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Nielsen theory of transversal fixed point sets

by

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With an appendix:

C^∞ and C^0 fixed point sets are the same

by

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Abstract. Examples exist of smooth maps on the boundary of a smooth manifold M which allow continuous extensions over M without fixed points but no such smooth extensions. Such maps are studied here in more detail. They have a minimal fixed point set when all transversally fixed maps in their homotopy class are considered. Therefore we introduce a Nielsen fixed point theory for transversally fixed maps on smooth manifolds without or with boundary, and use it to calculate the minimum number of fixed points in cases where continuous map extensions behave differently from smooth ones. In the appendix it is shown that a subset of a smooth manifold can be realized as the fixed point set of a smooth map in a given homotopy class if and only if it can be realized as the fixed point set of a continuous one. A special case of this result is used in a proof of the paper.

1. Introduction. An interesting phenomenon which had not been observed in Nielsen fixed point theory before was discovered in [2], namely the fact that smooth maps on manifolds may have larger minimal fixed point sets than maps which are only continuous. More precisely, we studied in [2] the least number of fixed points of extensions $f : M \to M$ of a given smooth map $\bar{f} : \partial M \to \partial M$ on the boundary ∂M of a smooth manifold M if $\bar{f}$ is kept fixed but f is allowed to vary in its homotopy class. As a lower bound for the number of fixed points on the interior $\operatorname{Int} M$ of M we introduced, for continuous extensions of $\bar{f}$, the extension Nielsen number $N(f|\bar{f})$, and for smooth extensions of $\bar{f}$, the extension Nielsen number $N^1(f|\bar{f})$. (See [2] or §4 below

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for the definitions.) Both $N(f|\bar{f})$ and $N^1(f|\bar{f})$ are optimal lower bounds, i.e. they can be realized by continuous resp. smooth maps, if the dimension of M is sufficiently high. It is always true that $N(f|\bar{f}) \le N^1(f|\bar{f})$, and $N(f|\bar{f}) \neq N^1(f|\bar{f})$ can occur. An easy and intriguing example, and the only one observed in [2], is the map $\bar{f}(z) = z^d$ on the boundary of the unit disk D in the complex plane. If $d > 1$, then $N(f|\bar{f}) = 0$ and $N^1(f|\bar{f}) = 1$, and $\bar{f}$ has continuous extensions over D which have no fixed points on its interior but every smooth extension of $\bar{f}$ must have at least one fixed point on $\operatorname{Int} D$.

It was crucial in [2] to assume that $\bar{f} : \partial M \to \partial M$ is transversally fixed (see the definition in §2), and this assumption is satisfied by $\bar{f}(z) = z^d$ if $d \neq 1$. But this property is not sufficient to lead to $N(f|\bar{f}) \neq N^1(f|\bar{f})$, as it is possible, for $d > 1$, to split one fixed point of $\bar{f}(z) = z^d$ into three fixed points, two of index -1 and one of index $+1$, and thus construct a transversally fixed map of ∂D of the same degree d which has not only continuous, but also smooth extensions over D without further fixed points. This behaviour is typical and will be made precise in Theorem 4.2. The map $\bar{f}(z) = z^d$ has, for all $d \neq 1$, a second property which is crucial: it has a minimal fixed point set if all maps in its homotopy class are considered which are transversally fixed. For this reason we study here such maps in more detail, and develop a Nielsen fixed point theory for transversally fixed maps.

This is first done in the non-relative setting. The "transversal Nielsen number" $N_{\pitchfork}(f)$ of $f : M \to M$, which parallels the classical Nielsen number $N(f)$, is introduced in §2, and we show that it has the usual basic properties of Nielsen numbers. In particular, it is a lower bound for the number of fixed points on M for all transversally fixed maps in the homotopy class of f, and can be realized by a map with precisely $N_{\pitchfork}(f)$ transversal fixed points if the dimension of M is sufficiently high. Some examples are given which illustrate how $N_{\pitchfork}(f)$ is calculated, and some transversally fixed maps with minimal fixed point sets are constructed. Such maps, which we call "sparse", play a role in the rest of this paper.

In order to apply our results to the extension problem of [2], we next study the case where $\partial M \neq 0$ and $f : (M, \partial M) \to (M, \partial M)$ is a map of pairs of spaces, i.e. we consider the setting of the most interesting special case of relative Nielsen theory [9] for transversally fixed maps. There are two assumptions on f which can be made, and which lead to different results.

In the first case we only assume that the restriction $\bar{f} : \partial M \to \partial M$ of f to ∂M is transversally fixed, but we allow arbitrary fixed points on $\operatorname{Int} M$. It is this case which is closely related to the work on fixed points of map extensions [2]. In §3 we define the "boundary transversal Nielsen number" $N(f; M, \partial M_{\pitchfork})$ of $f : (M, \partial M) \to (M, \partial M)$. Its structure is related to that of the relative Nielsen number $N(f; M, \partial M)$ [9] but uses the concept of "transversally common" fixed point classes. It has the usual properties

(§3 and §5). In particular, $N(f; M, \partial M_{\pitchfork})$ is an optimal lower bound for the number of fixed points on M for all maps in the homotopy class of $f : (M, \partial M) \to (M, \partial M)$ which are transversally fixed on ∂M if M is of sufficiently high dimension.

The relations between the boundary transversal Nielsen number and the extension Nielsen numbers are studied in §4. The extension Nielsen numbers $N(f|\bar{f})$ and $N^1(f|\bar{f})$ are not invariant under all homotopies of pairs of spaces (i.e. homotopies of the form $(M \times I, \partial M \times I) \to (M, \partial M)$), but we show that they are invariant under such homotopies if the restriction to ∂M at the beginning and end of the homotopy is sparse. For sparse maps $\bar{f} : \partial M \to \partial M$ we can calculate $N(f|\bar{f})$ and $N^1(f|\bar{f})$, and hence $N^1(f|\bar{f}) - N(f|\bar{f})$, in terms of the Nielsen numbers from §2, §3 and [9]. Some examples of calculations are included which show, e.g., that $N^1(f|\bar{f}) \neq N(f|\bar{f})$ for all $n \geq 2$ if $\bar{f} : \partial B^n \to \partial B^n$ is a sparse map of degree d on the boundary of the n-ball B^n and $(-1)^d n \geq 2$, and so the different fixed point behaviour of smooth and of continuous extensions of maps on the boundary of the disk occurs again in all higher dimensions. They also show that $N^1(f|\bar{f}) \neq N(f|\bar{f})$ can happen on many other manifolds, and that the difference $N^1(f|\bar{f}) - N(f|\bar{f})$ can be arbitrarily large. Clearly the map $\bar{f}(z) = z^d$ with $d \geq 2$ on the boundary of the disk for which $N^1(f|\bar{f}) \neq N(f|\bar{f})$ was first observed [2] does not illustrate an isolated phenomenon, but one which occurs quite frequently.

Finally, in §6, we study minimal fixed point sets of maps $f : (M, \partial M) \to (M, \partial M)$ which are transversally fixed on all of M, and describe this case with the help of the "relative transversal Nielsen number" $N(f; M_{\pitchfork}, \partial M)$.

We make the following *assumptions* throughout this paper: unless otherwise stated all manifolds are compact, connected and smooth (i.e. C^1) and all maps and homotopies are smooth (i.e. C^1). The definitions of the Nielsen numbers introduced here can be made in a continuous setting, but this would not be of interest. On the other hand, C^1 could be replaced throughout by C^k, for any $k = 1, 2, \ldots, \infty$, without changes [2, Remark at the end of §7]. We will repeat some definitions and results from relative Nielsen theory which are used, but the reader needs at least a superficial knowledge of [2] and [9] to understand the motivations and methods of proof. An important tool, and a basic reason for the difference of fixed point sets of smooth and of merely continuous maps of $(M, \partial M)$ is the "Index Theorem" [2, Theorem 5.1], which is repeated as Theorem 3.1 below.

In Theorem 2.13 sparse maps are used to characterize all possible fixed point sets of transversally fixed maps on manifolds without boundary. The proof of this theorem makes use of the fact that a continuous map on a smooth manifold with a singleton as fixed point set is homotopic to a smooth map with the same fixed point set. This follows from a much more general result which was recently proved by Robert E. Greene and which can be found in the Appendix to this paper. It shows that a subset of a smooth manifold can be realized as the fixed point set of a smooth selfmap if and only if it can be realized as the fixed point set of a continuous selfmap.

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2. The transversal Nielsen number. Let $T_p(M)$ be the tangent space at the point p of the manifold M, let $\mathrm{Id} : T_p(M) \to T_p(M)$ be the identity operator and $df_p : T_p(M) \to T_p(M)$ the derivative of the map $f : M \to M$ at p. As in [2, §6] we say that f is *transversally fixed* if the linear operator $df_p - \mathrm{Id} : T_p(M) \to T_p(M)$ is non-singular for each fixed point p of f. In other words, f is transversally fixed if and only if its graph in $M \times M$ is transversal to the diagonal. Hence each fixed point of a transversally fixed map is isolated and has index ± 1. We are interested in the least number of fixed points of all transversally fixed maps in a given homotopy class, and for this reason introduce a new Nielsen-type number. The fixed point index of the fixed point class F of $f : M \to M$ is denoted by $i(F)$.

DEFINITION 2.1. The *transversal Nielsen number* of a map $f : M \to M$ is $N_{\pitchfork}(f) = \sum(|i(F)| : F$ is a fixed point class of $f)$.

It is clear from the definition that $0 \le N(f) \le N_{\pitchfork}(f)$, and that $N_{\pitchfork}(f) = 0$ if and only if $N(f) = 0$. As each fixed point class of a transversally fixed map must contain $\ge |i(F)|$ fixed points, we immediately have

PROPOSITION 2.2 (Lower bound). *A transversally fixed map $f : M \to M$ has at least $N_{\pitchfork}(f)$ fixed points.*

The definition shows that $N_{\pitchfork}(f)$ has the following usual basic properties of Nielsen numbers.

