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Open subspaces of countable dense homogeneous spaces

by

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Abstract. We construct a completely regular space which is connected, locally connected and countable dense homogeneous but not strongly locally homogeneous. The space has an open subset which has a unique cut-point. We use the construction of a C^1-diffeomorphism of the plane which takes one countable dense set to another.

1. Introduction and history. In the 1895 volume of Mathematische Annalen, Georg Cantor and Paul Stäckel, colleagues at Halle, wrote consecutive articles. In Cantor's paper [8], an argument, the "back-and-forth" variation of which has become standard (see, for example, [11]), was used to show, among other things, that, for any countable dense subsets A and B of the real line $\mathbb{R}$, there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ which takes A onto B in a monotonic manner. In Stäckel's paper [33], an ordinary induction was used to show that, for any countable dense subsets A and B of the complex plane $\mathbb{C}$, there is an analytic function $h : \mathbb{C} \to \mathbb{C}$ which takes A into B. An English language translation and slight simplification of Stäckel's proof was given by Burckel and Saeki in 1983 [7]. These two papers represent the two main lines of inquiry into mappings of one countable dense set into another.

In 1925, Franklin [21], unaware of Stäckel's work, showed that, for any countable dense subsets A and B of $\mathbb{R}$, there is a function $f : \mathbb{C} \to \mathbb{C}$ which is analytic on a neighborhood of $\mathbb{R}$ and which takes A onto B in a monotonic manner. He did this using Cantor's back-and-forth technique and Stäckel's construction of a sequence of analytic functions converging uniformly on each closed disk (but using real coefficients). Unfortunately, he defined an equation between an infinite polynomial in x and an infinite polynomial in y and had to resort to the implicit function theorem to get a solution. Franklin also pointed out that the analytic homeomorphisms of $\mathbb{C}$ are the nonconstant linear transformations and thus that h cannot, in general, be taken to be a bijection.

In 1957, Erdős [13], [22] (see also [36]) asked whether there is a nonlinear real-analytic $h : \mathbb{R} \to \mathbb{R}$ such that $h^{-1}(\mathbb{Q}) = \mathbb{Q}$. In 1958, Melzak [26] answered Erdős without realizing it by showing that Franklin's proof can take care of countably many pairs A_n and B_n at once. In 1963, Neumann and Rado [28], without being aware of either Franklin or Stäckel showed that there is a nonpolynomial analytic $h : \mathbb{C} \to \mathbb{C}$ which is an order-preserving bijection on $\mathbb{R}$ such that $h(\mathbb{Q}) = \mathbb{Q}$. Neumann and Rado unfortunately relied on the field structure of $\mathbb{Q}$ but at least improved Melzak's result by getting h to be analytic everywhere. In 1970, Barth and Schneider [5] extended the result of Neumann and Rado to get $h(A) = B$ for arbitrary countable dense subsets of $\mathbb{R}$ but used some less elementary complex analysis. Their result answered another 1957 question of Erdős. Barth and Schneider knew of Franklin and Neumann and Rado but remained unaware of Stäckel.

Meanwhile, the problem for countable dense subsets of the complex plane continued to attract interest. In 1967, Erdős [12] asked if, whenever A and B are countable dense subsets of $\mathbb{C}$, there is an analytic $h : \mathbb{C} \to \mathbb{C}$ such that $h^{-1}(B) = A$. The same year, Maurer [25] showed that one can get $h(A) = B$ by defining y as an infinite polynomial of x by a simple back-and-forth argument which recovered the simplicity of Stäckel's method. Nevertheless, Maurer was unaware of Stäckel, Franklin and Neumann and Rado. In 1972 and 1974, Maurer's technique was applied to countable dense sets of $\mathbb{R}$ independently by Sato and Rankin [32] and Nienhuys and Thiemann [29] to obtain the result of Barth and Schneider: If A and B are countable dense subsets of $\mathbb{R}$, then there is an analytic $h : \mathbb{C} \to \mathbb{C}$ which is an order-preserving bijection on $\mathbb{R}$ such that $h(A) = B$. This problem even appeared in the American Mathematical Monthly in 1975 [23].

