B$. As τ consists of Borel sets, G is also Borel. We claim that $G \notin I^{\text{int}}$. Otherwise, as I has the covering property, there are closed sets $\{F_n\}$ in I such that $G \subseteq \bigcup_n F_n$. Then by the usual argument with the Baire category theorem we deduce that one of the F_n is not τ-meager, which is a contradiction.

(ii)⇒(i). This follows immediately from the previous lemma. ■

If we trace back how much of the covering property is needed to prove this theorem we see that it would be sufficient to have the covering property for G_δ sets. This is because the topologies used in the proof of 5.2 admit a basis consisting of G_δ sets in the original topology of X. In other words, the proof of 5.2 shows that

$$I = \bigcap\{\mathrm{Meager}(\tau_K) \cap \mathcal{K}(X) : K \text{ is an } I\text{-perfect closed set and } \tau_K \text{ is the canonical Polish topology for which } K \text{ is clopen}\}.$$

In fact, the conclusion of the previous theorem is equivalent to the covering property for G_δ sets, as we show next.

LEMMA 5.5. *Let I be a σ-ideal of meager subsets of X. Then I has the covering property for G_δ sets iff I-meager $= I^{\mathrm{int}}$ for sets with the Baire property with respect to every topology compatible with I.*

Proof. One direction follows from the previous theorem and the remark we did after it. For the other direction let $G \subseteq X$ be a G_δ set such that $\overline{G}$ is I-perfect. We want to show that $G \notin I^{\mathrm{int}}$. Let $K = \overline{G}$ and let $\tau = \tau_K$ be the canonical topology for K; then it is easy to see that τ is compatible with I. By the Baire category theorem G cannot be τ-meager, hence by hypothesis $G \notin I^{\mathrm{int}}$. ■

Let us recall the definition of Π^1_1-additivity (see [3]): A hereditary collection J of subsets of X is called *Π^1_1-additive* if for every sequence $\{A_\xi\}_{\xi<\theta}$ of sets in J such that the associated prewellordering

$$x \preceq y \quad \text{iff} \quad x, y \in \bigcup_{\xi<\theta} A_\xi \ \&\ \text{least } \xi(x \in A_\xi) \le \text{least } \xi(y \in A_\xi)$$

is $\boldsymbol{\Pi}^1_1$, we have $\bigcup_{\xi<\theta} A_\xi \in J$. As mentioned before, we have the following

COROLLARY 5.6. *Let I be a σ-ideal of closed meager subsets of X with the covering property. Then I^{int} is Π^1_1-additive.*

Proof. The proof is the same as in the case of the σ-ideal of closed meager sets (see [3]). Towards a contradiction, assume θ is the least ordinal such that there is a sequence $\{A_\xi\}_{\xi<\theta}$ of sets in I^{int} such that the associated prewellordering $\preceq$ is $\boldsymbol{\Pi}^1_1$, but $\bigcup_{\xi<\theta} A_\xi \notin I^{\mathrm{int}}$.

First we observe that by the same argument as in [3], θ is a limit ordinal.

Let $K \subseteq \bigcup_{\xi<\theta} A_\xi$ with $K \notin I$ and fix a Polish topology τ compatible with I such that K is not τ-meager. The restriction of $\preceq$ to $K \times K$ is $\boldsymbol{\Pi}^1_1$ and hence it has the Baire property with respect to τ. We can assume that we are working in (K,τ). For every $x \in K$ we have

$$S_x = \{y \in K : y \preceq x\} \subseteq \bigcup_{\xi<\eta} A_\xi$$

for some $\eta < \theta$ (as θ is limit). Hence $S_x \in I^{\rm int}$ by the minimality of θ. From the previous theorem we deduce that S_x is τ-meager. By the Kuratowski–Ulam theorem (see for instance [12]) we know that for τ-comeager many y's, $S^y = \{x \in K : y \preceq x\}$ is τ-meager. As $K = S_y \cup S^y$, we conclude that K is τ-meager, which is a contradiction. ■

Corollary 5.7. *Let I be a Π^1_1 σ-ideal of closed meager subsets of X with the covering property. Then there exists a largest Π^1_1 set in $I^{\rm int}$.*

Proof. In order to apply Theorem 1A-2 in [3] we need only show that the collection of $\boldsymbol{\Sigma}^1_1$ sets in $I^{\rm int}$ is Π^1_1 on the codes. This is a consequence of the fact that I is strongly calibrated, as shown in Section 3, Theorem 3.2. ■

When we work in 2^ω, the largest Π^1_1 set in $I^{\rm int}$ can be characterized in the same fashion as C_1, the largest Π^1_1 set without perfect subsets. The main theorem is the following

Theorem 5.8. *Let I be a Π^1_1 σ-ideal of meager subsets of 2^ω with the covering property. Then there is a largest Π^1_1 set $C_1(I)$ in $I^{\rm int}$ which is characterized by*

$$x \in C_1(I) \quad \textit{iff} \quad (\exists T \in L_{\omega_1^x})(T \textit{ is a tree on } 2 \ \& \ x \in [T] \ \& \ [T] \in I)\,.$$

From now on we fix a Π^1_1 σ-ideal I of closed meager subsets of 2^ω with the covering property.

There is a derivative operator on closed sets similar to the Cantor–Bendixson derivative which will provide us with canonical closed sets to cover a given Σ^1_1 set in $I^{\rm ext}$.