PROPOSITION 2.3 (Homotopy invariance). *If $f, g : M \to M$ are homotopic, then $N_{\pitchfork}(f) = N_{\pitchfork}(g)$.*

PROPOSITION 2.4 (Commutativity). *If $f : M \to N$ and $g : N \to M$ are maps between the manifolds M and N, then $N_{\pitchfork}(g \circ f) = N_{\pitchfork}(f \circ g)$.*

PROPOSITION 2.5 (Homotopy type invariance). *If $f : M \to M$ and $g : N \to N$ are maps of the same homotopy type, then $N_{\pitchfork}(f) = N_{\pitchfork}(g)$.*

The calculation of $N_{\pitchfork}(f)$ is usually no harder than the calculation of $N(f)$. In particular, it is often possible to obtain $N_{\pitchfork}(f)$ from the Lefschetz number $L(f)$.

Proposition 2.6. (i) $N_{\pitchfork}(f) \geq |L(f)|$, (ii) $N_{\pitchfork}(f) = |L(f)|$ *if and only if the indices of all essential fixed point classes of* f *have the same sign.*

P r o o f. (i) follows immediately from

$$N_{\pitchfork}(f) = \sum |i(F)| \geq \Big|\sum i(F)\Big| = |\mathrm{ind}\,(M, f, M)| = |L(f)|\,,$$

where $\mathrm{ind}(M, f, M)$ denotes the fixed point index of f on M and the summation is taken over all fixed point classes of f. We see that (ii) is true as $N_{\pitchfork}(f) = |L(f)|$ if and only if

$$\sum |i(F)| = \Big|\sum i(F)\Big|\,.$$

Note that Proposition 2.6(ii) is satisfied if f is a deformation, if M is simply connected, if M is a Jiang space (i.e. if $\pi_1(X)$ is equal to the Jiang group $J(X)$, see [6, Ch. II, §3]) or if M is a nilmanifold [1], as then all fixed point classes of f have the same index. So we have, e.g., for a selfmap $f : S^n \to S^n$ of the n-sphere S^n $(n \geq 1)$ of degree d

$$N_{\pitchfork}(f) = |1 + (-1)^n d|\,.$$

Next we show, in Theorems 2.7 and 2.8, that $N_{\pitchfork}(f)$ is an optimal lower bound for the number of fixed points of transversally fixed maps in the homotopy class of f if the dimension of the manifold satisfies the same assumptions as in the classical case of the Nielsen number $N(f)$.

Theorem 2.7 (Minimum theorem for $N_{\pitchfork}(f)$). *If* M *is a manifold of dimension* $\neq 2$, *then every map* $f : M \to M$ *is homotopic to a transversally fixed map* $g : M \to M$ *with* $N_{\pitchfork}(f)$ *fixed points.*

P r o o f. By transversality theory we can homotope f to a map $f' : M \to \mathrm{Int}\, M$ (where $\mathrm{Int}\, M$ denotes the interior of M) which is transversally fixed. If $\dim M \geq 3$ and if a fixed point class of f' contains one fixed point of index $+1$ and one fixed point of index -1, we can cancel these two fixed points with the help of the Whitney trick [5, §3] and thus obtain g after repeating this process as often as possible. If $\dim M = 1$, it is easy to see directly that Theorem 2.7 is still true.

Theorem 2.7 is not true for surfaces, as there exists on every surface of negative Euler characteristic a continuous selfmap with Nielsen number 0 so that every map in its homotopy class must have a fixed point [7, Theorem 1]. Hence any transversally fixed smooth approximation f of such a map has $N(f) = 0$ and at least two fixed points. As in the classical case Theorem 2.7 is, however, still true for surfaces if f is a deformation, i.e. homotopic to the identity $\mathrm{id} : M \to M$. By Proposition 2.6(ii) we see that $N_{\pitchfork}(\mathrm{id})$ is the absolute value of the Euler characteristic $\chi(M)$.

Theorem 2.8. *Every manifold admits a deformation which is transversally fixed and has* $N_{\pitchfork}(\mathrm{id}) = |\chi(M)|$ *fixed points.*

Proof. By Theorem 2.7 we can assume that $\dim M = 2$. Let $f : M \to \operatorname{Int} M$ be a small perturbation of the identity which is transversally fixed. If x_1, x_2 are two fixed points of f whose indices are of opposite sign, we choose open sets N and N_1 in M with $\{x_1, x_2\} \subset N \subset N_1$ and $f(\overline{N}) \subset \overline{N}_1$, where $\overline{N}$ denotes the closure, so that there exists a diffeomorphism $h : (\overline{N}_1, \overline{N}) \to (C_1, C)$ of $(\overline{N}_1, \overline{N})$ onto a pair of closed bounded convex sets in the plane. [2, Lemma 7.2] shows that $h \circ f \circ h^{-1}|C : C \to C_1$ is homotopic relative to ∂C to a fixed point free map $F : C \to C_1$, and so $h^{-1} \circ F \circ h|\overline{N} : \overline{N} \to \overline{N}_1$ can be used to cancel the fixed points x_1, x_2 of f. After repeating this process as often as possible we obtain a deformation on M with $N_{\pitchfork}(\mathrm{id})$ fixed points.

Maps $f : M \to M$ which are transversally fixed and which have a minimal fixed point set will play an important role later on, especially in §4. Therefore we make

Definition 2.9. A map $f : M \to M$ on a manifold M is called *sparse* if it is transversally fixed and has $N_{\pitchfork}(f)$ fixed points.

Here are some examples of sparse maps.

Example 2.10. If $S^1 = \{z : |z| = 1\}$ is the unit circle in the complex plane and $d \neq 1$, then the map $f : S^1 \to S^1$ given by $f(z) = z^d$ is a sparse map of degree d. To obtain, inductively, a sparse map of given degree $d \neq (-1)^{n-1}$ on the unit sphere S^n for all $n > 1$, assume that $\bar{f} : S^{n-1} \to S^{n-1}$ is a sparse map of degree $-d$, consider $S^n = S(S^{n-1})$ as the suspension of S^{n-1}, let $S(\bar{f}) : S^n \to S^n$ be the suspension of $\bar{f}$ and let $r : S^n \to S^n$ be the reflection of S^n in the "equator" S^{n-1}. Then $f' = r \circ S(\bar{f}) : S^n \to S^n$ has $N_{\pitchfork}(\bar{f}) = |1 + (-1)^{n-1}(-d)| = |1 + (-1)^n d|$ transversal fixed points and is smooth at every point of S^n apart from the two "north and south poles" of S^n. By standard theorems on smooth approximations (see e.g. [8] or [3]) we can approximate f' by a smooth map $f : S^n \to S^n$ which equals f' near S^{n-1} and has the same fixed point set as f', and so f is a sparse map of degree d. For $d = (-1)^{n-1}$ a sparse map is fixed point free, and so the antipodal map will do.

Example 2.11. To obtain a sparse deformation on a closed orientable surface S of genus $g > 1$, we start with the "hot fudge topping" map $f' : S \to S$ in [4, p. 125] which has one source, one sink and $2g$ saddles. It is easy to cancel the source and the top saddle, and also the sink and the bottom saddle, and thus obtain a deformation with $2g - 2$ saddles, i.e. a deformation with $N_{\pitchfork}(\mathrm{id}) = |\chi(S)|$ transversal fixed points.

We now show, in Theorem 2.13, how sparse maps can be used to construct maps with prescribed transversal fixed point sets. As the fixed point set of a transversally fixed map is finite and as each fixed point has index ± 1, possible fixed point sets are much tighter controlled than in the case of continuous maps or of smooth maps with a not necessarily transversal fixed point set. Fixed point sets of continuous maps in a given homotopy class are characterized in [11], and Robert E. Greene has shown that a subset K of M can be the fixed point set of a smooth selfmap in a given homotopy class if and only if it can be the fixed point set of a continuous selfmap in this (continuous) homotopy class. (See the Appendix.) The proof of Theorem 2.13 will use the next lemma which is stated in a more general form needed in the proof of Theorem 4.2 below.

LEMMA 2.12. *Let M be a manifold without boundary and let x_0 be an isolated fixed point of the map $f : M \to M$. Given an integer $n > 0$, there exist a neighbourhood U of x_0 with* $\mathrm{Fix}\, f \cap \overline{U} = x_0$ *and a map $g : M \to M$ which is homotopic to f relative $M - U$ so that*

(i) $\mathrm{Fix}\, g \cap U = \{x_0, x_1, \ldots, x_n,\ y_1, \ldots, y_n\}$,

(ii) *all fixed points of g on U are in the same fixed point class*,

(iii) $\mathrm{ind}(M, g, x_0) = \mathrm{ind}(M, f, x_0)$, $\mathrm{ind}(M, g, x_j) = 1$ *and* $\mathrm{ind}(M, g, y_j) = -1$ *for* $j = 1, \ldots, n$,

(iv) $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$ *are transversal fixed points of g*,

(v) *if f is transversally fixed*, *then g is transversally fixed.*

Proof. If $\dim M \geq 2$, we pick a (small) Euclidean neighbourhood U of x_0 with $\mathrm{Fix}\, f \cap \overline{U} = \{x_0\}$ and use [5, §4] to get a map $f' : M \to M$ homotopic to f relative $M - U$ with $\mathrm{Fix}\, f' \cap U = \{x_0, y_0, x_1, \ldots, x_n\}$, where $\mathrm{ind}(M, f', x_0) = \mathrm{ind}(M, f, x_0)$, $\mathrm{ind}(M, f', y_0) = -n$, $\mathrm{ind}(M, f', x_j) = 1$ for $j = 1, \ldots, n$ and $x_1, \ldots, x_n$ are transversal fixed points of f'. It follows from the construction of f' in [5, §4] that we can further require that $f'(\overline{U}) \subset \overline{W}$, where W is a Euclidean neighbourhood. According to [4, Splitting Proposition, p. 126] we can perturb f' slightly in a small Euclidean neighbourhood V of y_0 which is disjoint from the other fixed points of f' and contained in U, to a transversally fixed map f'' which is homotopic to f' relative $M - V$, and we can require that $f''(\overline{V}) \subset \overline{W}$. Hence we can use (as in the proof of Theorem 2.8) [2, Lemma 7.2] to cancel fixed points of f'' on V which are of opposite index ± 1. Thus we obtain a map g which satisfies (i)–(iv). As the construction of f' with the help of [5, §4] does not change f in a small neighbourhood of x_0, we see that g also satisfies (v). If the dimension of M is 1, then it is easy to see directly that Lemma 2.12 is true.

