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Planar rational compacta and universality

by

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Abstract. We prove that in some families of planar rational compacta there are no universal elements.

1. Introduction. The spaces considered in this paper are separable and metrizable and the ordinals are countable. For a subset Q of a space X we denote by $|Q|$ the cardinality of Q and by $\mathrm{diam}(Q)$ the diameter of Q when X is a metric space. A *compactum* is a compact metrizable space; a *continuum* is a connected compactum. A space is said to be *scattered* iff every non-empty subset has an isolated point. A space is said to be *planar* iff it is homeomorphic to a subset of the plane.

A space T is said to be *universal* for a class A of spaces iff both the following conditions are satisfied: (α) $T \in A$, (β) for every $X \in A$, there exists a homeomorphism of X onto a subset of T. If only condition (β) is satisfied, then T is called a *containing space* for the class A.

An ordinal α is called *isolated* iff $\alpha = \beta + 1$, where β is an ordinal. A non-isolated ordinal is called a *limit* ordinal. Hence, the ordinal zero is considered as a limit ordinal. By $\mathbb{N}$ we denote the set $\{0, 1, 2, \ldots\}$ of non-negative integers.

Let M be a topological space. Let $M^{(0)} = M$ and let $M^{(1)}$ be the set of all limit points of M in M. For every ordinal α we define the set $M^{(\alpha)}$ putting $M^{(\alpha)} = (M^{(\alpha-1)})^{(1)}$ if α is an isolated ordinal and $M^{(\alpha)} = \bigcap_{\beta<\alpha} M^{(\beta)}$ if α is a limit ordinal. $M^{(\alpha)}$ is called the α-*derivative* of M (see [Ku], Vol. I, §24.IV).

If $M^{(\alpha)} = \emptyset$, we say that M has *type* $\leq \alpha$, and we write $\mathrm{type}(M) \leq \alpha$. If α is the least such ordinal, we say that M has *type* α, and we write $\mathrm{type}(M) = \alpha$. Obviously, $\mathrm{type}(M) = 0$ iff $M = \emptyset$.

We note the following properties:

(1) A compactum is scattered iff it is countable.

(2) The type of a non-empty countable compactum is an isolated ordinal.

(3) For every isolated ordinal α there exist compacta having type α (see [M-S]).

(4) If M and Q are compacta having type $\le \alpha$, then $M \cup Q$ has type $\le \alpha$.

(5) If f is a continuous map of a scattered compactum M onto Q such that $f^{-1}(x)$ is finite for every $x \in Q$, then $f(M^{(\alpha)}) = Q^{(\alpha)}$ for every ordinal α and, hence, $\text{type}(Q) = \text{type}(M)$.

We recall that a closed subset M of a space X *separates* X iff there exists an integer $m > 1$ such that $X \setminus M = U_1 \cup \ldots \cup U_m$, where U_i, $i = 1, \ldots, m$, is a non-empty regular open set (that is, U_i is the interior of the closure of U_i) and $U_i \cap U_j = \emptyset$ if $i \neq j$. In this case we say that M separates X *into* m *parts* (see [I]).

For the notions of *upper semicontinuous partition*, *quotient space* and *natural projection* see, for example, [K].

A space X is said to have *rim-type* $\le \alpha$, and we write $\text{rim-type}(X) \le \alpha$, iff X has a basis of open sets whose boundaries have type $\le \alpha$. If α is the least such ordinal, then we say that X has *rim-type* α and we write $\text{rim-type}(X) = \alpha$.

A space X is said to be *rim-finite* iff X has a basis of open sets whose elements have finite boundaries. Let RF be the family of all rim-finite spaces.

Nöbeling (see [N]) proved that for every rim-finite space Y there exists a rim-finite continuum X which is not topologically contained in Y. From this it follows that in the family of all rim-finite spaces, or rim-finite compacta, or rim-finite continua, there is no universal element.

A generalization of the family RF of all rim-finite spaces is the family $R^{\text{rim-com}}(\alpha)$, where α is a given ordinal. A space X belongs to $R^{\text{rim-com}}(\alpha)$ iff X has a basis of open sets whose elements have compact boundaries of type $\le \alpha$. Obviously, if $\alpha = 1$, then $R^{\text{rim-com}}(\alpha) = RF$.

In [I] the result of Nöbeling is generalized for the family $R^{\text{rim-com}}(\alpha)$, $\alpha > 0$. For every $Y \in R^{\text{rim-com}}(\alpha)$ there exists a locally connected continuum X of rim-type $\le \alpha$ which is not topologically contained in Y. In general, X is not planar.

We also note the following result of [M-T]: There exists a planar locally connected continuum of rim-type $\alpha + 1$ which is a containing space for the family of all planar compacta of rim-type $\le \alpha$. This gives an affirmative answer to Problem 5 of [I].

The main result of the present paper is the following: for every $Y \in R^{\text{rim-com}}(\alpha)$, $\alpha > 0$, there exists a locally connected planar continuum of rim-type α which is not topologically contained in Y. In particular, in the family of all planar continua (or planar locally connected continua, or planar compacta) having rim-type $\le \alpha$, there is no universal element. This gives a negative answer to Problem 1 of [I].