Definition 5.9. Let S be a tree on $2\times\omega$; define a derivative as follows:

$$(s,u) \in S^{(1)} \quad \text{iff} \quad \overline{p[S_{(s,u)}]} \notin I\,.$$

By transfinite recursion we define S^η for every ordinal η:

$$S^{\eta+1} = (S^\eta)^{(1)}\,,$$

and for λ a limit ordinal

$$S^\lambda = \bigcap_{\eta<\lambda} S^\eta\,.$$

Notice that S^η is also a tree on $2\times\omega$ and $S^{\eta+1} \subseteq S^\eta$. Since S is countable there is a countable ordinal θ such that $S^{\theta+1} = S^\theta$. We denote this fixed point by S^∞.

Lemma 5.10. *$S^\infty = \emptyset$ iff $p[S] \in I^{\rm ext}$.*

Proof. Suppose that $S^\infty = \emptyset$. Let θ be a countable ordinal such that $S^\theta = \emptyset$. Since $([S^\eta])$ is sequence of subsets of $[S]$ that decreases to the empty set we have

$$p[S] \subseteq \bigcup\{\overline{p[S^\alpha_{(s,u)}]} : \overline{p[S^\alpha_{(s,u)}]} \in I \ \&\ \alpha < \theta \ \&\ (s,u) \in S\}\,.$$

This clearly shows that $p[S] \in I^{\rm ext}$.

On the other hand, suppose that $p[S] \in I^{\rm ext}$, say $p[S] \subseteq \bigcup K_n$ with $K_n \in I$. Let $L = [S^\infty]$. Then $L \subseteq \bigcup(K_n \times \omega^\omega)$. Suppose that $L \neq \emptyset$. By the Baire category theorem there are n and $(s,u) \in S^\infty$ such that $\emptyset \neq L \cap (N_s \times N_u) \subseteq K_n \times \omega^\omega$. Hence $\overline{p[S^\infty_{(s,u)}]} \in I$, contrary to $(s,u) \in S^\infty$. ∎

Before proving the lemmas necessary for Theorem 5.8 let us give an idea of how the proof goes. Fix a Π^1_1 set $A \in I^{\rm int}$. Let T be a recursive tree on $2 \times \omega$ such that

$$x \in A \quad \text{iff} \quad T(x) \text{ is well-founded.}$$

Let $x \in A$ and let $\xi = |T(x)|$. There is a canonical way of defining a tree S_ξ on $2 \times \xi$ such that

$$|T(x)| \le \xi \quad \text{iff} \quad S_\xi(x) \text{ is not well-founded.}$$

Put $S = S_\xi$. As $p[S]$ is a $\boldsymbol{\Sigma}^1_1$ subset of A and $A \in I^{\rm int}$, then $p[S] \in I^{\rm ext}$. We can easily translate the definition of the derivative to the space $2 \times \xi$. Hence by 5.10, $S^\infty = \emptyset$. Thus the closed sets $\overline{p[S^\alpha_{(s,u)}]}$ cover $p[S]$, as in the proof of 5.10. The key to the proof is the fact that for each of these closed sets we can find a tree $T^\alpha_{(s,u)}$ in the least admissible set containing ξ such that

$$\overline{p[S^\alpha_{(s,u)}]} \subseteq [T^\alpha_{(s,u)}] \in I\,.$$

Since clearly $\xi < \omega_1^x$, this tree belongs to $L_{\omega_1^x}$, and we are done.

We will define the trees S_ξ uniformly on the codes of ξ using the following

Lemma 5.11 (Shoenfield, see [11]). *Let T be a recursive tree on $2 \times \omega$. Let $A \subseteq 2^\omega$ be defined by*

$$x \in A \quad \textit{iff} \quad T(x) \textit{ is well-founded}.$$

Also, define for each countable ordinal ξ

$$x \in A_\xi \quad \textit{iff} \quad |T(x)| \le \xi\,.$$

There is a recursive relation $S \subseteq \omega^\omega \times 2^{<\omega} \times \omega^{<\omega}$ such that

(i) *If $\mathrm{w} \in WO$ and $|\mathrm{w}| = \xi$, then $S(\mathrm{w}) = \{(t,u) : S(\mathrm{w},t,u)\}$ is a tree on $2 \times \omega$ such that*

$$x \in A_\xi \quad \textit{iff} \quad S(\mathrm{w})(x) \textit{ is not well-founded.}$$

(ii) *There is a tree S_ξ on $2\times\xi$ (as mentioned above) such that $p[S_\xi] = A_\xi$, and this tree belongs to the least admissible set containing ξ. Moreover, given* *a sequence* $u \in \omega^{<\omega}$, *by using the wellorder of* ω *given by* w *we can think that* u *codes a sequence of ordinals* h (*and vice versa*, *given* h *we can find* u) *such that*

$$(t,u) \in S(\mathrm{w}) \quad \textit{iff} \quad (t,h) \in S_\xi\,.$$

Thus if $\mathrm{w},\mathrm{z} \in WO$ *and* $|\mathrm{w}| = |\mathrm{z}| = \xi$, *then* $S(\mathrm{w})$ *and* $S(\mathrm{z})$ *code essentially the same tree* S_ξ. ■

In the following lemma we compute the complexity of the derivative defined above.

LEMMA 5.12. *Let* I *be a* Π^1_1 σ*-ideal of closed subsets of* 2^ω *with the covering property. Let* T *and* S *be as in Lemma* 5.11.

(i) *There is a* Σ^1_1 *relation* P *on* $\omega \times \omega \times \omega^\omega$ *such that for* $\mathrm{v},\mathrm{w} \in WO$ *we have*

$$P(t,u,\mathrm{v},\mathrm{w}) \quad \textit{iff} \quad (t,u) \in [S(\mathrm{w})]^{|\mathrm{v}|}\,,$$

where $[S(\mathrm{w})]^{|\mathrm{v}|}$ *is defined as in* 5.9.