In 1972, Barth and Schneider [4] answered Erdős' 1967 question by using a little more complex analysis and showed that one can get $h^{-1}(B) = A$. In 1976, Dobrowolski [10] showed that, although analytic homeomorphisms have been ruled out, one can find real-analytic isomorphisms of $\mathbb{R}^2$ which map one arbitrary countable dense set to another. It is a C^1-diffeomorphism which we will need in this paper. In 1987, Morayne [27] showed that one can get analytic homeomorphisms of $\mathbb{C}^2$ taking one countable dense subset of $\mathbb{C}^2$ to another. He also showed that one can get analytic homeomorphisms of $\mathbb{C}^2$ taking one countable dense subset of $\mathbb{R}^2$ to another (thus providing another proof of Dobrowolski's theorem).

The study of the existence of homeomorphisms taking one countable dense set to another was extended to more general spaces by Fort [20] in 1962. He showed that the product of countably many manifolds with boundary (for example, the Hilbert cube) admit such homeomorphisms. In 1972, the abstract study was begun by Bennett [6] who called such spaces "countable dense homogeneous" or "CDH". Bennett established two key results.

In 1954, Ford [19] had defined a space X to be strongly locally homogeneous (SLH) if there is a base of open sets U such that, for each $x, y \in U$, there is a homeomorphism of X which takes x to y but is the identity outside U. Bennett showed, first, that CDH connected first countable spaces are homogeneous and, second, that SLH locally compact separable metric spaces are CDH. Bennett also asked a series of questions: (1) "Are CDH continua locally connected?", (2) "Are CDH continua n-homogeneous?" and (3) "Is the product of CDH spaces CDH?". Fitzpatrick [14], in the same year, answered question (1) by showing that CDH connected locally compact metric spaces are locally connected. Ungar [37], in 1978, showed that, for compact metric spaces, CDH is equivalent to n-homogeneous, answering question (2). Question (3) was answered in 1980 with a counterexample by Kuperberg, Kuperberg and Transue [24].

Bennett's influential theorems above have led to a series of results: In 1969, de Groot [9] weakened locally compact to complete (see proof on p. 317 of [2]) in Bennett's second theorem. In 1974, Ravdin [30] replaced SLH by "locally homogeneous of variable type" (which implies "representable" which is apparently weaker than SLH) for complete separable metric spaces. In 1974, Fletcher and McCoy [18] showed that "representable" complete separable metric spaces are CDH while in 1976, Bales [3] showed that "representable" is equivalent to SLH in any case.

In 1982, van Mill [38] put an end to the sequence of weakenings of Bennett's second theorem by constructing a subset of $\mathbb{R}^2$ which is connected locally connected and SLH but not CDH. In 1989, Saltsman [31] showed that the subspace of the plane consisting of those points whose coordinates are either both rational or both irrational is such a subset (this subspace had been studied in the early 1980's by Eric van Douwen). In 1985, Steprāns and Zhou [35] contributed another lower bound by constructing a separable manifold (thus SLH and locally compact) which is not CDH (see also [39]).

Meanwhile, in 1984, Fitzpatrick and Lauer [17] showed that the assumption of first countability in Bennett's first theorem was unnecessary.

Ungar raised a new question in 1978: Is each open subspace of a CDH space CDH? Jan van Mill also raised a question in 1982: Are connected CDH spaces SLH? In 1985, Fitzpatrick and Zhou [15] answered these two questions negatively for the class of Hausdorff spaces. Unfortunately, their spaces were not regular.

In this paper, we construct a completely regular space which answers the last two questions and which uses the classical tradition of constructing smooth functions on $\mathbb{R}^2$ which take one countable dense set to another. In fact, we need the construction of a C^1-diffeomorphism of the plane which takes one countable dense set to another.

We finish this section with a short list of open problems:

1. (a) Are CDH connected metric spaces SLH?
   (b) Are CDH continua SLH?
   (c) Are CDH connected compact Hausdorff spaces SLH?
2. (a) Are open subsets of CDH metric spaces CDH?
   (b) Are open subsets of CDH compact Hausdorff spaces CDH?
   (c) Are open subsets of CDH continua CDH (or even homogeneous)?
3. Are CDH connected spaces 2-homogeneous?

(*Added in proof*: W. L. Saltsman has recently constructed, under CH, a connected CDH subset of the plane which is not SLH.) We also warmly recommend the article by Ben Fitzpatrick and Zhou Hao-Xuan in *Open Problems in Topology* [16].