2. Definitions and notations. We denote by L_n, $n = 1, 2, \ldots$, the set of all ordered n-tuples $i_1 \ldots i_n$, where $i_t = 0$ or 1, $t = 1, \ldots, n$. Also, we set $L_0 = \{\emptyset\}$ and $L = \bigcup_{n=0}^{\infty} L_n$. For $n = 0$, by convention $i_1 \ldots i_n$ denotes the element $\emptyset$ of L. We write $i_1 \ldots i_n \le j_1 \ldots j_m$ if either $n = 0$, or $n \le m$ and $i_t = j_t$ for every $t \le n$. The elements of L are also denoted by $\bar{i}, \bar{j}, \bar{i}_1$, etc. If $\bar{i} = i_1 \ldots i_n$ and $\bar{j} = j_1 \ldots j_m$, then $\bar{i}\bar{j}$ or $\bar{i} j_1 \ldots j_m$ denotes the element $i_1 \ldots i_n j_1 \ldots j_m$ of L.

Let C denote by the Cantor ternary set. By $C_{\bar{i}}$, where $\bar{i} = i_1 \ldots i_n \in L$, $n \ge 1$, we denote the set of all points of C for which the t-th digit in the ternary expansion, $t = 1, \ldots, n$, is 0 if $i_t = 0$, and 2 if $i_t = 1$. Also, we set $C_{\emptyset} = C$.

We denote by $a(C_{\bar{i}})$ (respectively, $b(C_{\bar{i}})$) the element c of $C_{\bar{i}}$ for which $c \le x$ (respectively, $x \le c$) for every $x \in C_{\bar{i}}$. Also, we set $a(\bar{i}) = b(C_{\bar{i}0})$ and $b(\bar{i}) = a(C_{\bar{i}1})$.

Let $C^2 = C \times C$ and if $\bar{i}, \bar{j} \in L_n$, $n = 0, 1, \ldots$, then set $C^2(\bar{i}, \bar{j}) = C_{\bar{i}} \times C_{\bar{j}}$. For every $x \in C^2$ and $n \in \mathbb{N}$ we denote by $\mathrm{st}^2(x, n)$ the set $C^2(\bar{i}, \bar{j})$ for which $x \in C^2(\bar{i}, \bar{j})$ and $\bar{i}, \bar{j} \in L_n$.

For every $n \in \mathbb{N}$ and for every $\bar{i}, \bar{j} \in L_n$ the following pairs of sets: $C^2(\bar{i}0, \bar{j}0)$ and $C^2(\bar{i}0, \bar{j}1)$, $C^2(\bar{i}0, \bar{j}1)$ and $C^2(\bar{i}1, \bar{j}1)$, $C^2(\bar{i}1, \bar{j}1)$ and $C^2(\bar{i}1, \bar{j}0)$, $C^2(\bar{i}1, \bar{j}0)$ and $C^2(\bar{i}0, \bar{j}0)$ are called *adjacent*.

We denote by E^2 the plane with a Cartesian coordinate system. We consider C^2 as a subset of E^2. For any distinct points x and y of E^2 we denote by $[x, y]$ the straight line segment joining x and y (including the ends).

3. Partitions D of C^2 and $\widehat{D}$ of E^2. Let α be an isolated ordinal. For every $n \in \mathbb{N}$ and $\bar{i} \in L_n$, let $P_{\bar{i}}$ be a scattered compact subset of $C_{\bar{i}} \setminus \{a(\bar{i}), b(\bar{i}), a(C_{\bar{i}}), b(C_{\bar{i}})\}$.

For every $\bar{i}, \bar{j} \in L_n$ we define a collection $D(\bar{i}, \bar{j})$ of two-element subsets of C^2. Let $x = (a, b)$, $y = (c, e) \in C^2$. Then $\{x, y\} \in D(\bar{i}, \bar{j})$ iff either $a = c \in P_{\bar{i}}$ and $\{b, e\} = \{a(\bar{j}), b(\bar{j})\}$, or $b = e \in P_{\bar{j}}$ and $\{a, c\} = \{a(\bar{i}), b(\bar{i})\}$. Also we set $D_n = \bigcup_{\bar{i}, \bar{j} \in L_n} D(\bar{i}, \bar{j})$ and $D(1) = \bigcup_{n \in \mathbb{N}} D_n$.

We denote by D the collection of subsets of C^2 consisting of all elements of $D(1)$ and all singletons $\{x\}$, where $x \in C^2$ does not belong to any element of $D(1)$.

Let $\widehat{D}(1)$ denote the set of all straight line segments $[x, y]$, where $\{x, y\} \in D(1)$. We denote by $\widehat{D}$ the collection of subsets of E^2 consisting of all elements of $\widehat{D}(1)$ and all singletons $\{x\}$, where $x \in E^2$ does not belong to any element of $\widehat{D}(1)$.

It is not difficult to see that D is a partition of C^2 and $\widehat{D}$ a partition of E^2. We denote by p the natural projection of C^2 onto the quotient space D and by $\widehat{p}$ the natural projection of E^2 onto the quotient space $\widehat{D}$.

Let $\bar{i}, \bar{j} \in L_n$ and $\bar{i}_1 \in L_m$. We denote by $D^{\bar{i}_1}(\bar{i}, \bar{j})$ (respectively, $D_{\bar{i}_1}(\bar{i}, \bar{j})$) the set of all elements $\{(a,b),(c,e)\}$ of $D(\bar{i}, \bar{j})$ such that $a = c \in P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ (respectively, $b = e \in P_{\bar{j}} \cap C_{\bar{j}\,\bar{i}_1}$).