THEOREM 2.13. *Let M be a manifold without boundary of dimension ≥ 3, let K be a subset of M and $f : M \to M$ a given map. Then there exists* *a map $g : M \to M$ homotopic to f which is transversally fixed and has K as its fixed point set if and only if the cardinality of K is $N_{\pitchfork}(f) + 2n$ for some integer $n \geq 0$.*

Proof. (i) *Necessity.* Let $g : M \to M$ be a transversally fixed map homotopic to f. If G is the fixed point class of g which corresponds to the fixed point class F of f, then $\operatorname{ind}(M, g, G) = \operatorname{ind}(M, f, F)$, and so, with # denoting the cardinality,

$$\#G = |\operatorname{ind}(M, g, G)| + 2k = |\operatorname{ind}(M, f, G)| + 2k$$

for some integer $k \geq 0$. Hence we have, for some integer $n \geq 0$,

$$\begin{aligned}\#\operatorname{Fix} g &= \sum(\#G : G \text{ is a fixed point class of } g)\\ &= \sum(|\operatorname{ind}(M, f, F)| : F \text{ is a fixed point class of } 2) + 2n\\ &= N_{\pitchfork}(f) + 2n.\end{aligned}$$

(ii) *Sufficiency.* It follows from Theorem 2.7 that there exists a sparse map $f' : M \to M$ homotopic to f. If $\#K = N_{\pitchfork}(f)$, we are done. If $\#K = N(f) + 2n$ with $n > 0$ and $N_{\pitchfork}(f) \neq 0$, the existence of g follows from Lemma 2.12(i) and (v). Finally, if $n > 0$ but $N_{\pitchfork}(f) = 0$, i.e. if f' is fixed point free, then there exists a continuous map $f_1 : M \to M$ which is continuously homotopic to f and has one fixed point x_0 [11, Theorem 3.2]. It follows from the Theorem in the Appendix that f_1 is homotopic to a smooth map $f_2 : M \to M$ with x_0 as its fixed point set, and as $N(f_2) = N_{\pitchfork}(f) = 0$, we have $\operatorname{ind}(M, f_2, x_0) = 0$. Thus we can use Lemma 2.12 to obtain a (smooth) map f_3 homotopic to f_2 which has the fixed point set $\{x_0, x_1, \ldots, x_n, y_1, \ldots, y_n\}$, where x_0 is of index zero and the other points are transversal fixed points. If we remove x_0 as in [5] (or by using [2, Lemma 7.2]) we obtain a (smooth) map g which has $2n$ fixed points and is continuously homotopic to the (smooth) map f, and hence also smoothly homotopic [8, §3 and §4]. So Theorem 2.13 holds in this case as well.

3. The boundary transversal Nielsen number. We now consider a manifold M with boundary ∂M and a map $f : (M, \partial M) \to (M, \partial M)$. We write $\bar{f} = f|\partial M : \partial M \to \partial M$ and say that f is *boundary transversally fixed* if $\bar{f} : \partial M \to \partial M$ is transversally fixed. If $\bar{f}$ is kept fixed, then this is the setting studied in [2, §6 and §7], but we are interested here in the minimum number of fixed points on M if f is allowed to vary under a homotopy. By a *homotopy* of a map of pairs $f : (M, \partial M) \to (M, \partial M)$ we mean in this paper always a homotopy of pairs, i.e. a map of the form $H : (M \times I,\ \partial M \times I) \to (M, \partial M)$, and so we want to determine the least number of fixed points on M for all maps $g : (M, \partial M) \to (M, \partial M)$ which are related to a given map $f : (M, \partial M) \to (M, \partial M)$ by a homotopy H

and which are boundary transversally fixed. Note that we do not ask that $H(\cdot,t)$ is a boundary transversally fixed map for all $0 \le t \le 1$, but only that $H(\cdot,1) = g$ is. If $H(\cdot,t)$ is boundary transversally fixed for all $0 \le t \le 1$, then the fixed point set on ∂M can only vary its location but the number of fixed points and their indices remain constant, and so this assumption leads to a setting which is practically identical to that of [2, §6 and §7]. The definitions which we shall introduce are motivated by the next theorem from [2].

THEOREM 3.1 (Index Theorem [2, Theorem 5.1]). *Let $p \in \partial M$ be an isolated fixed point of a map $f : (M,\partial M) \to (M,\partial M)$ so that $d\bar{f}_p - \mathrm{Id} : T_p(\partial M) \to T_p(\partial M)$ is a non-singular linear operator. Then either* $\mathrm{ind}(M,f,p) = 0$ *or* $\mathrm{ind}(M,f,p) = \mathrm{ind}(\partial M,\bar{f},p)$.

We write $i(\overline{F}) = \mathrm{ind}(\partial M,\bar{f},\overline{F})$ and $i(F) = \mathrm{ind}(M,f,F)$ for the index of a fixed point class $\overline{F}$ of $\bar{f} : \partial M \to \partial M$ and F of $f : M \to M$. Recall that a fixed point class F of $f : M \to M$ is called a *common* fixed point class of f and $\bar{f}$ if there exists an essential fixed point class $\overline{F}$ of $\bar{f} : \partial M \to \partial M$ which is contained in F [9, Definition 2.1 and Lemma 2.2].

DEFINITION 3.2. Let $f : (M,\partial M) \to (M,\partial M)$ be a map and F a fixed point class of $f : M \to M$.

(i) $u(F) = \max\{0, \sum(\bar{i}(\overline{F}) : \overline{F} \subset F \text{ and } \bar{i}(\overline{F}) > 0)\}$,
(ii) $l(F) = \min\{0, \sum(\bar{i}(\overline{F}) : \overline{F} \subset F \text{ and } \bar{i}(\overline{F}) < 0)\}$,
(iii) F is a *transversally common* fixed point class of f and $\bar{f}$ if $l(F) \le i(F) \le u(F)$.

Clearly $l(F)$ and $u(F)$, which serve as lower and upper bounds of $i(F)$ if F is transversally common, are always integers with $l(F) \le 0 \le u(F)$. By definition a transversally common essential fixed point class of f and $\bar{f}$ is common, as an essential fixed point class F is common if and only if not both $l(F)$ and $u(F)$ are zero. If $N(f,\bar{f}_{\pitchfork})$ denotes the number of essential and transversally common fixed point classes of f and $\bar{f}$, and $N(f,\bar{f})$ as in [9, §2] the number of essential common fixed point classes of f and $\bar{f}$, then

$$0 \le N(f,\bar{f}_{\pitchfork}) \le N(f,\bar{f}) \le N(f)\,.$$

We now introduce the Nielsen type number which will characterize minimal fixed point sets of boundary transversally fixed maps.

DEFINITION 3.3. The *boundary transversal Nielsen number* of $f : (M,\partial M) \to (M,\partial M)$ is

$$N(f;M,\partial M_{\pitchfork}) = N_{\pitchfork}(\bar{f}) + N(f) - N(f,\bar{f}_{\pitchfork})\,.$$

The definition is obviously of a similar structure as that of the relative Nielsen number in [9, §2], and the next results are easy consequences of this definition.

PROPOSITION 3.4.

(i) $\qquad N(f;M,\partial M_{\pitchfork}) \geq N(f;M,\partial M) \geq 0\,,$

(ii) $\qquad N(f;M,\partial M_{\pitchfork}) = N_{\pitchfork}(\bar f) + N(f) \qquad$ *if* $N(f,\bar f)=0,$

(iii) $\qquad N(f;M,\partial M_{\pitchfork}) = N(f) \qquad$ *if* $N(\bar f)=0\,,$

(iv) $\qquad N(f;M,\partial M_{\pitchfork}) = N_{\pitchfork}(\bar f) \qquad$ *if* $N(f)=0.$

We shall look at the relations between $N(f;M,\partial M_{\pitchfork})$ and the extension Nielsen numbers in the next section. Here we establish, in Propositions 3.6, 3.8, 3.9 and 3.10, that the boundary transversal Nielsen number has the usual basic properties of Nielsen type numbers. Some of the proofs will depend on the next lemma.

LEMMA 3.5. *Let $f:(M,\partial M)\to(M,\partial M)$ be boundary transversally fixed. If F is an essential fixed point class of $f: M\to M$ which is not transversally common and if $F\subset\partial M$, then*

$$\#F \geq \sum(|\bar i(\overline F)| : \overline F\subset F) + 2\,.$$

Proof. As f is boundary transversally fixed, $F\subset\partial M$ is finite, and as F is not transversally common, either $0\leq u(F) < i(F)$ or $i(F) < l(F) \leq 0$. In the first case $F = F\cap\partial M$ must contain at least $i(F)\geq u(F)+k$, for some $k>0$, fixed points x with $\operatorname{ind}(M,f,x)=+1$, and hence it follows from Theorem 3.1 that F also contains at least $i(F)\geq u(F)+k$ fixed points with $\operatorname{ind}(\partial M,\ \bar f,x)=+1$. Now $i(F)=u(F)+l(F)=u(F)-|l(F)|$, and so F must contain at least $|l(F)|+k$ fixed points x with $\operatorname{ind}(M,f,x)=-1$. Therefore F contains at least

$$u(F)+|l(F)|+2k \geq \sum(|\bar i(\overline F)| : \overline F\subset F)+2$$

fixed points in all. The case $i(F)<l(F)$ is similar, as $F=F\cap\partial M$ must then contain $\geq |i(F)| \geq |l(F)|+k$, for some $k>0$, fixed points x with $\operatorname{ind}(M,f,x)=-1$.

PROPOSITION 3.6 (Lower bound). *A boundary transversally fixed map $f:(M,\partial M)\to(M,\partial M)$ has at least $N(f;M,\partial M_{\pitchfork})$ fixed points on M.*

Proof. As $\bar f:\partial M\to\partial M$ is transversally fixed, f has at least $N_{\pitchfork}(\bar f)$ fixed points on ∂M. Now let F be an essential fixed point class of $f: M\to M$ which is not transversally common. Then Lemma 3.5 shows that either $F\cap\operatorname{Int} M\neq 0$ or $F\cap\partial M$ contains $\geq\sum(|\bar i(\overline F)| : \overline F\subset F)+2$ fixed points, and so F contributes at least either a fixed point on $\operatorname{Int} M$ or 2 fixed points on ∂M beyond the minimal fixed point set of $\bar f$. Hence Proposition 3.6 follows from the definition of $N(f;M,\partial M_{\pitchfork})$.

The next lemma will not only prove the homotopy invariance of the boundary transversal Nielsen number, but also of the relative transversal

Nielsen number which will be defined in §6. The proof is related to the proof of [10, Theorem 4.1].