2. The example

THEOREM 1. *There is a completely regular connected locally connected space which is CDH and which has an open subset with a unique cut-point.*

Proof. The space is the Euclidean plane with a topology which is larger than the Euclidean topology. Whenever $z:[0,1]\to\mathbb{R}^2$ is a continuously differentiable simple closed curve (that is, $z(0)=z(1)=p$ for some p) and $z'(0)=-z'(1)\neq 0$, we write $\mathbb{R}^2-\operatorname{rng}(z)$ as the free union of two nonempty open sets and denote the one with compact closure as $A(z)$. We let each $\overline{A(z)}-\{z(0)\}$ be closed and denote the new topology by τ.

This topology is completely regular since for any such curve z_0, there is another such curve z_1 "inside" z_0 and a homotopy $m:[0,1]\times[0,1]\to\mathbb{R}^2$ such that $m_0=z_0$, $m_1=z_1$, $m(\{0,1\}\times[0,1])=p$ and m is one-to-one. Let π be the projection of an ordered pair onto the second coordinate. The real-valued continuous function $\pi\circ m^{-1}$ can be extended to a function t which is zero outside z_0 and one inside z_1. We also define $t(p)=1$. Thus t is the continuous real-valued function which witnesses complete regularity for these additional sub-basic open sets. The reader may wish to compare this construction with that of Fitzpatrick and Lauer [17]. Indeed, if we had chosen instead to construct a larger topology in which we remove the simple closed curves but not their interiors (or simply remove arcs), we would have a Hausdorff space with all the other properties of τ but this topology would not be completely regular. The space defined in this paper can be viewed as "fattening up" the sets which can be removed in order to "squeeze in" the necessary continuous real-valued functions.

If $h:\mathbb{R}^2\to\mathbb{R}^2$ is a one-to-one onto mapping with continuous nonzero partial derivatives, then elementary considerations (see pp. 71 and 107 of [1]) imply that h is a homeomorphism and that h^{-1} also has continuous nonzero partial derivatives. Continuously differentiable simple closed curves $z : [0,1] \to \mathbb{R}^2$ are preserved by h and $z'(0) = -z'(1) \neq 0$ implies $h(z)'(0) = -h(z)'(1) \neq 0$ by a simple calculation. Thus if h is a C^1-diffeomorphism, then h^{-1} is also a C^1-diffeomorphism and the set of new base elements is closed under h and h^{-1}. Thus we deduce that h is a homeomorphism.

A countable set is dense in τ if and only if it is dense in the Euclidean topology. Using Dobrowolski's result that there is a C^1-diffeomorphism which takes arbitrary countable dense sets to arbitrary countable dense sets, we conclude that the plane with the topology τ is CDH.

Let $U = \{(a,b) \in \mathbb{R}^2 : |b| > a^2\} \cup \{(0,0)\}$. The open set U has a point $(0,0)$ whose removal leaves the set disconnected. We shall prove that $(\mathbb{R}^2, \tau)$ is connected. We will then deduce that, since each open ball of $\mathbb{R}^2$ is C^1-diffeomorphic to $\mathbb{R}^2$, U is connected. This also implies that U has no other cut points and also that τ is a locally connected topology. We can then deduce that U is not CDH and so that CDH is not open hereditary. Furthermore, the plane with the topology τ is not SLH answering van Mill's question.

3. The proof of connectedness. We prove a lemma which is clearly sufficient:

LEMMA 1. *Let e denote the usual Euclidean topology of the plane $\mathbb{R}^2$. Let τ be another topology. If τ satisfies the following*:

(1) $(\forall V \in \tau)\, V \subset \mathrm{cl}_e(\mathrm{int}_e(V))$,

(2) $(\forall x \in \mathbb{R}^2)(\forall V \in \tau)\, x \in V \Rightarrow \lim_{r\to 0^+} \mu(\mathrm{int}_e(V) \cap B(x,r)) = \pi r^2$,

where μ denotes Lebesgue measure and $B(x,r)$ denotes $\{y \in \mathbb{R}^2 : d(x,y)<r\}$, then $(\mathbb{R}^2, \tau)$ is connected.

Proof. Aiming for a contradiction, assume that A and B are disjoint nonempty τ-open subsets which cover $\mathbb{R}^2$. By (1), $\mathrm{int}_e(A) \neq \emptyset \neq \mathrm{int}_e(B)$, hence there are $a_1 \in \mathrm{int}_e(A)$ and $b_1 \in \mathrm{int}_e(B)$ and $0 < r_1 < 1$ such that $B(a_1, r_1) \subset \mathrm{int}_e(A)$ and $B(b_1, r_1) \subset \mathrm{int}_e(B)$.