The following properties are easily proved:

(1) If F is a closed subset of C^2 (respectively, of E^2), $n \in \mathbb{N}$ and $\bar{i}, \bar{j} \in L_n$, then the union $(D_F(\bar{i}, \bar{j}))^*$ (respectively, $(\widehat{D}_F(\bar{i}, \bar{j}))^*$) of all elements of $D(\bar{i}, \bar{j})$ (respectively, of $\widehat{D}(\bar{i}, \bar{j})$) which intersect F is a closed subset of C^2 (respectively, of E^2).

(2) If $\bar{i}_n, \bar{j}_n \in L_n$ and $(D(\bar{i}_n, \bar{j}_n))^*$ (respectively, $(\widehat{D}(\bar{i}_n, \bar{j}_n))^*$) is the union of all elements of $D(\bar{i}_n, \bar{j}_n)$ (respectively, of $\widehat{D}(\bar{i}_n, \bar{j}_n)$), then we have $\lim_{n\to\infty} \operatorname{diam}(D(\bar{i}_n, \bar{j}_n))^* = 0$ (respectively, $\lim_{n\to\infty} \operatorname{diam}(\widehat{D}(\bar{i}_n, \bar{j}_n))^* = 0$).

4. Lemma. (1) *D and $\widehat{D}$ are upper semicontinuous partitions of C^2 and E^2, respectively.*

(2) *The quotient space $\widehat{D}$ is homeomorphic to E^2 and the quotient space D is a planar compactum.*

(3) *For every $n, m \in \mathbb{N}$, $\bar{i}, \bar{j} \in L_n$ and $\bar{i}_1 \in L_m$ the subsets $D^{\bar{i}_1}(\bar{i}, \bar{j})$ and $D_{\bar{i}_1}(\bar{i}, \bar{j})$ of D are homeomorphic to $P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ and $P_{\bar{j}} \cap C_{\bar{j}\,\bar{i}_1}$, respectively.*

(4) *If for every $n \in \mathbb{N}$ and $\bar{i} \in L_n$, $\operatorname{type}(P_{\bar{i}}) \leq \alpha$, then $\operatorname{rim-type}(D) \leq \alpha$.*

(5) *Let $n_0 \in \mathbb{N}$ and $\bar{i}_0, \bar{j}_0 \in L_{n_0}$. If for every $\bar{i} \geq \bar{i}_0$ and for every $\bar{j} \geq \bar{j}_0$, $P_{\bar{i}} \cap C_{\bar{i}0} \neq \emptyset$, $P_{\bar{i}} \cap C_{\bar{i}1} \neq \emptyset$, $P_{\bar{j}} \cap C_{\bar{j}0} \neq \emptyset$ and $P_{\bar{j}} \cap C_{\bar{j}1} \neq \emptyset$, then $p(C^2(\bar{i}_0, \bar{j}_0))$ is connected and locally connected.*

Proof. (1) We prove that D is an upper semicontinuous partition of C^2. Let $d \in D$ and let $d \subseteq U$, where U is an open subset of C^2. For every $n \in \mathbb{N}$ and for every $\bar{i}, \bar{j} \in L_n$ by property (1) of Section 3 the set $F \cup (D_F(\bar{i}, \bar{j}))^*$ is a closed subset of C^2, where $F = C^2 \setminus U$. By property (2) of Section 3 so is $F_1 = \bigcup_{n\in\mathbb{N}} \bigcup_{\bar{i},\bar{j}\in L_n} F \cup (D_F(\bar{i}, \bar{j}))^*$. Thus $V = C^2 \setminus F_1$ is an open subset of C^2 which is a union of elements of D. Obviously, $d \subseteq V \subseteq U$, that is, D is an upper semicontinuous partition of C^2. The proof for $\widehat{D}$ is similar.

(2) Since every element of $\widehat{D}$ is a singleton or an arc, the quotient space $\widehat{D}$ is homeomorphic to the plane E^2 (see [Ku], Vol. II, §61.IV).

Obviously, D is a compactum. To prove that D is planar, we construct a homeomorphism i of D onto a subset of $\widehat{D}$.

Let $d \in D$. If $d = \{x\}$, then we set $i(d) = d$, and if $d = \{x, y\}$, then $i(d) = d'$, where $d' = [x, y]$. Obviously, i is a one-to-one map of D onto a subset of $\widehat{D}$.

To prove that i is continuous, let $i(d) = d'$ and let U be an open neighbourhood of d' in $\widehat{D}$. Then $\widehat{p}^{-1}(U)$ is an open subset of E^2 which is a union of elements of $\widehat{D}$. Hence, $W = \widehat{p}^{-1}(U) \cap C^2$ is an open subset of C^2 which is a union of a collection, say V, of elements of D. Thus, V is an open subset of D, $d \in V$ and $i(V) \subseteq U$. Hence, i is continuous. Since D is a compactum, i is a homeomorphism onto a subset of E^2.

(3) Let $f : P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1} \to C^2$ be defined by $f(a) = (a, a(\bar{j}))$ for $a \in P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$. Obviously, f is continuous and one-to-one. Also, $p \circ f$ is a continuous one-to-one map of $P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ onto $D^{\bar{i}_1}(\bar{i}, \bar{j})$. Since $P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ is a compactum, $p \circ f$ is a homeomorphism. Similarly, we can prove that $D_{\bar{i}_1}(\bar{i}, \bar{j})$ is homeomorphic to $P_{\bar{j}} \cap C_{\bar{j}\,\bar{i}_1}$.