LEMMA 3.7. *If* $H : (M \times I,\ \partial M \times I) \to (M, \partial M)$ *is a homotopy between the maps* $f, g : (M, \partial M) \to (M, \partial M)$ *and if the essential fixed point class* F *of* $f : M \to M$ *is* H*-related to the essential fixed point class* G *of* $g : M \to M$, *then* $u(F) = u(G)$ *and* $l(F) = l(G)$.

Proof. Let

$$F \cap \partial M = \overline{F}_1 \cup \ldots \cup \overline{F}_r \cup \overline{F}_{r+1} \cup \ldots \cup \overline{F}_s \cup \overline{F}' ,$$

where $\overline{F}_j$, $j = 1, \ldots, r$, is an essential fixed point class of $\bar{f}\partial M \to \partial M$ with $\bar{i}(\overline{F}_j) > 0$, and for $j = r+1, \ldots, s$, an essential fixed point class of $\bar{f}$ with $\bar{i}(\overline{F}_j) < 0$, and where $\overline{F}'$ is the union of inessential fixed point classes of $\bar{f}$. (Not all sets need occur.) As H restricts to a homotopy $\overline{H} : \partial M \times I \to \partial M$ from $\bar{f}$ to $\bar{g}$, we have

$$G \cap \partial M = \overline{G}_1 \cup \ldots \cup \overline{G}_r \cup \overline{G}_{r+1} \cup \ldots \cup \overline{G}_s \cup \overline{G}' ,$$

where $\overline{G}_j$ is the essential fixed point class of $\bar{g}$ which is $\overline{H}$-related to $\overline{F}_j$ and $\overline{G}'$ the union of inessential fixed point classes of $\bar{g}$. From $\bar{i}(\overline{F}_j) = \bar{i}(\overline{G}_j)$ for $j = 1, \ldots, s$ we see that Lemma 3.7 is true.

PROPOSITION 3.8 (Homotopy invariance). *If* $f, g : (M, \partial M) \to (M, \partial M)$ *are homotopic*, *then* $N(f; M, \partial M_{\pitchfork}) = N(g; M, \partial M_{\pitchfork})$.

Proof. This follows immediately from Definitions 3.2 and 3.3, Proposition 2.3 and Lemma 3.7.

We omit the proofs of the next two results, as they can easily be obtained in the standard way. (See e.g. [6, Ch. I, Theorems 5.2 and 5.4, pp. 20–21].) Homotopy type for pairs of spaces is defined by using homotopies of maps of pairs, i.e. as it was done in the continuous case in [9, p. 465].

PROPOSITION 3.9 (Commutativity). *If* $f : (M, \partial M) \to (N, \partial N)$ *and* $g : (N, \partial N) \to (M, \partial M)$ *are maps between manifolds with boundary* M *and* N, *then* $N(g \circ f; M, \partial M_{\pitchfork}) = N(f \circ g; N, \partial N_{\pitchfork})$.

PROPOSITION 3.10 (Homotopy type invariance). *If* $f : (M, \partial M) \to (M, \partial M)$ *and* $g : (N, \partial N) \to (N, \partial N)$ *are maps of the same homotopy type*, *then* $N(f; M, \partial M_{\pitchfork}) = N(g; N, \partial N_{\pitchfork})$.

We will show in §5 that the boundary transversal Nielsen number is usually an optimal lower bound for the number of fixed points on M of boundary transversally fixed maps. Here we illustrate Definitions 3.2 and 3.3 with two examples.

EXAMPLE 3.11. Let M be the unit ball B^n with its bounding $(n-1)$-sphere S^{n-1}, where $n \geq 2$, and let $f : (B^n, S^{n-1}) \to (B^n, S^{n-1})$ be a map so that $\bar{f} : S^{n-1} \to S^{n-1}$ is of degree d. Then $f : B^n \to B^n$ has one fixed point class F with $i(F) = 1$, and so $l(F) \le 0 < i(F)$. If $n = 2$, then $\bar{f}$ has $|1-d|$ essential fixed point classes, each of the same index, and $\sum \bar{i}(\overline{F}) = L(\bar{f}) = 1 - d$. Hence $i(F) \le u(F)$ if and only if $d \le 0$, and

$$N(f, \bar{f}_{\pitchfork}) = \begin{cases} 1 & \text{if } d \le 0, \\ 0 & \text{if } d > 0. \end{cases}$$

If $n \ge 3$, then $\bar{f}$ has one fixed point class $\overline{F}$ with $\bar{i}(\overline{F}) = 1 + (-1)^{n-1} d$, and so

$$N(f, \bar{f}_{\pitchfork}) = \begin{cases} 1 & \text{if } (-1)^{n-1} d \ge 0, \\ 0 & \text{if } (-1)^{n-1} d < 0, \end{cases}$$

a formula which still holds for $n = 2$. As $N(f) = 1$ and $N_{\pitchfork}(\bar{f}) = |1-(-1)^n d|$ for all d and all $n \ge 2$, we obtain

$$N(f; B^n, \partial B^n_{\pitchfork}) = \begin{cases} |1-(-1)^n d| & \text{if } (-1)^n d \le 0, \\ |1-(-1)^n d| + 1 & \text{if } (-1)^n d > 0, \end{cases}$$

i.e. we have for all $n > 2$

$$N(f; B^n, \partial B^n_{\pitchfork}) = \begin{cases} 1 + |d| & \text{if } (-1)^n d \le 0, \\ |d| & \text{if } (-1)^n d > 0. \end{cases}$$

Example 3.12. Let $f : (M, \partial M) \to (M, \partial M)$ be a deformation of an n-dimensional manifold M with boundary ∂M. First assume that n is even. Then $N(\bar{f}) = 0$, and so Proposition 3.4(iii) gives

$$N(\mathrm{id}; M, \partial M_{\pitchfork}) = \begin{cases} 1 & \text{if } \chi(M) \ne 0, \\ 0 & \text{if } \chi(M) = 0 \end{cases} \qquad (n \text{ is even}).$$

Now let n be odd. As $\chi(M) = \chi(\partial M) + \chi(M, \partial M)$, Lefschetz duality implies $\chi(\partial M) = 2\chi(M)$. If $\chi(M) \ne 0$, then $f : M \to M$ has one fixed point class F with $i(F) = \chi(M)$. For $\chi(M) > 0$ we see from Definition 3.2 that $u(F) \ge \chi(\partial M)$, and for $\chi(M) < 0$ that $l(F) \le \chi(\partial M)$, so in either case F is transversally common. Thus $N(f) = N(f, \bar{f}_{\pitchfork}) = 1$. If $\chi(M) = 0$, then $N(f) = N(f, \bar{f}_{\pitchfork}) = N_{\pitchfork}(\bar{f}) = 0$. Thus Definition 3.3 gives

$$N(\mathrm{id};\, M, \partial M_{\pitchfork}) = N_{\pitchfork}(\overline{\mathrm{id}}) = \sum |\chi(\partial M_j)| \qquad (n \text{ is odd}),$$

where the summation is taken over the components ∂M_j of ∂M.

4. Fixed points of extensions of sparse maps. We now relate our results to those of [2]. So let $f : (M, \partial M) \to (M, \partial M)$ be a map of a manifold M with boundary ∂M. We studied in [2] the least number of fixed points on the interior of M for all maps in the homotopy class of f if $\bar{f} : \partial M \to \partial M$ is kept fixed during the homotopy or, in other words, we studied the least number of fixed points on $\operatorname{Int} M$ of extensions of a given map $\bar{f} : \partial M \to \partial M$ to a map $f : (M, \partial M) \to (M, \partial M)$ in a given homotopy class. A lower bound for the number of fixed points on $\operatorname{Int} M$ of continuous extensions of a continuous map $\bar{f}$ is the *extension Nielsen number* $N(f|\bar{f})$ which (if $\bar{f}$ is a map on the boundary of M) equals the number of essential fixed point classes F of $f : M \to M$ with $F \cap \partial M = \emptyset$ [2, §2 and §6]. A lower bound for the number of fixed points in $\operatorname{Int} M$ of smooth extensions of a smooth and transversally fixed map $\bar{f}$ is the *smooth extension number* $N^1(f|\bar{f})$ which equals the number of essential fixed point classes F of $f : M \to M$ which are not representable on ∂M, where F is called *representable on* ∂M if there exists a subset $F' \subset F \cap \partial M$ with $\operatorname{ind}(M, f, F) = \operatorname{ind}(\partial M, \bar{f}, F')$ [2, §6]. Both $N(f|\bar{f})$ and $N^1(f|\bar{f})$ are sharp lower bounds (i.e. they can be realized by continuous resp. smooth extensions) if $\dim M \geq 3$. As outlined in the introduction, there exists smooth and transversally fixed maps $\bar{f}$ on the boundary of the disk D so that for any extension f we have $N(f|\bar{f}) = 0$ but $N^1(f|\bar{f}) = 1$, and again these Nielsen numbers are sharp. But $\bar{f}$ can be homotoped to a transversally fixed map $\bar{g}$ which has smooth extensions over D with no fixed points on the interior, and so $N(g|\bar{g}) = N^1(g|\bar{g})$. We will show in Theorem 4.2 that this behaviour is typical. The following lemma, which is well known in the continuous case, will be used in the proof of this theorem, and again in the proofs of Theorems 5.1, 5.3 and 7.2. Recall that we assume that a map is a smooth function.

LEMMA 4.1. *If $f : (M, \partial M) \to (M, \partial M)$ is a map of a manifold M with boundary ∂M and if $\bar{g} : \partial M \to \partial M$ is homotopic to $\bar{f} : \partial M \to \partial M$, then there exists a map $g : (M, \partial M) \to (M, \partial M)$ which extends $\bar{g}$ and is homotopic to f.*

Proof. It follows from the homotopy extension property that $\bar{g}$ extends to a continuous selfmap of M which is homotopic to f, and hence from [2, Remark at the beginning of §6], that $\bar{g}$ has a smooth extension g homotopic to f.