(ii) *Let* A *and* A_ξ *be defined as in* 5.11 *and suppose that* $A \in I^{\mathrm{int}}$. *For every* $\xi < \omega_1$ *and every* $\mathrm{w} \in WO$ *with* $|\mathrm{w}| = \xi$, *the closure ordinal of* $S(\mathrm{w})$ *is* $< \xi^+$ (*the least admissible ordinal greater than* ξ).

Proof. (i) Let D be the following relation on $\omega \times \omega \times \omega^\omega$:

$$D(t,u,J) \quad \text{iff} \quad J \text{ is a tree on } 2 \times \omega \ \&\ (t,u) \in J^{(1)}\,.$$

We claim that D is Σ^1_1. To see this, consider the following relation:

$$B(x,J) \quad \text{iff} \quad J \text{ is a tree on } 2 \times \omega \ \&\ x \in \overline{\mathrm{proj}[J]}\,.$$

B is clearly Σ^1_1 and $D(t,u,J)$ iff $B(J_{(t,u)}) \notin I^{\mathrm{int}}$. We have shown in Section 3 (Theorem 3.2) that the collection of $\boldsymbol{\Sigma}^1_1$ sets in I^{int} is Π^1_1 on the codes of $\boldsymbol{\Sigma}^1_1$ sets; this easily implies that D is Σ^1_1.

We will use the recursion theorem to define P. Let $\mathcal{U}$ be a Σ^1_1 universal set on $\omega \times \omega \times \omega^\omega \times \omega^\omega \times \omega^\omega$. Consider the following relation:

$$\begin{aligned} Q(t,u,\mathrm{v},\mathrm{w},\varrho) \quad \text{iff} \quad & \mathrm{v} \notin WO \text{ or } (\mathrm{v} \in LO \ \&\ \mathrm{v} \equiv \emptyset \ \&\ S(t,u,\mathrm{w})) \\ & \text{or } (\exists \mathrm{z})(\mathrm{v},\mathrm{z} \in LO \ \&\ \mathrm{v} \equiv \mathrm{z}+1 \ \&\ D(t,u,\{(l,k) : \mathcal{U}(l,k,\mathrm{z},\mathrm{w},\varrho)\})) \\ & \text{or v is limit } \&\ (\forall n)\,\mathcal{U}(t,u,\mathrm{v}\lceil n,\mathrm{w},\varrho) \end{aligned}$$

where $\mathrm{v} \equiv \emptyset$ means that v codes the empty order; $\mathrm{v} \equiv \mathrm{z}+1$ means that the linear order coded by v has a last element and z is the linear order obtained by deleting this last element; and $\mathrm{v}\lceil n$ is the linear order obtained by restricting v to $\{m : m <_{\mathrm{v}} n\}$. "v is limit" means that for all n there is m such that $n <_{\mathrm{v}} m$.

Notice that $D(t,u,A)$ holds iff $(\exists B)(B \subseteq A \ \&\ A \text{ is a tree } \&\ D(t,u,B))$ (i.e., it is a monotone operator), hence Q is Σ^1_1. By the recursion theorem there is a recursive ϱ^* such that

$$Q(t,u,\mathrm{v},\mathrm{w},\varrho^*) \leftrightarrow \mathcal{U}(t,u,\mathrm{v},\mathrm{w},\varrho^*)\,.$$

As usual, put

$$P(t,u,\mathrm{v},\mathrm{w}) \leftrightarrow \mathcal{U}(t,u,\mathrm{v},\mathrm{w},\varrho^*)\,.$$

By induction on the length of $\mathrm{v} \in WO$ one can easily show that if $\mathrm{w} \in WO$, then

$$P(t,u,\mathrm{v},\mathrm{w}) \leftrightarrow (t,u) \in [S(\mathrm{w})]^{|\mathrm{v}|}\,.$$

(ii) Let $\mathrm{w} \in WO$ with $|\mathrm{w}| = \xi$ and let $S = S(\mathrm{w})$. Then $A_\xi = p[S]$ is a $\boldsymbol{\Sigma}^1_1$ set in I^{int}. As I has the covering property, by Lemma 5.10, $S^\infty = \emptyset$. Since the derivative operator is Σ^1_1 it is a standard fact that in this case the closure ordinal of S is recursive in S, hence recursive in w.

From 5.11 we also get the following: Let $\mathrm{z} \in WO$ with $|\mathrm{w}| = |\mathrm{z}| = \xi$ and let $u, v \in \omega^{<\omega}$. If u, v code the same sequence of ordinals with respect to the wellorders of ω given by w and z respectively, then

$$(t,u) \in S(\mathrm{w})^{(1)} \quad \text{iff} \quad (t,v) \in S(\mathrm{z})^{(1)}\,.$$

In particular, the closure ordinals of $S(\mathrm{w})$ and of $S(\mathrm{z})$ are the same. Let then z be a generic (with respect to the partial order that collapses ξ to ω) ordinal code for ξ. It is a standard fact that $\omega_1^{\mathrm{z}} = \xi^+$. This finishes the proof of (ii). ■

A key fact in the proof is that the trees $S(\mathrm{w})$ in the previous lemma have an invariant definition in the following sense.

Definition 5.13. Let $\sim$ be an equivalence relation on ω^ω and let Γ be a pointclass. We say that a set A is *$\sim$-invariantly-$\Gamma(\alpha)$* if there is a Γ relation R on $X \times \omega^\omega$ such that for every $\beta \sim \alpha$ we have

$$x \in A \quad \text{iff} \quad R(x,\beta)\,.$$

In particular, A is called *$\sim$-invariantly-$\Delta^1_1(\alpha)$* if A is both $\sim$-invariantly-$\Sigma^1_1(\alpha)$ and $\sim$-invariantly-$\Pi^1_1(\alpha)$.