(4) Let $d \in D$ and let U be an open neighbourhood of d in D. If $d = \{x\}$, then there exist $n \in \mathbb{N}$ and $\bar{i}, \bar{j} \in L_n$ such that $d \subseteq C^2(\bar{i}, \bar{j}) \subseteq p^{-1}(U)$. If $d = \{x, y\}$, then there exist $n \in \mathbb{N}$ and $\bar{i}_1, \bar{i}_2, \bar{j}_1, \bar{j}_2 \in L_n$ such that $d \subseteq C^2(\bar{i}_1, \bar{j}_1) \cup C^2(\bar{i}_2, \bar{j}_2) \subseteq p^{-1}(U)$.

Let V be the set of all elements of D which are contained in $C^2(\bar{i}, \bar{j})$ (respectively, in $C^2(\bar{i}_1, \bar{j}_1) \cup C^2(\bar{i}_2, \bar{j}_2)$). Obviously, $d \in V \subseteq U$. We prove that the boundary $\mathrm{Bd}(V)$ has type $\le \alpha$. Indeed, it is easy to verify that $\mathrm{Bd}(V) \subseteq \bigcup_{k=0}^{n} D_k$. By the assumption and (3), $\mathrm{type}(\bigcup_{k=0}^{n} D_k) \le \alpha$. Hence, $\mathrm{type}(\mathrm{Bd}(V)) \le \alpha$. Thus, rim-type$(D) \le \alpha$.

(5) Suppose that $D^0 = p(C^2(\bar{i}_0, \bar{j}_0))$ is not connected. Then $D^0 = D^1 \cup D^2$, where D^1 and D^2 are simultaneously open and closed non-empty subsets of D^0 with empty intersection. Hence, $C_1^2 = p^{-1}(D^1) \cap C^2(\bar{i}_0, \bar{j}_0)$ and $C_2^2 = p^{-1}(D^2) \cap C^2(\bar{i}_0, \bar{j}_0)$ are simultaneously open and closed in $C^2(\bar{i}_0, \bar{j}_0)$. Moreover, it is easy to see that C_1^2 and C_2^2 are not empty and if $d \in D^0$ and $d \subseteq C^2(\bar{i}_0, \bar{j}_0)$, then either $d \subseteq C_1^2$ or $d \subseteq C_2^2$.

There exists an integer $n \ge n_0$ such that if $\bar{i}, \bar{j} \in L_n$, $\bar{i} \ge \bar{i}_0$ and $\bar{j} \ge \bar{j}_0$, then $C^2(\bar{i}, \bar{j})$ is contained either in C_1^2 or in C_2^2. We can suppose that n is the minimal such ordinal.

If $n \le n_0$, then either $C^2(\bar{i}_0, \bar{j}_0) \subseteq C_1^2$ or $C^2(\bar{i}_0, \bar{j}_0) \subseteq C_2^2$. Hence, either $C_1^2 = \emptyset$ or $C_2^2 = \emptyset$, which is impossible. Thus, $n > n_0$.

There exist $\bar{i}, \bar{j} \in L_{n-1}$ such that $\bar{i} \ge \bar{i}_0$, $\bar{j} \ge \bar{j}_0$, $C^2(\bar{i}, \bar{j}) \not\subseteq C_1^2$ and $C^2(\bar{i}, \bar{j}) \not\subseteq C_2^2$. On the other hand, each of the sets $C^2(\bar{i}0, \bar{j}0)$, $C^2(\bar{i}0, \bar{j}1)$, $C^2(\bar{i}1, \bar{j}0)$ and $C^2(\bar{i}1, \bar{j}1)$ is contained either in C_1^2 or in C_2^2. It is easy to see that there are two of them which are adjacent with one being contained in C_1^2 and the other in C_2^2. Suppose $C^2(\bar{i}0, \bar{j}0)$ and $C^2(\bar{i}0, \bar{j}1)$ have this property.

By the assumption there exists $a \in P_{\bar{i}} \cap C_{\bar{i}0}$. Then by the definition of D we have $\{x, y\} = d \in D$, where $x = (a, a(\bar{j}))$ and $y = (a, b(\bar{j}))$. Obviously, $x \in C^2(\bar{i}0, \bar{j}0)$ and $y \in C^2(\bar{i}0, \bar{j}1)$ and, hence, $d \subseteq C^2(\bar{i}_0, \bar{j}_0)$, $d \not\subseteq C_1^2$ and $d \not\subseteq C_2^2$, which is a contradiction.

For another pair of adjacent sets the argument is similar. Hence, D^0 is connected.

Now, we prove that D^0 is locally connected. It is sufficient to prove that if $d \in D^0$ and U is an open neighbourhood of d in D, then there exist an open neighbourhood V of d in D^0 and a connected subset W of D^0 such that $d \in V \subseteq W \subseteq U$.

Let $d \in D^0 \cap U$, where U is an open subset of D. If $d = \{x\}$, then $x \in C^2(\bar{i}_0, \bar{j}_0)$ and, hence, there exist an integer $n \geq n_0$ and $\bar{i}, \bar{j} \in L_n$ such that $d \subseteq C^2(\bar{i}, \bar{j}) \subseteq C^2(\bar{i}_0, \bar{j}_0) \cap p^{-1}(U)$. The set V of all elements of D which are contained in $C^2(\bar{i}, \bar{j})$ is an open neighbourhood of d in D, which is contained in D^0. On the other hand, by the above, $W = p(C^2(\bar{i}, \bar{j}))$ is connected. Obviously, $d \in V \subseteq W \subseteq U$.