THEOREM 4.2. *Every map $f : (M, \partial M) \to (M, \partial M)$ is homotopic to a boundary transversally fixed map $g : (M, \partial M) \to (M, \partial M)$ with $N(g|\bar{g}) = N^1(g|\bar{g})$.*

Proof. Let $\bar{f}' : \partial M \to \partial M$ be a transversally fixed map which is homotopic to $\bar{f} : \partial M \to \partial M$. Lemma 4.1 shows that $\bar{f}'$ has a smooth extension $f' : (M, \partial M) \to (M, \partial M)$ which is homotopic to $f : (M, \partial M) \to (M, \partial M)$. If $N(f'|\bar{f}') = N^1(f'|\bar{f}')$, we are done. Otherwise there exist $N^1(f'|\bar{f}') - N(f'|\bar{f}')$ essential fixed point classes of $f' : M \to M$ which intersect ∂M but are not representable on ∂M. Let F be one of them. We pick $q \in F \cap \partial M$ and use Lemma 2.12 to change $\bar{f}'$ near q to obtain a transversally fixed map $\bar{f}'' : \partial M \to \partial M$ which agrees with f' outside a small neighbourhood V of q in ∂M and has on V the fixed point q and $2|i(F)|$

further fixed points so that $|i(F)|$ have index $+1$ and $|i(F)|$ have index -1 with respect to the map $\bar{f}'' : \partial M \to \partial M$. Thus F is now representable for a smooth extension of $\bar{f}''$ which is in the homotopy class of $f : (M, \partial M) \to (M, \partial M)$ and equals f' outside a small neighbourhood of q in M. After dealing in this manner with all such essential fixed point classes of $f' : M \to M$ which intersect ∂M but are not representable on ∂M we obtain a transversally fixed map $\bar{g} : \partial M \to \partial M$ which extends to a smooth map $g : M \to M$ in the homotopy class of f so that $N(g|\bar{g}) = N^1(g|\bar{g})$.

The map $\bar{g}$ in Theorem 4.2 is still transversally fixed, but its fixed point set has been increased. In particular, $\bar{g}$ is usually not a map with a minimal transversal fixed point set, i.e. it is usually not sparse (Definition 2.9). Our next aim is to calculate, in Corollary 4.4, the smooth extension Nielsen number of sparse maps with the help of the Nielsen numbers introduced in §2 and §3. First we clarify the relation between the particular kinds of fixed point classes which enter the definitions.

Proposition 4.3. *Let $f : (M, \partial M) \to (M, \partial M)$ be boundary transversally fixed and let F be an essential fixed point class of $f : M \to M$.*

(i) *If F is transversally common, then F is representable on ∂M.*
(ii) *If $\bar{f} : \partial M \to \partial M$ is sparse, then F is transversally common if and only if it is representable on ∂M.*

Proof. (i) If F is transversally common and $\bar{f} : \partial M \to \partial M$ transversally fixed, then Definition 3.2 shows that $F \cap \partial M$ must contain $\geq u(F)$ fixed points x with $\operatorname{ind}(\partial M, \bar{f}, x) = +1$ and $\geq |l(F)|$ fixed points x with $\operatorname{ind}(\partial M, \bar{f}, x) = -1$. So $l(F) \leq i(F) \leq u(F)$ implies that F is representable on ∂M.

(ii) If $\bar{f} : \partial M \to \partial M$ is sparse, then $F \cap \partial M$ consists of precisely $u(F)$ fixed points x of $\operatorname{ind}(\partial M, \bar{f}, x) = +1$ and precisely $|l(F)|$ fixed points x of $\operatorname{ind}(\partial M, \bar{f}, x) = -1$. Hence F is representable on ∂M if and only if $l(F) \leq i(F) \leq u(F)$.

From Proposition 4.3 and the definitions of the Nielsen numbers involved we obtain

Corollary 4.4. *Let $f : (M, \partial M) \to (M, \partial M)$ be boundary transversally fixed.*

(i) $N^1(f|\bar{f}) \leq N(f; M, \partial M_{\pitchfork}) - N_{\pitchfork}(\bar{f})$,
(ii) *if $\bar{f}$ is sparse, then* $N^1(f|\bar{f}) = N(f; M, \partial M_{\pitchfork}) - N_{\pitchfork}(\bar{f})$.

A result for $N(f|\bar{f})$ which parallels Corollary 4.4 can be obtained from [2, Theorem 2.4 and Corollary 2.6].

Proposition 4.5. *Let $f : (M, \partial M) \to (M, \partial M)$ be boundary transversally fixed.*

(i) $N(f|\bar f) \le N(f; M, \partial M) - N(\bar f)$,

(ii) *if all fixed point classes of* $\bar f : \partial M \to \partial M$ *are essential*, *then* $N(f|\bar f) = N(f; M, \partial M) - N(\bar f)$.

Note that a sparse map $\bar f$ satisfies the condition in Proposition 4.5(ii), but that Corollary 4.4(ii) need not hold if $\bar f$ is a transversally fixed map with only essential fixed point classes which is not sparse. (This can be seen by using Theorem 4.2 to get a boundary transversally fixed map $f : (B^3, \partial B^3) \to (B^3, \partial B^3)$ so that $\bar f : \partial B^3 \to \partial B^3$ has degree -2 and $N^1(f|\bar f) = 0$.)

From Corollary 4.4 and Proposition 4.5 we obtain a precise formula for the number of additional fixed points which must occur in smooth and not only continuous extensions of sparse maps.

THEOREM 4.6. *If* $\bar f : \partial M \to \partial M$ *is sparse*, *then*

$$N^1(f|\bar f) - N(f|\bar f) = N(f, \bar f) - N(f, \bar f_{\pitchfork}),$$

i.e. it equals the number of essential fixed point classes of $f : M \to M$ *which are common but not transversally common.*

Proof. If $\bar f$ is sparse, then all its fixed point classes are essential, and so the theorem follows immediately from Corollary 4.4(ii), Proposition 4.5(ii) and the definitions of the relative and the boundary transversal Nielsen number.

In the next two examples we use Theorem 4.6 to calculate $N^1(f|\bar f) - N(f|\bar f)$. The first example shows that the fixed point behaviour observed on the disk in [2], where $N^1(f|\bar f) - N(f|\bar f) = 1$, occurs again on all higher-dimensional balls, and the second one shows that the difference $N^1(f|\bar f) - N(f|\bar f)$ can be arbitrarily large.

EXAMPLE 4.7. Let $n \ge 2$ and let $f : (B^n, \partial B^n) \to (B^n, \partial B^n)$ be a map so that $\bar f : \partial B^n \to \partial B^n$ is a sparse map of degree d. Then, by definition,

$$N(f|\bar f) = \begin{cases} 1 & \text{if } d \ne (-1)^n, \\ 0 & \text{if } d = (-1)^n, \end{cases}$$

and so Example 3.11 gives

$$N^1(f|\bar f) - N(f|\bar f) = \begin{cases} 0 & \text{if } (-1)^n d \le 1, \\ 1 & \text{if } (-1)^n d \ge 2. \end{cases}$$

We see that a sparse map of S^{n-1} of degree d with $(-1)^n d \ge 2$ has no smooth extension over B^n without a fixed point on $\operatorname{Int} B^n$. But continuous extensions without a fixed point on $\operatorname{Int} B^n$ exist [2, Theorem 4.6]. Note that a sparse map of S^{n-1} of a given degree has been constructed in Example 2.10.

EXAMPLE 4.8. Let $M = S^1 \times B^2$ be the solid torus and let $f : (M, \partial M) \to$

$(M, \partial M)$ be the map given by

$$f(e^{i\phi}, re^{i\theta}) = (e^{id\phi}, r^2 e^{2i\theta}),$$

where $0 \le \phi < 2\pi,\ 0 \le \theta < 2\pi,\ 0 \le r \le 1$ and $d \ge 2$. Then $\bar{f} : S^1 \times S^1 \to S^1 \times S^1$ is a product map with $d-1$ fixed points and $L(\bar{f}) = (1-d)(1-2) = d-1 > 0$. As $N_{\pitchfork}(\bar{f}) = d-1$ by Proposition 2.6, the map $\bar{f}$ is sparse. Clearly $N(f, \bar{f}) = 0$. But $L(f) = 1 - d < 0$, and so none of the $d-1$ essential fixed point classes of $f : S^1 \times B^2 \to S^1 \times B^2$ are transversally common. Thus we see that

$$N^1(f|\bar{f}) - N(f|\bar{f}) = N(f, \bar{f}) - N(f, \bar{f}_{\pitchfork}) = d - 1$$

can take any value in $\mathbb{Z}_+$. The existence of continuous (smooth) extensions of $\bar{f}$ with $N(f|\bar{f})$ $(N^1(f|\bar{f}))$ fixed points on $\operatorname{Int} M$ follows from [2, Theorems 4.2 and 7.5].

If $\bar{f}$ is not sparse, then precise results for $N^1(f|\bar{f}) - N(f|\bar{f})$ are hard to formulate, but we can obtain from Corollary 4.4(i), Proposition 4.5(i) and [2, Theorem 6.1]

Theorem 4.9. *If all fixed point classes of $\bar{f} : \partial M \to \partial M$ are essential, then*

$$0 \le N^1(f|\bar{f}) - N(f|\bar{f}) \le N(f, \bar{f}) - N(f, \bar{f}_{\pitchfork}).$$

The next example shows that the assumption that all fixed point classes of $\bar{f}$ are essential cannot be omitted in Theorem 4.9. It also shows that there exist boundary transversal deformations with $N^1(f|\bar{f}) \ne N(f|\bar{f})$ on many even-dimensional manifolds.

Example 4.10. Let M be any compact connected even-dimensional manifold with boundary such that $|\chi(M)| \ge 2$ and $f : (M, \partial M) \to (M, \partial M)$ a deformation with $\operatorname{Fix} f = \{x_0, x_1\}$, where x_0 and x_1 lie in the same boundary component, $\operatorname{ind}(\partial M, \bar{f}, x_0) = +1$, $\operatorname{ind}(\partial M, \bar{f}, x_1) = -1$ and $\bar{f}$ is transversally fixed. Then $N(f, \bar{f}) - N(f, \bar{f}_{\pitchfork}) = 0$ as there exists no common essential fixed point class of f and $\bar{f}$. But $N^1(f|\bar{f}) - N(f|\bar{f}) = 1 - 0 = 1$, as $f : M \to M$ has one fixed point class F with $i(F) = \chi(M)$, and so F is not representable on ∂M although $F \cap \partial M \ne \emptyset$. Note that the idea of this example can be used to construct a map with $N^1(f|\bar{f}) \ne N(f|\bar{f})$ on any manifold with boundary which admits a map with an essential fixed point class F so that $|i(F)| \ge 2$, F is not common, but F is weakly common in the sense of X. Zhao [12].