Let $f : [0,1] \to [0,1]$ be defined by

$$f(t) = \frac{\mu(\mathrm{int}_e(A) \cap B((1-t)a_1 + tb_1, r_1))}{\pi r_1^2}.$$

We can check that $f(0) = 1$ and $f(1) = 0$ and that f is continuous. Hence there is some $t_1 \in (0,1)$ with $f(t_1) = 1/2$. Let $x_1 = (1-t_1)a_1 + t_1 b_1$.

The ball $B(x_1, r_1)$ cannot be contained in A, because otherwise it is contained in $\mathrm{int}_e(A)$, which is impossible by $f(t_1) = 1/2 < 1$. The ball $B(x_1, r_1)$ cannot be contained in B, because then it would be disjoint from $\mathrm{int}_e(A)$, which is again impossible by $f(t_1) = 1/2 > 0$. But by (1), we now have $\mathrm{int}_e(A) \cap B(x_1, r_1) \neq \emptyset$ and $\mathrm{int}_e(B) \cap B(x_1, r_1) \neq \emptyset$. Therefore there are $a_2 \in \mathrm{int}_e(A) \cap B(x_1, r_1)$, $b_2 \in \mathrm{int}_e(B) \cap B(x_1, r_1)$ and $r_2 < 1/2$ such that $B(a_2, r_2) \subset \mathrm{int}_e(A) \cap B(x_1, r_1)$ and $B(b_2, r_2) \subset \mathrm{int}_e(B) \cap B(x_1, r_1)$. Apply the same reasoning as before and continue. As a result, one finds a nested sequence $\{B(x_n, r_n) : n \in \omega\}$ of open balls with a one-point intersection z such that

$$(\forall n \in \omega)\ \frac{\mu(\mathrm{int}_e(A) \cap B(x_n, r_n))}{\pi r_n^2} = \frac{1}{2}\,.$$

Since $A \cup B = \mathbb{R}^2$, let us assume that $z \in A$ (the other case is analogous). Choose some τ-neighborhood V of z such that $V \subset A$. By (2), there is some $R > 0$ such that

$$(\forall r > 0)\ r < R\ \Rightarrow\ \frac{\mu(\mathrm{int}_e(V) \cap B(z, r))}{\pi r^2} > \frac{7}{8}\,.$$

Choose $n \in \omega$ large enough to make sure that $d(z, x_n) < R/2$ and $r_n < R/2$. Let $r = d(z, x_n) + r_n$. Since $z \in B(x_n, r_n)$, $d(z, x_n) < r_n$ and consequently $r_n > r/2$. Now $B(z, r) - \mathrm{int}_e(V) \supset B(x_n, r_n) - \mathrm{int}_e(V)$ and so

$$\mu(B(z, r) - \mathrm{int}_e(V)) \ge \mu(B(x_n, r_n) - \mathrm{int}_e(V)) \ge \mu(B(x_n, r_n) - \mathrm{int}_e(A))\,.$$

Thus we can calculate

$$\frac{\mu(B(z, r) - \mathrm{int}_e(V))}{\pi r^2} \ge \frac{\mu(B(x_n, r_n) - \mathrm{int}_e(A))}{\pi r^2} \ge \frac{\frac{1}{2}\pi r_n^2}{\pi r^2} > \frac{\frac{1}{2} \cdot \frac{1}{4} r^2}{r^2} = \frac{1}{8}\,.$$

Notice that $r < R$ and that therefore

$$1 = \frac{\mu(B(z, r))}{\pi r^2} = \frac{\mu(B(z, r) \cap \mathrm{int}_e(V))}{\pi r^2} + \frac{\mu(B(z, r) - \mathrm{int}_e(V))}{\pi r^2} > \frac{7}{8} + \frac{1}{8} = 1\,,$$

which is a contradiction and completes the proof.

The first author wishes to thank Zhou Hao-Xuan for suggesting this problem. This space was first presented in Chengdu, China in August, 1986 by the first author. The connectedness was first proved by the second author. Connectedness was also proved independently by Jo Heath, Ben Fitzpatrick and Michael Wage. The open subspace was found while discussing the space with E. K. van Douwen and Alan Dow.

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