Let $d = \{x, y\}$ and $d \subseteq C^2(\bar{i}_0, \bar{j}_0)$. There exist $n \in \mathbb{N}$ and elements $\bar{i}_1, \bar{j}_1, \bar{i}_2, \bar{j}_2 \in L_n$ such that $x \in C^2(\bar{i}_1, \bar{j}_1)$, $y \in C^2(\bar{i}_2, \bar{j}_2)$ and $C^2(\bar{i}_1, \bar{j}_1) \cup C^2(\bar{i}_2, \bar{j}_2) \subseteq C^2(\bar{i}_0, \bar{j}_0) \cap p^{-1}(U)$. As above, the set V of all elements of D which are contained in $C^2(\bar{i}_1, \bar{j}_1) \cup C^2(\bar{i}_2, \bar{j}_2)$ is an open neighbourhood of d in D, which is contained in D^0. Also, $W = p(C^2(\bar{i}_1, \bar{j}_1)) \cup p(C^2(\bar{i}_2, \bar{j}_2))$ is a connected subset of D^0 because $p(C^2(\bar{i}_1, \bar{j}_1))$ and $p(C^2(\bar{i}_2, \bar{j}_2))$ are connected and intersect each other. Obviously, $d \in V \subseteq W \subseteq U$.

Finally, let $d = \{x, y\}$ such that $x \in C^2(\bar{i}_0, \bar{j}_0)$ and $y \notin C^2(\bar{i}_0, \bar{j}_0)$. There exist an integer n and $\bar{i}_1, \bar{i}_2, \bar{j}_1, \bar{j}_2 \in L_n$ such that $x \in C^2(\bar{i}_1, \bar{j}_1) \subseteq C^2(\bar{i}_0, \bar{j}_0) \cap p^{-1}(U)$ and $y \in C^2(\bar{i}_2, \bar{j}_2) \subseteq (C^2 \setminus C^2(\bar{i}_0, \bar{j}_0)) \cap p^{-1}(U)$. The set V' of all elements of D which are contained in $C^2(\bar{i}_1, \bar{j}_1) \cup C^2(\bar{i}_2, \bar{j}_2)$ is an open neighbourhood of d in D, and $V = V' \cap D^0$ is an open neighbourhood of d in D^0. It is easy to see that V consists of all elements d' of V' for which $d' \cap C^2(\bar{i}_1, \bar{j}_1) \neq \emptyset$. Hence, $d \in V \subseteq W \subseteq U$, where W is the connected set $p(C^2(\bar{i}_1, \bar{j}_1))$. Thus, $p(C^2(\bar{i}_0, \bar{j}_0))$ is locally connected.

5. LEMMA. *Let M be a scattered compact subset of D, $n_0 \in \mathbb{N}$ and $\bar{i}_0, \bar{j}_0 \in L_{n_0}$. Suppose that for every $n \in \mathbb{N}$, $n \geq n_0$, for every $\bar{i}, \bar{j} \in L_n$ with $\bar{i} \geq \bar{i}_0$ and $\bar{j} \geq \bar{j}_0$ and for every $\bar{i}_1 \in L_2$ we have either* $\mathrm{type}(Q) > \mathrm{type}(M)$, *or* $0 < \mathrm{type}(Q) = \mathrm{type}(M) = \alpha$ *and* $|Q^{(\alpha-1)}| > |M^{(\alpha-1)}|$ *(hence, $Q \neq \emptyset$), for both $Q = P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ and $Q = P_{\bar{j}} \cap C_{\bar{j}\,\bar{i}_1}$. Then $p(C^2(\bar{i}_0, \bar{j}_0)) \setminus M$ is connected.*

Proof. We prove the lemma by induction on the ordinal $\mathrm{type}(M) = \alpha$. If $\alpha = 0$, then $M = \emptyset$ and by Lemma 4(5), $p(C^2(\bar{i}_0, \bar{j}_0))$ is connected.

Suppose that the lemma is proved for all $\alpha < \beta$. Since the type of a scattered compact space is an isolated ordinal we may suppose that β is also isolated.

Let M be a scattered compact subset of D having type β. Then $p^{-1}(M)$ is a compactum having type β and, hence, $p^{-1}(M^{(\beta-1)})$ is finite. It is easy to see that $p^{-1}(M^{(\beta-1)}) = (p^{-1}(M))^{(\beta-1)}$.

Hence, there exists an integer $n_1 \geq n_0$ such that if x and y are distinct elements of $p^{-1}(M^{(\beta-1)})$, then $\mathrm{st}^2(x, n_1) \cap \mathrm{st}^2(y, n_1) = \emptyset$.