The map $\bar{f}$ in Example 4.10 is not sparse. If f is a boundary transversal deformation with $N^1(f|\bar{f}) \ne N(f|\bar{f})$, then this is necessary, for we have

Theorem 4.11. *Let $f : (M, \partial M) \to (M, \partial M)$ be a boundary transversally fixed deformation of an n-dimensional manifold with boundary ∂M. If n is odd, or if n is even and $\bar{f}$ is sparse, then $N^1(f|\bar{f}) = N(f|\bar{f})$.*

Proof. If n is odd, then $N(\mathrm{id};\, M, \partial M_{\pitchfork}) = N_{\pitchfork}(\overline{\mathrm{id}})$ by Example 3.12, and so Corollary 4.4(i) and $0 \le N(f|\bar f) \le N^1(f|\bar f)$ imply $N^1(f|\bar f) = N(f|\bar f) = 0$. If n is even and $\bar f$ is sparse, then $\mathrm{Fix}\,\bar f = \emptyset$, and so $N^1(f|\bar f) = N(f|\bar f)$ follows from the definitions.

5. The minimum theorem for the boundary transversal Nielsen number. We now want to show that $N(f; M, \partial M_{\pitchfork})$ is an optimal lower bound for the number of fixed points on M for boundary transversally fixed maps in the homotopy class of $f : (M, \partial M) \to (M, \partial M)$ if the dimension of M is sufficiently large. This can easily be done with the help of the Minimum Theorem for $N^1(f|\bar f)$ from [2, Theorem 7.5] if we use Corollary 4.4(ii).

THEOREM 5.1 (Minimum Theorem for $N(f; M, \partial M_{\pitchfork})$). *Let* $f : (M, \partial M) \to (M, \partial M)$ *be a map so that either* $\dim M \ge 4$, *or so that* $\dim M = 3$ *and* $\bar f : \partial M \to \partial M$ *is homotopic to a sparse map. Then f is homotopic to a map* $g : (M, \partial M) \to (M, \partial M)$ *which is boundary transversally fixed and has* $N(f; M, \partial M_{\pitchfork})$ *fixed points.*

Proof. By assumption $\bar f : \partial M \to \partial M$ is homotopic to a sparse map $\bar f_1 : \partial M \to \partial M$, and by Lemma 4.1 we can extend $\bar f_1$ to a map $f_1 : (M, \partial M) \to (M, \partial M)$ which is homotopic to $f : (M, \partial M) \to (M, \partial M)$. Corollary 4.4(ii) and Propositions 2.3 and 3.8 show that

$$N^1(f_1|\bar f_1) = N(f_1; M, \partial M_{\pitchfork}) - N_{\pitchfork}(\bar f_1) = N(f; M, \partial M_{\pitchfork}) - N_{\pitchfork}(\bar f)\,,$$

and so [2, Theorem 7.5] implies that $\bar f_1 : \partial M \to \partial M$ extends to a map $g : (M, \partial M) \to (M, \partial M)$ which is homotopic to $f : (M, \partial M) \to (M, \partial M)$ and has $N^1(f_1|\bar f_1)$ fixed points on $\mathrm{Int}\, M$. As $\bar g = \bar f_1$ the map g is boundary transversally fixed, and as $\bar g$ is sparse, g has $N^1(f_1|\bar f_1) + N_{\pitchfork}(\bar f) = N(f; M, \partial M_{\pitchfork})$ fixed points on M.

The proof constructs the map g so that there are $N_{\pitchfork}(\bar f)$ fixed points on ∂M and $N(f; M, \partial M_{\pitchfork}) - N_{\pitchfork}(\bar f)$ fixed points on $\mathrm{Int}\, M$. Lemma 3.5 shows that this distribution of the fixed points is necessary:

THEOREM 5.2 (Location). *If* $f : (M, \partial M) \to (M, \partial M)$ *is boundary transversally fixed and has* $N(f; M, \partial M_{\pitchfork})$ *fixed points, then f has* $N_{\pitchfork}(\bar f)$ *fixed points on* ∂M *and* $N(f) - N(f, \bar f_{\pitchfork}) = N^1(f|\bar f)$ *fixed points on* $\mathrm{Int}\, M$.

(The last equation follows from Corollary 4.4(ii).) As usual we can weaken the assumptions on the dimension of M if we deal with deformations.

THEOREM 5.3. *If M is a manifold with boundary, then there exists a deformation* $f : (M, \partial M) \to (M, \partial M)$ *which is boundary transversally fixed and has* $N(\mathrm{id}; M, \partial M_{\pitchfork})$ *fixed points on M.*

Proof. If $\dim M \ge 3$, then Theorem 5.3 follows from Theorems 5.1 and 2.8. If $\dim M = 2$, then ∂M is the disjoint union of circles, and so there exists a fixed point free smooth deformation $\bar{f} : \partial M \to \partial M$ arbitrarily close to the identity. We can use Lemma 4.1 and the construction of g in its proof to extend $\bar{f}$ to a deformation $f' : (M, \partial M) \to (M, \partial M)$ which is arbitrarily close to the identity, and we can further assume, by [2, Lemma 7.1 and its proof] and by the argument at the beginning of the proof of [2, Theorem 7.5], that f' is transversally fixed on M. If f' has more than one fixed point on $\operatorname{Int} M$, we can unite these fixed points as in the proof of Theorem 2.8 until we obtain a deformation of M which extends $\bar{f}$ and has on $\operatorname{Int} M$ only one fixed point p with $\operatorname{ind}(M, f', p) = \operatorname{ind}(M, f', M) = \chi(M)$. After cancelling this fixed point if $\chi(M) = 0$ as in [2, Lemma 7.1] (see also [5]) we obtain a deformation of $(M, \partial M)$ which has $N(\mathrm{id})$ fixed points, and Proposition 3.4 (or Example 3.12) shows that $N(\mathrm{id}; M, \partial M_{\pitchfork}) = N(\mathrm{id})$. If $\dim M = 1$, i.e. if M is a closed interval, then f can easily be constructed directly.

6. The relative transversal Nielsen number. Finally, we study maps $f : (M, \partial M) \to (M, \partial M)$ which are transversally fixed on all of M and not only on ∂M, where we call a map $f : (M, \partial M) \to (M, \partial M)$ *transversally fixed* if $df_p - \mathrm{Id} : T_p(M) \to T_p(M)$ is a non-singular linear operator on the tangent space $T_p(M)$ at each fixed point p of $f : M \to M$. Note that this implies that if $p \in \partial M$ is a fixed point of $\bar{f} : \partial M \to \partial M$, then $d\bar{f}_p - \overline{\mathrm{Id}} : T_p(\partial M) \to T_p(\partial M)$ is non-singular, and so a transversally fixed map $f : (M, \partial M) \to (M, \partial M)$ is boundary transversally fixed.

To obtain a lower bound for the number of fixed points for a transversally fixed map of $(M, \partial M)$ we shall use

DEFINITION 6.1. Given a map $f : (M, \partial M) \to (M, \partial M)$ and an essential fixed point class F of $f : M \to M$, the *boundary index* of F is

$$b(F) = \begin{cases} \min\{i(F), u(F)\} & \text{if } i(F) > 0, \\ \max\{i(F), l(F)\} & \text{if } i(F) < 0. \end{cases}$$

The reason for the name is contained in Theorem 7.3 below, which shows that if $f : (M, \partial M) \to (M, \partial M)$ is a transversally fixed map with a minimal fixed point set, then $\operatorname{ind}(M, f, F \cap \partial M) = b(F)$. Clearly $\operatorname{sgn} i(F) = \operatorname{sgn} b(F)$ and $0 \le |b(F)| \le |i(F)|$. A straightforward consequence of the definition is

PROPOSITION 6.2. *An essential fixed point class is transversally common if and only if* $i(F) = b(F)$.

If we put

$$N(f_{\pitchfork}, \bar{f}) = \sum (|b(F)| : F \text{ is an essential fixed point class of } f : M \to M),$$

then $0 \le N(f_{\pitchfork}, \bar{f}) \le N_{\pitchfork}(f)$.

DEFINITION 6.3. The *relative transversal Nielsen number* of $f : (M, \partial M)$

$\to (M, \partial M)$ is

$$N(f; M_{\pitchfork}, \partial M) = N_{\pitchfork}(\bar{f}) + N_{\pitchfork}(f) - N(f_{\pitchfork}, \bar{f})\,.$$

We see immediately from the definition that

$$N(f; M_{\pitchfork}, \partial M) = N_{\pitchfork}(\bar{f}) + \sum(|i(F) - b(F)| : \text{F is a fixed point class which is not transversally common}).$$

In order to show that $N(f; M_{\pitchfork}, \partial M)$ is a lower bound for the number of fixed points of a transversally fixed map of $(M, \partial M)$ we use a lemma which is in the same spirit as Lemma 3.5.

LEMMA 6.4. *Let $f : (M, \partial M) \to (M, \partial M)$ be transversally fixed. If F is an essential fixed point class of $f : M \to M$ and if $\#(F \cap \mathrm{Int} M) < |i(F) - b(F)|$, then*

$$\#(F \cap \partial M) \geq \sum(|\bar{i}(\overline{F})| : \overline{F} \subset F) + 2\,.$$

Proof. We first consider the case $i(F) > 0$, where F has at least $i(F)$ fixed points x with $\mathrm{ind}(M, f, x) = +1$. If

$$\#(F \cap \mathrm{Int} M) < |i(F) - b(F)| = i(F) - b(F)\,,$$

then $i(F) > b(F) \geq 0$, and so $b(F) = u(F)$. As $F \cap \partial M$ must contain $\geq b(F) + 1$ fixed points x with $\mathrm{ind}(M, f, x) = +1$ it follows from Theorem 3.1 that $F \cap \partial M$ must also contain $\geq b(F) + 1$ fixed points x with $\mathrm{ind}(\partial M, \bar{f}, x) = +1$. Now

$$\begin{aligned}\mathrm{ind}(\partial M, \bar{f}, F \cap \partial M) = &\sum(\bar{i}(\overline{F}) : \overline{F} \subset F \text{ and } \bar{i}(\overline{F}) > 0)\\ &+ \sum(\bar{i}(\overline{F}) : \overline{F} \subset F \text{ and } \bar{i}(\overline{F}) < 0) = u(F) + l(F).\end{aligned}$$

So if $F \cap \partial M$ contains $\geq b(F) + 1 = u(F) + 1$ fixed points x with $\mathrm{ind}(\partial M, \bar{f}, x) = +1$, it must contain $\geq |l(F)| + 1$ fixed points x with $\mathrm{ind}(\partial M, \bar{f}, x) = -1$, and hence $F \cap \partial M$ contains at least

$$u(F) + |l(F)| + 2 = \sum(|\bar{i}(\overline{F})| : \overline{F} \subset F) + 2$$

fixed points.