Consider the following equivalence relation on ω^ω: Let LO be the collection of codes of linear orders of ω. We say that two codes α and β in LO are *isomorphic* if the linear orders coded by them are isomorphic. Define $\equiv$ by

$$\alpha \equiv \beta \quad \text{iff} \quad \alpha,\beta \in \mathrm{LO}\ \&\ \alpha \text{ and } \beta \text{ are isomorphic.}$$

It is a standard fact that $\equiv$ is a Σ^1_1 relation (see [11]). The following two lemmas make it clear why the notion of $\equiv$-invariantly definable sets is interesting.

LEMMA 5.14. *Let ξ be a countable ordinal and* w *an ordinal code for* ξ. *Let* $T \subseteq \omega$ *be a* $\equiv$*-invariantly-*$\Delta^1_1(\mathrm{w})$ *set. Then* T *belongs to the least admissible set containing* ξ.

Proof. Let M denote the least admissible set containing ξ. We will show that T is $\underset{\sim}{\Delta}_1$ definable over M. Let $R \subseteq \omega \times \omega^\omega$ be a Π^1_1 set such that for every ordinal code w with $|\mathrm{w}| = \xi$, we have

$$s \in T \quad \text{iff} \quad R(s, \mathrm{w}).$$

Let ψ be a Σ_1 formula (in ZF) such that if N is an admissible set and w$\in N$, then

$$(*) \qquad R(s, \mathrm{w}) \quad \text{iff} \quad N \vDash \psi(s, \mathrm{w}).$$

Consider the notion of forcing $\mathbf{P}$ that collapses ξ to ω. If G is $\mathbf{P}$-generic, let w_G be the corresponding ordinal code, i.e.,

$$\mathrm{w}_G(n, m) = 0 \quad \text{iff} \quad (\exists p \in G)(p(n) < p(m)).$$

Consider the following name:

$$\tau = \{\langle \sigma, p\rangle : \sigma = \langle (\check{n, m}), 0\rangle \text{ and for some ordinals } \alpha < \beta, \langle n, \alpha\rangle, \langle m, \beta\rangle \in p\}.$$

Then for every $\mathbf{P}$-generic G, $i_G(\tau) = \mathrm{w}_G$. Since for every admissible set N, $N[G]$ is also admissible, from $(*)$ we get

$$(**) \qquad R(s, \mathrm{w}_G) \quad \text{iff} \quad M[G] \vDash \psi(s, \mathrm{w}_G).$$

As $(**)$ holds for every $\mathbf{P}$-generic G, we have

$$s \in T \quad \text{iff} \quad \Vdash \psi(\check{s}, \tau).$$

Since ψ is Σ_1, the relation $B(s, \tau)$ iff $\Vdash \psi(\check{s}, \tau)$ is Σ_1 over M. Hence T is $\underset{\sim}{\Sigma}_1$ over M. Similarly, $s \notin T$ is $\underset{\sim}{\Sigma}_1$ over M. ∎

There is another basic fact about Σ^1_1 equivalence relations and Π^1_1 sets that we are going to use.

DEFINITION 5.15 (Solovay [5]). Let $\sim$ be an equivalence relation on ω^ω and let $P \subseteq \omega^\omega$ be a $\sim$-invariant set, i.e., if $x \in P$ and $y \sim x$ then $y \in P$. A norm $\varphi : P \to$ ordinals is called $\sim$-*invariant* if

$$x \sim y \ \&\ x \in P \Rightarrow \varphi(x) = \varphi(y).$$

Let Γ be a pointclass. We say that Γ is *invariantly normed* if for every equivalence relation $\sim$ in $\check{\Gamma}$ and every $\sim$-invariant set P in Γ, P admits a $\sim$-invariant norm.

It was proved by Solovay (see [5]) that Π^1_1 is invariantly normed.

The following result is the "invariant" version of 4.2.

Lemma 5.16 (see Barua–Srivatsa [1]). *Let $\sim$ be a Σ^1_1 equivalence relation on ω^ω and let T be a $\sim$-invariantly-$\Sigma^1_1(\alpha)$ binary tree with $[T] \in I$. There is a $\sim$-invariantly-$\Delta^1_1(\alpha)$ tree S such that $[T] \subseteq [S]$ and $[S] \in I$.*

Proof. The proof is entirely similar to the one of 4.2. We will sketch it, to point out where we use the notion of $\sim$-invariant sets.

Put a $\sim$-invariant $\Pi^1_1(\alpha)$ norm on $\mathrm{Seq} - T$. Define, as in 4.2, the sets A_s and the set M. We claim that A_s is $\sim$-invariantly-$\Sigma^1_1(\alpha)$ and M is $\sim$-invariantly-$\Pi^1_1(\alpha)$. Assuming the claim we finish the proof.

The separation theorem holds in an invariant form, i.e. given two disjoint $\sim$-invariant Σ^1_1 sets there is a $\sim$-invariant Δ^1_1 set separating them. Thus, as in the proof of 4.2, let B be a $\sim$-invariant $\Delta^1_1(\alpha)$ set such that $T \subseteq B \subseteq M$. Let S be the tree generated by B. Then S is easily seen to be $\sim$-invariantly-$\Delta^1_1(\alpha)$.