First, we prove the following assertions:

(1) *If $n \geq n_0$, $\bar{i}, \bar{j} \in L_n$, $\bar{i} \geq \bar{i}_0$, $\bar{j} \geq \bar{j}_0$ and $C^2(\bar{i}, \bar{j}) \cap p^{-1}(M^{(\beta-1)}) = \emptyset$, then $p(C^2(\bar{i}, \bar{j})) \setminus M$ is connected.*

(2) *If* $n \geq n_1$, $x \in p^{-1}(M^{(\beta-1)})$ *and* $\mathrm{st}^2(x,n) \subseteq C^2(\bar{\imath}_0, \bar{\jmath}_0)$, *then* $p(\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1)) \setminus M$ *is connected.*

(3) *For* x *and* n *as in* (2), $p(\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+2)) \setminus M$ *is connected.*

Obviously, in (1), $p(C^2(\bar{\imath},\bar{\jmath})) \cap M^{(\beta-1)} = \emptyset$. Thus, $\mathrm{type}(p(C^2(\bar{\imath},\bar{\jmath})) \cap M) < \beta$. Hence, by induction, $p(C^2(\bar{\imath},\bar{\jmath})) \setminus M$ is connected.

To prove (2), let $\mathrm{st}^2(x,n) = C^2(\bar{\imath},\bar{\jmath})$. Then $\mathrm{st}^2(x,n+1)$ is either $C^2(\bar{\imath}0,\bar{\jmath}0)$, $C^2(\bar{\imath}0,\bar{\jmath}1)$, $C^2(\bar{\imath}1,\bar{\jmath}0)$, or $C^2(\bar{\imath}1,\bar{\jmath}1)$. Suppose that $\mathrm{st}^2(x,n+1) = C^2(\bar{\imath}0,\bar{\jmath}0)$; the other cases are treated similarly.

Let $k = |M^{(\beta-1)}|$. Since $C^2(\bar{\imath},\bar{\jmath}) \subseteq C^2(\bar{\imath}_0,\bar{\jmath}_0)$ we have $\bar{\imath} \geq \bar{\imath}_0$ and $\bar{\jmath} \geq \bar{\jmath}_0$. Hence, since $P_{\bar{\imath}} \cap C_{\bar{\imath}10} \subseteq P_{\bar{\imath}} \cap C_{\bar{\imath}1}$, we have either $\mathrm{type}(P_{\bar{\imath}} \cap C_{\bar{\imath}1}) = \beta$ and $|(P_{\bar{\imath}} \cap C_{\bar{\imath}1})^{(\beta-1)}| > k$, or $\mathrm{type}(P_{\bar{\imath}} \cap C_{\bar{\imath}1}) > \beta$. Also, either $\mathrm{type}(P_{\bar{\jmath}} \cap C_{\bar{\jmath}1}) = \beta$ and $|(P_{\bar{\jmath}} \cap C_{\bar{\jmath}1})^{(\beta-1)}| > k$, or $\mathrm{type}(P_{\bar{\jmath}} \cap C_{\bar{\jmath}1}) > \beta$. From the above and Lemma 4(3) it follows that $D^{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \not\subseteq M$ and $D_{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \not\subseteq M$, where $\bar{\imath}_1 = 1 \in L_1$. Let $d \in D^{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \setminus M$. If $d = \{y,z\}$, then either $y \in C^2(\bar{\imath}1,\bar{\jmath}0)$ and $z \in C^2(\bar{\imath}1,\bar{\jmath}1)$, or $z \in C^2(\bar{\imath}1,\bar{\jmath}0)$ and $y \in C^2(\bar{\imath}1,\bar{\jmath}1)$. In both cases $(p(C^2(\bar{\imath}1,\bar{\jmath}0)) \setminus M) \cap (p(C^2(\bar{\imath}1,\bar{\jmath}1)) \setminus M) \neq \emptyset$. Similarly, we have $(p(C^2(\bar{\imath}0,\bar{\jmath}1)) \setminus M) \cap (p(C^2(\bar{\imath}1,\bar{\jmath}1)) \setminus M) \neq \emptyset$.

Since $n \geq n_1$ and $x \in C^2(\bar{\imath}0,\bar{\jmath}0)$ we have $\mathrm{type}(C^2(\bar{\imath}0,\bar{\jmath}1) \cap p^{-1}(M)) < \beta$, $\mathrm{type}(C^2(\bar{\imath}1,\bar{\jmath}0) \cap p^{-1}(M)) < \beta$ and $\mathrm{type}(C^2(\bar{\imath}1,\bar{\jmath}1) \cap p^{-1}(M)) < \beta$. Hence, by (1), the sets $p(C^2(\bar{\imath}0,\bar{\jmath}1)) \setminus M$, $p(C^2(\bar{\imath}1,\bar{\jmath}0)) \setminus M$ and $p(C^2(\bar{\imath}1,\bar{\jmath}1)) \setminus M$ are connected.

Since the first two of them intersect the third, the union of the three sets, equal to $p(C^2(\bar{\imath}1,\bar{\jmath}0) \cup C^2(\bar{\imath}0,\bar{\jmath}1) \cup C^2(\bar{\imath}1,\bar{\jmath}1)) \setminus M$, is connected.

But $\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1) = C^2(\bar{\imath}1,\bar{\jmath}0) \cup C^2(\bar{\imath}0,\bar{\jmath}1) \cup C^2(\bar{\imath}1,\bar{\jmath}1)$. Hence, $p(\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1)) \setminus M$ is connected.

Now we prove (3). Obviously, $\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+2) = (\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1)) \cup (\mathrm{st}^2(x,n+1) \setminus \mathrm{st}^2(x,n+2))$. By (2), the sets $p(\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1)) \setminus M$ and $p(\mathrm{st}^2(x,n+1) \setminus \mathrm{st}^2(x,n+2)) \setminus M$ are connected. Hence, in order to prove (3) it is sufficient to show that they intersect each other.