If $i(F) < 0$ and $\#(F \cap \mathrm{Int}\, M) < |i(F) - b(F)| = b(F) - i(F)$, then $b(F) = l(F)$, and the proof is similar, with indices $+1$ and -1 interchanged.

PROPOSITION 6.5 (Lower bound). *A transversally fixed map $f : (M, \partial M) \to (M, \partial M)$ has at least $N(f; M_{\pitchfork}, \partial M)$ fixed points on M.*

Proof. As $\bar{f} : \partial M \to \partial M$ is also transversally fixed, f has at least $N_{\pitchfork}(\bar{f})$ fixed points on ∂M. Now let F be an essential fixed point class of $f : M \to M$. If $\#(F \cap \mathrm{Int}\, M) < |i(F) - b(F)|$, then Lemma 6.4 shows that $F \cap \partial M$ contains at least 2 fixed points beyond the $\sum(|\bar{i}(\overline{F})| : \overline{F} \subset F)$

fixed points which F contributes to a minimal fixed point set on ∂M. So F contributes then at least $|i(F) - b(F)|$ fixed points beyond the minimal number $N_{\pitchfork}(\bar{f})$ on ∂M, and Proposition 6.5 follows.

PROPOSITION 6.6 (Homotopy invariance). *If* $f, g : (M, \partial M) \to (M, \partial M)$ *are homotopic, then* $N(f; M_{\pitchfork}, \partial M) = N(g; M_{\pitchfork}, \partial M)$.

Proof. From the definition and Proposition 2.3, as $N(f_{\pitchfork}, \bar{f})$ is homotopy invariant by Lemma 3.7.

As usual, $N(f; M_{\pitchfork}, \partial M)$ also has the properties of commutativity and homotopy type invariance, i.e. the properties corresponding to Propositions 3.9 and 3.10. We omit the details.

The calculation of $N(f; M_{\pitchfork}, \partial M)$ in the cases which correspond to Examples 3.11 and 3.12 is straightforward. Here are the results.

EXAMPLE 6.7. If $f : (B^n, S^{n-1}) \to (B^n, S^{n-1})$ is a map so that $n \geq 2$ and $\bar{f} : S^{n-1} \to S^{n-1}$ is of degree d, then f has one fixed point class F with $i(F) = 1$, and so $b(F) = \min(1, u(F))$. Inspection shows that $u(F) = 0$ if and only if $(-1)^n d \geq 1$, and hence

$$b(F) = \begin{cases} 0 & \text{if } (-1)^n d \geq 1, \\ 1 & \text{if } (-1)^n d < 1. \end{cases}$$

Thus Definition 6.3 and Example 3.11 show that

$$N(f; B^n_{\pitchfork}, S^{n-1}) = N(f; B^n, S^{n-1}_{\pitchfork}) \qquad \text{for all } n \geq 2\,.$$

EXAMPLE 6.8. Let $f : (M, \partial M) \to (M, \partial M)$ be a deformation of an n-dimensional manifold M with boundary ∂M. If $\chi(M) \neq 0$, then $f : M \to M$ has one fixed point class F with $i(F) = \chi(M)$. If n is even, then $\chi(\partial M) = 0$ and $b(F) = 0$, so $N(f_{\pitchfork}, \bar{f}) = N_{\pitchfork}(\bar{f}) = 0$ implies

$$N(\mathrm{id}; M_{\pitchfork}, \partial M) = N_{\pitchfork}(f) = |\chi(M)| \qquad (n \text{ is even})\,.$$

For odd n we have $\chi(\partial M) = 2\chi(M)$ (see Example 3.12) and so $i(F) = b(F)$, thus $N(f) = N(f_{\pitchfork}, \bar{f}) = |\chi(M)|$ and

$$N(\mathrm{id}; M_{\pitchfork}, \partial M) = N_{\pitchfork}(\bar{f}) = \sum |\chi(\partial M_j)| \qquad (n \text{ is odd})\,.$$

Both formulae are clearly still true if $\chi(M) = 0$.

By comparing Examples 6.7 and 6.8 with Examples 3.11 and 3.12 we see that in these cases $N(f; M, \partial M_{\pitchfork}) \leq N(f; M_{\pitchfork}, \partial M)$. Proposition 6.2 and the definitions of the various Nielsen numbers show that this is always true, i.e. we have

PROPOSITION 6.9. $N(f; M, \partial M_{\pitchfork}) \leq N(f; M_{\pitchfork}, \partial M)$.

Proposition 6.9 corresponds to Proposition 3.4(i) for boundary transversal maps. We can also extend the other results of Proposition 3.4.

PROPOSITION 6.10.

(i) $N(f;M_{\pitchfork},\partial M)=N_{\pitchfork}(\bar f)+N_{\pitchfork}(f)$ *if* $N(f,\bar f)=0$,

(ii) $N(f;M_{\pitchfork},\partial M)=N_{\pitchfork}(f)$ *if* $N(\bar f)=0$,

(iii) $N(f;M_{\pitchfork},\partial M)=N_{\pitchfork}(\bar f)=N(f;M,\partial M_{\pitchfork})$ *if* $N(f)=0$.

Proof. (i) If $N(f,\bar f)=0$ and if F is an essential fixed point class of $f:M\to M$, then F is not common, and so $u(F)=l(F)=b(F)=0$ and $N(f_{\pitchfork},\bar f)=0$.

(ii) If $N(\bar f)=0$, then $N_{\pitchfork}(\bar f)=0$, and also $b(F)=0$ for every essential fixed point class of f. Hence again $N(f_{\pitchfork},\bar f)=0$.

(iii) If $N(f)=0$, then $N_{\pitchfork}(\bar f)-N(f_{\pitchfork},\bar f)\le N_{\pitchfork}(f)=0$.

7. The minimum theorem for the relative transversal Nielsen number. In §5 we have shown that $N(f;M,\partial M_{\pitchfork})$ is an optimal lower bound for the number of fixed points of boundary transversally fixed maps if the dimension of M is sufficiently large. We now prove a corresponding result for $N(f;M_{\pitchfork},\partial M)$. The proof will use the following refinement of [2, Theorem 7.3].

LEMMA 7.1. *In* [2, *Theorem* 7.3] *the map F can be constructed so that p is a transversal fixed point of F.*

Proof. An inspection of the proof of [2, Theorem 7.3] shows that p is a transversal fixed point of F_1 and hence (as $F_2=F_1$ in a neighbourhood V of p) a transversal fixed point of F_2. The change from F_2 to F, which is carried out with the help of [2, Lemma 7.2], does not move points in a small neighbourhood of $\partial M\cap V$, and so p is a transversal fixed point of F.

THEOREM 7.2 (Minimum Theorem for $N(f;M_{\pitchfork},\partial M)$). *Let $f:(M,\partial M)\to(M,\partial M)$ be a map so that either $\dim M\ge 4$, or so that $\dim M=3$ and $\bar f:\partial M\to\partial M$ is homotopic to a sparse map. Then f is homotopic to a map $g:(M,\partial M)\to(M,\partial M)$ which is transversally fixed and has $N(f;M_{\pitchfork},\partial M)$ fixed points.*

Proof. By Lemma 4.1 we can homotope $f:(M,\partial M)\to(M,\partial M)$ to a map $f_1:(M,\partial M)\to(M,\partial M)$ so that $\bar f_1:\partial M\to\partial M$ is sparse, and by using [2, Lemma 7.1] we can further assume that there exists an open neighbourhood U of ∂M so that

(1a) $f_1(x)\ne x$ for all $x\in U-\partial M$,
(1b) $\operatorname{ind}(M,f_1,q)=0$ for all $q\in\operatorname{Fix}\bar f_1$,
(1c) every $q\in\operatorname{Fix}\bar f_1$ is a transversal fixed point of f_1.

As $\bar{f}_1$ is sparse, we have (as in the proof of Theorem 5.1)

$$N^1(f_1|\bar{f}_1) = N(f) - N(f, f_{\pitchfork}) .$$

Next we proceed as in the proof of [2, Theorem 7.5] with the help of Lemma 7.1 to obtain a map $f_2 : (M, \partial M) \to (M, \partial M)$ which is homotopic to f_1 and extends $\bar{f}_1$ so that

(2a) $f_2(x) \neq x$ for all $x \in U - \partial M$,

(2b) f_2 is transversally fixed on U,

(2c) f_2 has $N^1(f_1|\bar{f}_1)$ fixed points on $\operatorname{Int} M$, one for each essential fixed point class which is not representable on ∂M,

(2d) $\operatorname{ind}(M, f_2, p) = 0$ for all $p \in \operatorname{Fix} \bar{f}_2$ which are not contained in a representable fixed point class.

Now let F be an essential fixed point class of $f_2 : M \to M$ which is not representable on ∂M. Then $F \cap \operatorname{Int} M = \{p_F\}$ is a singleton, and (2d) shows that $\operatorname{ind}(M, f_2, p_F) = i(F)$. As $\bar{f}_2 = \bar{f}_1$ is sparse, we know from Proposition 4.3(ii) that F is not transversally common, so either $0 \leq u(F) < i(F)$ and $b(F) = i(F)$, or $i(F) < l(F) \leq 0$ and $b(F) = l(F)$. If $q \in F \cap \partial M$, with $i(F) > 0$ and $\operatorname{ind}(\partial M, \bar{f}_2, q) = +1$, we use Lemma 7.1 and [2, Theorem 7.3] to homotope f_2 to a map which extends $\bar{f}_1$ and so that q has index $+1$ (rather than 0) on M with respect to this map, and then unite the fixed point created on $\operatorname{Int} M$ in this way with p_F as in [2, Theorem 7.3]. This will reduce the index of p_F by 1, and if we carry out this procedure for all such q, we can reduce the index of p_F to $i(F) - u(F) = |i(F) - b(F)|$. If $q \in F \cap \partial M$ with $i(F) < 0$, we can in the same way change the index of p_F to $i(F) - l(F)$, and in this case $|i(F) - l(F)| = |i(F) - b(F)|$. We now have a map $f_3 : (M, \partial M) \to (M, \partial M)$ homotopic to f_1 which extends $\bar{f}_1$ so that

(3a) $f_3(x) \neq x$ for all $x \in U - \partial M$,

(3b) f_3 is transversally fixed on U,

(3c) f_3 has $N^1(f_1|\bar{f}_1)$ fixed points on $\operatorname{Int} M$, namely one fixed point p_F for each essential fixed point class F which is not representable on ∂M, and $|\operatorname{ind}(M, f_3, p_F)| = |i(F) - b(F)|$.