It is clear that A_s is $\sim$-invariantly-$\Sigma^1_1(\alpha)$, because of the definition of the Π^1_1 norm. Now for M we have

$$\begin{aligned} s \in M \quad &\text{iff} \quad (\forall K)(K \subseteq [A_s] \Rightarrow K \in I) \\ &\text{iff} \quad (\forall K)\{[(\exists t)(N_t \cap K \neq \emptyset) \ \&\ t \notin A_s] \text{ or } K \in I\}. \end{aligned}$$

It is clear that the relation inside curly brackets is $\sim$-invariantly-$\Pi^1_1(\alpha)$ (because the only thing that depends on α is A_s). This proves the claim. ∎

Proof of Theorem 5.8. First we want to show that $C_1(I)$ is a Π^1_1 set in I^{int}. We have

$$x \in C_1(I) \quad \text{iff} \quad (\exists T \in L_{\omega_1^x})(T \text{ is a tree} \ \&\ x \in [T] \ \&\ [T] \in I)\,.$$

It is clearly Π^1_1, since

$$T \in L_{\omega_1^x} \quad \text{iff} \quad (\exists \gamma, \beta \in \Delta^1_1(x))[\gamma \in WO \ \&\ \beta \in L_{|\mathrm{w}|} \ \&\ \beta = T]\,.$$

Now we show that $C_1(I) \in I^{\mathrm{int}}$. Put $C = C_1(I)$. By 5.4 it suffices to show that C is τ-meager for every topology τ compatible with I. Fix such a topology τ. Define the following prewellordering on C:

$$x \le y \quad \text{iff} \quad \omega_1^x \le \omega_1^y\,.$$

Since this prewellordering is in the σ-algebra generated by the $\boldsymbol{\Sigma}^1_1$ sets, it has the Baire property with respect to τ. Now, for every $y \in C$,

$$\{x \in C : x \le y\} \subseteq \bigcup\{[T] : T \in L_{\omega_1^y} \ \&\ [T] \in I\}\,.$$

As every $L_{\omega_1^y}$ is countable, $\{x \in C : x \le y\}$ is τ-meager. Thus by the Kuratowski–Ulam theorem, $\{y \in C : x \le y\}$ is τ-meager except for a τ-meager set of x's. Thus C is τ-meager.

Finally, we need only show that every Π^1_1 set A in I^{int} is a subset of $C_1(I)$. Fix such an A and let T be a recursive tree on $2 \times \omega$ such that

$$x \in A \quad \text{iff} \quad T(x) \text{ is well-founded.}$$

Fix $x \in A$ and let $|T(x)| = \xi$. Notice that $\xi^+ < \omega_1^x$. Let S be as in 5.11. Then for every ordinal code w with $|\mathrm{w}| = \xi$ we have

$$A_\xi = p[S(\mathrm{w})]\,.$$

As $A_\xi \in I^{\mathrm{int}}$ and I has the covering property, from Lemma 5.10 we get $S(\mathrm{w})^\infty = S(\mathrm{w})^\theta = \emptyset$ for some countable ordinal θ. Hence as in the proof of 5.10,

$$A_\xi \subseteq \bigcup\{\overline{p[S(\mathrm{w})^\alpha_{(s,u)}]} : \overline{p[S(\mathrm{w})^\alpha_{(s,u)}]} \in I \ \&\ \alpha < \theta \ \&\ (s,u) \in S(\mathrm{w})\}\,.$$

We want to show that the sets $[S(\mathrm{w})^\alpha_{(s,u)}]$ have an invariant definition in order to apply 5.16. Let P be as in 5.12. Consider the following relations:

$$(\mathrm{z}_1, \ldots, \mathrm{z}_m) \equiv_{\mathrm{w}} r \quad \text{iff} \quad (r \in \omega^{<\omega})\ \& \\ (\forall i \le m)(\mathrm{z}_i \in LO\ \&\ \mathrm{w} \in LO\ \&\ \mathrm{w}\lceil r(i) \equiv \mathrm{z}_i)$$

where $\mathrm{w}\lceil r(i)$ is the initial segment of the linear order coded by w determined by $r(i)$, i.e.,

$$\mathrm{w}\lceil r(i) = \{(l,k) : \mathrm{w}(l,k) = \mathrm{w}(l, r(i)) = \mathrm{w}(k, r(i)) = 0\}\,.$$

Put

$$R(s,u,t,\mathrm{z},\mathrm{w},\mathrm{v}) \quad \text{iff} \quad t \in 2^{<\omega}\ \&\ \mathrm{lh}(t) = n\ \&\ t \prec s\ \& \\ (\exists r \in \omega^{<\omega})((\mathrm{z}_1, \ldots, \mathrm{z}_n) \equiv_{\mathrm{w}} r\ \&\ r \prec u\ \&\ P(s,u,\mathrm{v},\mathrm{w}))\,.$$

Now consider the following equivalence relation on $\omega^\omega \times \omega^\omega \times \omega^\omega$:

$$(\mathrm{z},\mathrm{w},\mathrm{v}) \sim (\mathrm{z}',\mathrm{w}',\mathrm{v}') \quad \text{iff} \quad \mathrm{z}_0(0) = \mathrm{z}'_0(0)\ \& \\ (\forall 0 \le i \le \mathrm{z}_0(0))(\mathrm{z}_i, \mathrm{z}'_i \in LO\ \&\ \mathrm{z}_i \equiv \mathrm{z}'_i\ \&\ \mathrm{w}_i \equiv \mathrm{w}'_i\ \&\ \mathrm{v}_i \equiv \mathrm{v}'_i)\,.$$

Let $(t,r) \in S(\mathrm{w})$ such that

$$x \in \overline{p[S(\mathrm{w})^{|\mathrm{v}|}_{(t,r)}]}$$

and put

$$B = p[S(\mathrm{w})^{|\mathrm{v}|}_{(t,r)}]\,.$$

Now, by 5.11(ii), if z codes a sequence of ordinals such that $(\mathrm{z}_1, \ldots, \mathrm{z}_m) \equiv_{\mathrm{w}} r$, then

$$x \in B \quad \text{iff} \quad (\exists \alpha)(\forall n) R(x\lceil n, \alpha\lceil n, t, \mathrm{z}, \mathrm{w}, \mathrm{v})\,.$$

Hence B is $\sim$-invariantly-Σ^1_1 with respect to the variables $(\mathrm{z},\mathrm{w},\mathrm{v})$. Let L be the tree of $\overline{B}$. Clearly $[L] \in I$, thus by Lemma 5.16 there is a $\sim$-invariantly-Δ^1_1 tree K on 2 such that $[L] \subseteq [K]$ and $[K] \in I$.