As in (2), without loss of generality we can suppose that $\mathrm{st}^2(x,n+1) = C^2(\bar{\imath}0,\bar{\jmath}0)$. Then $\mathrm{st}^2(x,n) \setminus \mathrm{st}^2(x,n+1) = C^2(\bar{\imath}0,\bar{\jmath}1) \cup C^2(\bar{\imath}1,\bar{\jmath}0) \cup C^2(\bar{\imath}1,\bar{\jmath}1)$.

The set $\mathrm{st}^2(x,n+2)$ is either $C^2(\bar{\imath}00,\bar{\jmath}00)$, $C^2(\bar{\imath}01,\bar{\jmath}00)$, $C^2(\bar{\imath}00,\bar{\jmath}01)$, or $C^2(\bar{\imath}01,\bar{\jmath}01)$. By the assumption of the lemma and by Lemma 4(3) the sets $D^{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \setminus M$ and $D^{\bar{\imath}_2}(\bar{\imath},\bar{\jmath}) \setminus M$, where $\bar{\imath}_1 = 00 \in L_2$ and $\bar{\imath}_2 = 01 \in L_2$, are not empty.

Let $d \in D^{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \setminus M$ if $\mathrm{st}^2(x,n+2) \neq C^2(\bar{\imath}00,\bar{\jmath}01)$ and $d \in D^{\bar{\imath}_2}(\bar{\imath},\bar{\jmath}) \setminus M$ otherwise. If $d = \{x,y\}$, then one of the two points, say x, belongs to $C^2(\bar{\imath}0,\bar{\jmath}1)$. Then $y \in C^2(\bar{\imath}00,\bar{\jmath}01)$ if $d \in D^{\bar{\imath}_1}(\bar{\imath},\bar{\jmath}) \setminus M$, and $y \in C^2(\bar{\imath}01,\bar{\jmath}01)$ if $d \in D^{\bar{\imath}_2}(\bar{\imath},\bar{\jmath}) \setminus M$.

In both cases, the intersection considered is non-empty. Thus, (3) is proved.

From (1)–(3) the next statement follows:

(4) *Let* $\bar{\imath}, \bar{\jmath} \in L_{n_1}$ *and* $C^2(\bar{\imath}, \bar{\jmath}) \subseteq C^2(\bar{\imath}_0, \bar{\jmath}_0)$. *Then* $p(C^2(\bar{\imath}, \bar{\jmath})) \setminus M$ *is connected.*

Indeed, if $C^2(\bar{\imath}, \bar{\jmath}) \cap p^{-1}(M^{(\beta-1)}) = \emptyset$, then this is assertion (1). Otherwise, by the choice of the integer n_1, this intersection is a singleton $\{x\}$ and, hence, $\mathrm{st}^2(x, n_1) = C^2(\bar{\imath}, \bar{\jmath})$.

Obviously, $p(\mathrm{st}^2(x, n_1)) \setminus M = \bigcup_{m=0}^{\infty} (p(\mathrm{st}^2(x, n_1+m) \setminus \mathrm{st}^2(x, n_1+m+2)) \setminus M)$. By (3) each term of the union is connected. Since any two consecutive terms intersect each other, $p(\mathrm{st}^2(x, n_1)) \setminus M$ is also connected.

Finally, we prove that $p(C^2(\bar{\imath}_0, \bar{\jmath}_0)) \setminus M$ is connected. By (4) it is sufficient to prove the following statement:

Let $m \in \mathbb{N}$ *and let* $\bar{\imath}, \bar{\jmath} \in L_m$ *be such that* $n_0 \le m$, $m+1 \le n_1$, $C^2(\bar{\imath}, \bar{\jmath}) \subseteq C^2(\bar{\imath}_0, \bar{\jmath}_0)$ *and the sets* $p(C^2(\bar{\imath}0, \bar{\jmath}0)) \setminus M$, $p(C^2(\bar{\imath}0, \bar{\jmath}1)) \setminus M$, $p(C^2(\bar{\imath}1, \bar{\jmath}0)) \setminus M$ *and* $p(C^2(\bar{\imath}1, \bar{\jmath}1)) \setminus M$ *are connected. Then* $p(C^2(\bar{\imath}, \bar{\jmath})) \setminus M$ *is connected.*

Consider the sets $D^{\bar{\imath}_1}(\bar{\imath}, \bar{\jmath}) \setminus M$ and $D_{\bar{\imath}_1}(\bar{\imath}, \bar{\jmath}) \setminus M$, where $\bar{\imath}_1 \in L_1$. By the assumption of the lemma and by Lemma 4(3) all these sets are not empty. Let $d = \{x, y\} \in D^{\bar{\imath}_1}(\bar{\imath}, \bar{\jmath}) \setminus M$, where $\bar{\imath}_1 = 0 \in L_1$. Then one of the points x and y belongs to $C^2(\bar{\imath}0, \bar{\jmath}0)$ and the other to $C^2(\bar{\imath}0, \bar{\jmath}1)$. Hence, the first of the connected sets in the assumption of the statement intersects the second. Similarly we can prove that it also intersects the third, and that the third set intersects the fourth. Since $p(C^2(\bar{\imath}, \bar{\jmath})) \setminus M$ is the union of the four sets, it is connected. The proof of the lemma is complete.

6. Theorem. *Let* $0 < k(2), k(3), \ldots, k(n), \ldots$ *be an increasing sequence of integers and let* α *be an isolated ordinal. There exists a locally connected planar continuum* X *having rim-type* $\le \alpha$ *such that if a compact scattered subset* M *of* X *has type* α *and separates* X *into* $m \ge 2$ *parts*, *then* $|M^{(\alpha-1)}| \ge k(m)$.

Proof. Let $k_0, k_1, \ldots, k_n, \ldots$ be an increasing sequence of integers such that $k_n \ge k(4^{n+1})$. We construct a partition D of C^2 by defining the sets $P_{\bar{\imath}}$ as follows: if $\bar{\imath} \in L_n$, $n \in \mathbb{N}$, then $P_{\bar{\imath}}$ is a scattered compact subset of $C_{\bar{\imath}} \setminus \{a(\bar{\imath}), b(\bar{\imath}), a(C_{\bar{\imath}}), b(C_{\bar{\imath}})\}$ such that $\mathrm{type}(P_{\bar{\imath}} \cap C_{\bar{\imath}\bar{\imath}_1}) = \alpha$ and $|(P_{\bar{\imath}} \cap C_{\bar{\imath}\bar{\imath}_1})^{(\alpha-1)}| \ge k_n$, where $\bar{\imath}_1 \in L_2$. We prove that the quotient space D is the required space X.

Indeed, by (2), (4) and (5) of Lemma 4, D is a locally connected planar continuum having rim-type $\le \alpha$. Let M be a compact scattered subset of D which has type α and separates D into $m \ge 2$ parts.

Suppose that $|M^{(\alpha-1)}| < k(m)$. There exists an integer $q \geq 1$ such that $4^{q-1} < m \leq 4^q$. Then, if $|M^{(\alpha-1)}| = k$, we have $k < k(m) \leq k(4^q) \leq k_{q-1}$.

Let $n_0 = q-1$ and let $\bar{i}_0, \bar{j}_0 \in L_{n_0}$. Let $n \geq n_0$ and let $\bar{i}, \bar{j}$ belong to L_n with $\bar{i} \geq \bar{i}_0$ and $\bar{j} \geq \bar{j}_0$. If Q denotes any of the sets $P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}$ or $P_{\bar{j}} \cap C_{\bar{j}\,\bar{i}_1}$, where $\bar{i}_1 \in L_2$, then by the construction of the partition D and the choice of the integer n_0 we have $\mathrm{type}(Q) = \alpha$ and $|Q^{(\alpha-1)}| \geq k_n \geq k_{n_0} = k_{q-1} > k$. By Lemma 5 it follows that $p(C^2(\bar{i}_0, \bar{j}_0)) \setminus M$ is connected.

On the other hand, $D \setminus M = \bigcup_{\bar{i},\bar{j} \in L_{n_0}} (p(C^2(\bar{i}, \bar{j})) \setminus M)$. Since each term of the last union is connected, it follows that $m \leq |L_{n_0} \times L_{n_0}|$. Obviously, $|L_{n_0} \times L_{n_0}| = 4^{n_0} = 4^{q-1} < m$, which is a contradiction. Hence, $|M^{(\alpha-1)}| \geq k(m)$ and D is the required space X.

7. THEOREM. *Let α be a limit ordinal and let $\alpha(2), \alpha(3), \ldots, \alpha(n), \ldots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha(n) = \alpha$. There exists a locally connected planar continuum X having rim-type $\leq \alpha$ such that if a compact scattered subset M of X has type $\leq \alpha$ and separates X into $m \geq 2$ parts, then M has type $\geq \alpha(m)$.*

Proof. Let $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ be an increasing sequence of isolated ordinals such that $\alpha_n \geq \alpha(4^{n+1})$ and $\alpha_n < \alpha$. We construct a partition D of C^2 by defining the sets $P_{\bar{i}}$ as follows: if $\bar{i} \in L_n$, $n \in \mathbb{N}$, then $P_{\bar{i}}$ is a scattered compact subset of $C_{\bar{i}} \setminus \{a(\bar{i}), b(\bar{i}), a(C_{\bar{i}}), b(C_{\bar{i}})\}$ such that $\mathrm{type}(P_{\bar{i}} \cap C_{\bar{i}\,\bar{i}_1}) = \alpha_n$, where $\bar{i}_1 \in L_1$. The quotient space D is the required locally connected planar continuum X. The proof is the same as the corresponding part of the proof of Theorem 6.

8. THEOREM. *Let $Y \in R^{\text{rim-com}}(\alpha)$, where $\alpha > 0$. There exists a locally connected planar continuum X having rim-type $\leq \alpha$ which is not topologically contained in Y.*

The proof is the same as the proof of the Theorem of [I]. Instead of Lemmas 6 and 7 of [I] we here use Theorems 6 and 7.

COROLLARY. *The following families of spaces have no universal elements*:

(1) *the family of all* (*locally connected*) *planar elements of* $R^{\text{rim-com}}(\alpha)$,

(2) *the family of all* (*locally connected*) *planar compacta having rim-type* $\leq \alpha$,

(3) *the family of all* (*locally connected*) *planar continua having rim-type* $\leq \alpha$.

This corollary gives a negative answer to Problem 1 of [I]. It should be understood so that each "locally connected" may be either disregarded or considered part of the statement, thus giving 6 families without universal elements.

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