By a similar argument to the proof of Lemma 5.14 we know that K belongs to the least admissible set containing all the ordinals coded by w, z, v (we just use the product of the notion of forcing defined in 5.14, one for each of the m ordinals coded in $(\mathrm{z},\mathrm{w},\mathrm{v})$, where $m = \mathrm{lh}(r) + 2$).

But from Lemma 5.12(ii) we know that these ordinals are less than $\xi^+ < \omega_1^x$. Therefore $K \in L_{\omega_1^x}$. This finishes the proof of Theorem 5.8. ∎

Remark. This proof clearly works for σ-ideals on $(2^\omega)^m$.

6. On the strength of the covering property for Σ^1_2 sets. It is well known that the perfect set theorem for $\boldsymbol{\Pi}^1_1$ sets is equiconsistent with the existence of an inaccessible cardinal (Solovay). In fact, $\omega_1^L < \omega_1$ iff the perfect set theorem holds for Π^1_1 sets. In this section we will show that under the assumption that there are only countable many reals in L, any Π^1_1 σ-ideal of closed meager subsets of 2^ω with the covering property also has the covering property for Σ^1_2 sets. Also, we will see that for some σ-ideals, the covering property for Π^1_1 sets fails in L and thus it cannot be proved in ZFC.

THEOREM 6.1. *Let I be a Π^1_1 σ-ideal of meager closed subsets of 2^ω with the covering property. If $\omega_1^L < \omega_1$, then I has the covering property for Π^1_1 sets. By relativization, given $x \in \omega^\omega$, if $\omega_1^{L(x)} < \omega_1$, then the covering property holds for $\Pi^1_1(x)$ sets.*

The same holds for σ-ideals of closed meager subsets of $(2^\omega)^m$.

Proof. It clearly suffices to show that the largest Π^1_1 set $C_1(I)$ in I^{int} belongs to I^{ext}. But if $\omega_1^L < \omega_1$, then there are only countably many binary trees in L. Hence from Theorem 5.8 we easily conclude that $C_1(I) \in I^{\text{ext}}$. ∎

The next result is a generalization of the result of Solovay that says that if there are only countably many reals in L, then $\omega^\omega \cap L$ is the largest countable Σ^1_2 set. A similar result holds for some σ-ideals defined by games (see [4]).

THEOREM 6.2. *Under the hypothesis of* 6.1 *the largest Σ^1_2 set in I^{ext} and in I^{int} is*

$$C_2(I) = \{x \in 2^\omega : (\exists T \in L)(T \text{ is a tree on } 2 \ \& \ x \in [T] \ \& \ [T] \in I)\}.$$

In particular, the covering property holds for Σ^1_2 sets. By relativization, given $x \in \omega^\omega$, if $\omega_1^{L(x)} < \omega_1$, then the covering property holds for $\Sigma^1_2(x)$ sets.

Proof. If there are only countably many reals in L, then there are only countably many binary trees in L. Thus $C_2(I)$ is clearly a Σ^1_2 set in I^{ext}.

Let A be a Σ^1_2 set in I^{int} and let $B \subseteq X \times 2^\omega$ be a Π^1_1 set such that $x \in A$ iff $(\exists \alpha)(x,\alpha) \in B$. Let J be the σ-ideal of closed subsets of $2^\omega \times 2^\omega$ defined by

$$(*) \qquad K \in J \quad \text{iff} \quad \text{proj}(K) \in I.$$

By Lemma 3.7, J has the covering property and clearly J is a Π^1_1 σ-ideal of meager sets. Hence by the previous theorem J has the covering property for Π^1_1 sets. As $A \in I^{\rm int}$, $B \in J^{\rm int}$. Let $C_1(J)$ be the largest Π^1_1 set in $J^{\rm int}$, i.e.,

$$C_1(J) = \{(x,\alpha) : (\exists S \in L_{\omega_1^{(x,\alpha)}})(S \text{ is a tree on } 2\times 2 \ \& \\ (x,\alpha) \in [S] \ \& \ \mathrm{proj}([S]) \in I)\}\,.$$

It is clear that $A \subseteq \mathrm{proj}(C_1(J))$. Now, let K be a closed subset of $2^\omega \times 2^\omega$ and let S be the tree of K. Put $T = \{t : (\exists s)(t,s) \in S\}$. Then T is clearly a tree and by using König's lemma it is easy to check that $[T] = \mathrm{proj}([S])$. Clearly if $S \in L$, then so does T. Hence

$$A \subseteq \mathrm{proj}(C_1(J)) \subseteq \{x \in 2^\omega : (\exists T \in L)(x \in [T] \ \& \ [T] \in I)\}\,. \ \blacksquare$$

The next lemma will be used in the proof that for some σ-ideals the covering property for Π^1_1 set fails in L. These results are due to Dougherty and Kechris; we include the proof with their permission.

Let us denote by $\le_{\rm T}$ the relation of Turing reducibility, i.e., $x \le_{\rm T} y$ iff x is recursive in y.

LEMMA 6.3 (Dougherty, Kechris). *Let μ be the product probability measure on 2^ω and let I be the σ-ideal of closed μ-measure zero subsets of 2^ω. Then for every $x \in 2^\omega$, $\{y : x \le_{\rm T} y\} \notin I^{\rm ext}$.*

Proof. Let $\{K_n\}$ be a countable collection of sets in I. We will define $y \notin \bigcup_n K_n$ such that $x \le_{\rm T} y$.

By the nth *block* we mean the interval $[2^n, 2^{n+1})$. Call $z \in 2^\omega$ *good* if for infinitely many n's, z is constant in the nth block. If z is good, let $\widetilde{z}$ be defined as follows: Let $n_0 < n_1 < \ldots$ be an enumeration of the blocks on which z is constant; put $\widetilde{z}(i) = j$ if z is constantly equal to j in the n_ith block.

We will define, by induction, a good $y \notin \bigcup_n K_n$ such that $\widetilde{y} = x$. Clearly $x \le_{\rm T} y$ and we will be done. For every n and k with $k > n$ and every sequence $s \in 2^{2^n}$, let

$$F^s_k = \{z \in 2^\omega : z \text{ is not constant in the } j\text{th block for } n \le j \le k \ \& \ s \prec z\}\,.$$

There are exactly $2^{2^n} - 2$ non-constant sequences of length 2^n. Therefore, if $z \in F^s_n$, then z can take $2^{2^j} - 2$ possible values in the jth block. From this, one easily gets

$$\mu(F^s_k) = (2^{2^n} - 2)(2^{2^{n+1}} - 2)\ldots(2^{2^k} - 2)/2^{2^{k+1}}\,.$$

Hence

$$(*) \qquad \mu(F^s_k) = \frac{1}{2^{2^n}} \prod_{j=n}^{k} \left(1 - \frac{2}{2^{2^j}}\right).$$

If $k \to \infty$, the infinite product $(*)$ is equiconvergent with $\sum_{j=n}^{\infty} 1/2^{2^j}$. Hence, for every $s \in 2^n$ we have

$$\mu\Big(\bigcap_{k=n}^{\infty} F_k^s\Big) > 0\,.$$

Let $F_s = \bigcap_{k=n}^{\infty} F_k^s$. Now we start defining y. As $\mu(F_\emptyset) > 0$, there is a $z \in F_\emptyset - K_0$. Choose n_0 large enough such that if $z\lceil 2^{n_0} \prec w$, then $w \notin K_0$. Define $t_0 \in 2^{n_0+1}$ by $t_0\lceil 2^{n_0} = z\lceil 2^{n_0}$ and $t(i) = x(0)$ for every $i \in [2^{n_0}, 2^{n_0+1})$. Put $y\lceil 2^{n_0+1} = t_0$. Notice that t_0 is not constant in any jth block for $j < n_0$. Clearly we can repeat this for K_1 and F_{t_0}. So let $z \in F_{t_0} - K_1$ and let $n_1 > n_0 + 1$ be large enough that if $z\lceil 2^{n_1} \prec w$, then $w \notin K_1$. Define as before $t_1 \in 2^{n_1+1}$ by $t_1\lceil 2^{n_1} = z\lceil 2^{n_1}$ and $t_1(i) = x(1)$ for every $i \in [2^{n_1}, 2^{n_1+1})$. Put $y\lceil 2^{n_1+1} = t_1$. The induction step should now be clear. So we get $y \notin \bigcup_n K_n$ and $\widetilde{y} = x$. ■

As mentioned before, for the σ-ideal of countable closed subsets of 2^ω the largest Π_1^1 set without perfect subsets is characterized by

$$C_1 = \{\alpha \in 2^\omega : \alpha \in L_{\omega_1^\alpha}\}\,.$$

The next theorem shows that (in L) C_1 cannot be covered by countably many closed sets of (Lebesgue) measure zero. However, observe that as C_1 has no perfect subsets, it clearly has measure zero and also belongs to I^{int} for every σ-ideal containing all singletons.

THEOREM 6.4 (Dougherty, Kechris). *Let μ and I be as in* 6.3. *In L, $C_1 \notin I^{\text{ext}}$. Therefore, if J is a σ-ideal on 2^ω such that J contains all singletons and $J \subseteq I$, then* (*in L*) *J does not have the covering property for Π_1^1 sets.*

Proof. Let $\{K_n\}$ be a countable collection of closed sets of μ-measure zero. We will show that there is a $y \in C_1$ with $y \notin \bigcup_n K_n$.

Let $\{T_n\}$ be the corresponding trees and let $\alpha < \omega_1^L$ be an ordinal such that each $T_n \in L_\alpha$. We can assume without loss of generality that α is an index (i.e., there is an $x \in \omega^\omega$ such that $x \in L_{\alpha+1} - L_\alpha$). Let x be a complete set of index α (that is, $x \in L_{\alpha+1} - L_\alpha$ and every $y \in \omega^\omega \cap L_{\alpha+1}$ is arithmetical in x), in particular $\alpha < \omega_1^x$.

Let y be as in the proof of the previous lemma. It is easy to check that y can be found in $L_{\alpha+\omega}$. As $\omega_1^x \le \omega_1^y$ (because $x \le_{\mathrm{T}} y$), we have $\alpha + \omega \le \omega_1^y$. Hence $y \in L_{\omega_1^y}$, so $y \in C_1$. By construction $y \notin \bigcup_n K_n$. ■

These theorems can be easily transferred to compact intervals of the real line as follows: Say we are working on $[0, 1]$ and consider the function

$f : 2^\omega \to [0,1]$ defined by

$$f(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon(i) 2^{-(i+1)}\,;$$

then f is continuous and surjective. Now, given a σ-ideal I of closed meager subsets of $[0,1]$, define a σ-ideal J of closed subsets of 2^ω as follows:

$$K \in J \quad \text{iff} \quad f[K] \in I\,.$$

Observe that J consists of meager sets (because for every nbhd N_s on 2^ω, $f[N_s]$ contains an interval).

Lemma 6.5. *If I has the covering property, then so does J.*

Proof. First we show that if A is a $\boldsymbol{\Sigma}^1_1$ set, then $A \in J^{\text{int}}$ iff $f[A] \in I^{\text{int}}$. The direction $\Leftarrow$ is obvious by the definition of J.

Let A be a $\boldsymbol{\Sigma}^1_1$ set such that $f[A] \notin I^{\text{int}}$, say $K \subseteq f[A]$ is a closed set and $K \notin I$. Define R as follows:

$$R(x,\alpha) \quad \text{iff} \quad \alpha \in A \ \&\ x \in K \ \&\ f(\alpha) = x\,.$$

Then $x \in K$ iff $(\exists\alpha)R(x,\alpha)$. Hence, as I is strongly calibrated, there is a closed set $F \subseteq R$ such that

$$K_0 = \{x : (\exists\alpha)(x,\alpha) \in F\} \notin I\,.$$

Notice that $K_0 \subseteq K$. Put $L = \{\alpha : (\exists x)(x,\alpha) \in F\}$. Then $f[L] = K_0$ and $L \subseteq A$, so $A \notin J^{\text{int}}$.

The covering property for J now follows: If $A \in J^{\text{int}}$ is a $\boldsymbol{\Sigma}^1_1$ set, then $f[A] \in I^{\text{int}}$. Hence $f[A] \in I^{\text{ext}}$, which clearly implies that $A \in J^{\text{ext}}$. ■

Theorem 6.6. *Let I be a Π^1_1 σ-ideal of closed meager subsets of $[0,1]$ with the covering property. Let f be the function defined above. The largest Π^1_1 set in I^{int} is*

$$C_1(I) = \{x \in [0,1] : (\exists T \in L_{\omega_1^x})(T \text{ is a tree on } 2 \ \&\ x \in f[T] \ \&\ f[T] \in I)\}$$

and the largest $\Sigma^1_2 \in I^{\text{ext}}$ is characterized by

$$C_2(I) = \{x \in [0,1] : (\exists T \in L)(T \text{ is a tree on } 2 \ \&\ x \in f[T] \ \&\ f[T] \in I)\}\,.$$

In particular, if $\omega_1^L < \omega_1$, then I has the covering property for Σ^1_2 sets. By relativization, given $x \in \omega^\omega$, if $\omega_1^{L(x)} < \omega_1$, then the covering property holds for $\Sigma^1_2(x)$ sets.

Proof. First, as in the proof of Theorem 5.8, $C_1(I)$ is a Π^1_1 set in I^{int}. To see that it is the largest, consider the σ-ideal J defined on 2^ω as in 6.5.

Then J has the covering property. Let $C_1(J)$ be the largest Π^1_1 set in J^{int} given by Theorem 5.8, i.e.,

$$C_1(J) = \{\alpha \in 2^\omega : (\exists T \in L_{\omega_1^\alpha})(T \text{ is a tree on } 2 \ \& \ \alpha \in [T] \ \& \ [T] \in J)\}.$$

Let A be a Π^1_1 set in I^{int}. Put $B = f^{-1}(A)$. Then B is a Π^1_1 set in J^{int}. So $B \subseteq C_1(J)$, hence it suffices to show that $f(C_1(J)) \subseteq C_1(I)$. Let $\alpha \in C_1(J)$ and let $T \in L_{\omega_1^\alpha}$ such that $\alpha \in [T]$ and $[T] \in J$. As f is Δ^1_1, $\omega_1^\alpha = \omega_1^{f(\alpha)}$. So $T \in L_{\omega_1^{f(\alpha)}}$. Thus $f(\alpha) \in f[T]$ and also $f[T] \in I$.

The proof for $C_2(I)$ is similar. ∎

Theorem 6.4 can also be transferred to $[0,1]$ as follows: Observe that for every basic nghd N_s in 2^ω we have $\mu(N_s) = \lambda(f[N_s])$, where μ is the standard product measure on 2^ω and λ is the Lebesgue measure on $[0,1]$. One easily checks that if $f[C_1]$ can be covered by countably many closed sets of Lebesgue measure zero, then C_1 can also be covered by countably many closed sets of μ-measure zero. It is also clear that this set does not contain a perfect subset. We collect these facts in the following

THEOREM 6.7. *Let I be a σ-ideal of closed subsets of* $[0,1]$ *such that every set in I has Lebesgue measure zero. In* L, *I does not have the covering property for Π^1_1 sets.* ∎

Remark. (1) As already mentioned, the σ-ideal of closed set of extended uniqueness has the covering property (see [2]). Hence, from 6.6 and 6.7 we see that the covering property for Π^1_1 sets of extended uniqueness is not provable in ZFC, but can be proved from the hypothesis that there are only countably many reals in L. Also, we get a characterization of the largest Π^1_1 set of extended uniqueness as in 6.6.

(2) It is an open question whether the covering property for $\boldsymbol{\Pi}^1_1$ sets (for instance for the σ-ideal of closed sets of extended uniqueness) implies (the consistence of) the existence of an inaccessible cardinal.

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DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE LOS ANDES
MÉRIDA 5101, VENEZUELA

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