Now let p_F be one of the fixed points of f_3 on $\operatorname{Int} M$. We can use [4, Splitting Proposition, p. 126], to split p_F into transversal fixed points which lie in a small neighbourhood of p_F, and if we obtain thus pairs of fixed points of opposite index ± 1, we can cancel these with the help of the Whitney trick as in [5, §3]. After carrying out this procedure for all fixed points of f_3 on $\operatorname{Int} M$ we are left with a transversally fixed map $g : (M, \partial M) \to (M, \partial M)$ which extends $\bar{f}_1$, is homotopic to f_1 and has

$$\sum (|i(F) - b(F)| : F \text{ is not representable on } \partial M)$$

fixed points on $\operatorname{Int} M$ and $N_{\pitchfork}(\bar{f})$ fixed points on ∂M. As $\bar{g}$ is sparse, Propositions 4.3(ii) and 6.2 show that F is representable on ∂M if and only if $i(F) = b(F)$, and so g has

$$N(f; M_{\pitchfork}, \partial M) = N_{\pitchfork}(\bar{f}) + \sum(|i(F) - b(F)| : \\ F \text{ is a fixed point class which is not transversally common})$$

fixed points on M.

The $N(f; M_{\pitchfork}, \partial M)$ fixed points of the map g constructed in the proof of Theorem 7.2 are located so that $N_{\pitchfork}(\bar{f})$ lie on ∂M and $N_{\pitchfork}(f) - N(f_{\pitchfork}, \bar{f})$ lie in $\operatorname{Int} M$. We can show, as was done in Theorem 5.2 for boundary transversal maps, that such a location is necessary, and we can also determine the indices of the intersections of the fixed point classes with ∂M and $\operatorname{Int} M$.

THEOREM 7.3 (Location). *If $f : (M, \partial M) \to (M, \partial M)$ is a transversally fixed map which has $N(f; M_{\pitchfork}, \partial M)$ fixed points, then*

(i) *f has $N_{\pitchfork}(\bar{f})$ fixed points on ∂M and $N_{\pitchfork}(f) - N(f_{\pitchfork}, \bar{f})$ fixed points on* $\operatorname{Int} M$,

(ii) *for every essential fixed point class F of $f : M \to M$ we have*

$$\operatorname{ind}(M, f, F \cap \partial M) = b(F)$$

and

$$\operatorname{ind}(M, f, F \cap \operatorname{Int} M) = i(F) - b(F).$$

Proof. (i) follows immediately from Lemma 6.4.

(ii) According to Lemma 6.4 every essential fixed point class F of $f : M \to M$ has $|i(F) - b(F)|$ fixed points on $\operatorname{Int} M$. No two of these fixed points can have opposite index, as they could otherwise be cancelled as in [5, §3] to obtain a transversally fixed map with fewer than $N(f; M_{\pitchfork}, \partial M)$ fixed points, which is impossible. Hence

$$|\operatorname{ind}(M, f, F \cap \operatorname{Int} M)| = |i(F) - b(F)|.$$

So we are done unless

$$\operatorname{ind}(M, f, F \cap \operatorname{Int} M) = b(F) - i(F) \neq 0.$$

But in this case we would deduce from

$$i(F) = \operatorname{ind}(M, f, F) = \operatorname{ind}(M, f, F \cap \operatorname{Int} M) + \operatorname{ind}(M, f, F \cap \partial M)$$

that

$$\operatorname{ind}(M, f, F \cap \partial M) = i(F) + (i(F) - b(F)).$$

If $i(F) > 0$ this implies (as $i(F) \neq b(F)$) that $b(F) = u(F) < i(F)$ and so

$$\operatorname{ind}(M, f, F \cap \partial M) > i(F) > u(F),$$

which contradicts the Index Theorem 3.1. For $i(F) < 0$ this implies $b(F) = l(F) > i(F)$ and so

$$\mathrm{ind}(M, f, F \cap \partial M) < i(F) < l(F)\,,$$

which again contradicts Theorem 3.1. Hence Theorem 7.3(ii) must hold.

As usual we can weaken the assumptions of Theorem 7.2 for deformations to obtain our final result. The adjustments to the proof of Theorem 7.2 for the case $\dim M \leq 2$ are similar to the ones made in the proof of Theorem 5.3 and use again [2, Lemma 7.2]. We omit the details.

THEOREM 7.4. *If M is a manifold with boundary, then there exists a deformation $f : (M, \partial M) \to (M, \partial M)$ which is transversally fixed and has $N(f; M_{\pitchfork}, \partial M)$ fixed points.*

We also omit an extension of Theorem 2.13 to transversally fixed maps which map the boundary of a manifold into itself as the result is complicated to state but easy to prove.

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Appendix: C^∞ and C^0 fixed point sets are the same

by

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This note presents the proof of the following

THEOREM. *If $f : M \to M$ is a continuous mapping from a C^∞ paracompact manifold* (*with or without boundary*) *to itself, then there exists a C^∞ mapping $\widehat{f} : M \to M$ such that*

(a) *f and $\widehat{f}$ are homotopic,*

(b) $\{p \in M : \widehat{f}(p) = p\} = \{p \in M : f(p) = p\}$, *i.e. f and $\widehat{f}$ have the same fixed point sets.*

Since a C^k manifold, $k \geq 1$, always admits a subordinate C^∞ structure, the theorem yields on a given C^k manifold ($k = 1, 2, \ldots, \infty$) a C^k map satisfying (a) and (b).

In outline, the proof proceeds as follows: Near its fixed point set f can be written as the Riemannian exponentiation of a C^0 vector field, relative to an arbitrary but fixed Riemannian metric. This vector field can be approximated by a C^∞ vector field with the same zeros. Exponentiation of that vector field yields a C^∞ mapping defined on a neighborhood of the fixed point set of f and with the same fixed point set as f. Away from the fixed point set, f can be approximated by a C^∞ map without a fixed point. Then the two approximations of f can be C^∞ patched together via a partition of unity to yield the required mapping $\widehat{f}$.

For the purpose of this patching argument it is convenient to suppose, and it shall hereafter be supposed, that the injectivity radius of M is at least 1. In particular, there exists for every pair of points $p_1, p_2 \in M$ with Riemannian distance $\mathrm{dis}(p_1, p_2)$ from p_1 to p_2 less than 1 a unique arc-length-parameter geodesic from p_1 to p_2 with length $= \mathrm{dis}(p_1, p_2)$, and this geodesic depends C^∞ on its endpoints. For a compact manifold M without boundary, it is obvious that such a metric exists. For the proof that such a metric exists on a noncompact manifold without boundary see [2].

For such metrics, maps can be patched as follows: Suppose U and V are open subsets of M and $f_1 : U \to M$, $f_2 : V \to M$ are C^∞ mappings. Suppose also that $\{\varrho, 1 - \varrho\}$ is a partition of unity on $U \cup V$ subordinate to $\{U, V\}$, i.e. ϱ is C^∞, $0 \leq \varrho \leq 1$, and $\operatorname{supp} \varrho \subset U$, $\operatorname{supp}(1 - \varrho) \subset V$, where $\operatorname{supp} \varrho$ is the closure of the set $\{x : \varrho(x) \neq 0\}$. Finally, suppose $\mathrm{dis}(f_1(x), f_2(x)) < 1$ for all $U \cap V$. Then a new C^∞ mapping $\widehat{f}$ can be defined from $U \cup V$ into M by setting

$$\widehat{f}(x) = \begin{cases} f_1(x) & \text{if } x \in U - V, \\ f_2(x) & \text{if } x \in V - U, \\ \text{the point on the minimal geodesic from } f_1(x) \text{ to } f_2(x) \text{ that divides} \\ \text{this geodesic in the proportion } 1-\varrho(x) \text{ to } \varrho(x) \text{ if } x \in U \cap V. \end{cases}$$

(I.e. for $x \in U \cap V$ we have $\widehat{f}(x) = f_1(x)$ if $\varrho(x) = 1$ and $\widehat{f}(x) = f_2(x)$ if $\varrho(x) = 0$.) The map $\widehat{f}$ is then C^∞.

Turning now to the proof of the Theorem itself, suppose first that M is a compact manifold without boundary; the proof for the case that M is noncompact (but paracompact) or has a boundary is similar, and the modifications necessary for these cases are easy to describe once the case of a compact manifold without boundary is clear. First, choose an open neighborhood of the fixed point set $\operatorname{Fix} f = \{p \in M : f(p) = p\}$ such that $\operatorname{dis}(x, f(x)) < \frac{1}{4}$ for each $x \in U$. It can and will be supposed that U contains only finitely many components: this follows from the fact that $\operatorname{Fix} f$ is compact, and so finitely many components of any open set containing it will cover it. It will also be supposed without loss of generality that each component of U contains at least one fixed point of f. Next, choose an open set U_1 with $\operatorname{Fix} f \subset U_1 \subset \overline{U}_1 \subset U$, and set $V = M - \overline{U}_1$.

The map $f|U$ can be represented by a continuous vector field X on U in the sense that $f(x) = \exp_x X(x)$ for each $x \in U$: this vector field is just chosen so that $t \to \exp_x tX$, for $t \in [0,1]$, is the (unique) minimal $[0,1]$-parameter geodesic from x to $f(x)$.

Now, by standard approximation results, there is a C^∞ vector field $\widehat{X}$ on U such that

(a) $\operatorname{dis}(\exp_x X, \exp_x \widehat{X}) < \frac{1}{8}\|X\|$ for all x in $U - U_1$, and $\|\widehat{X}\| < \frac{9}{32}$ on U,

(b) on each component U_λ of U, $\widehat{X}$ has no zeros outside $U_\lambda \cap \operatorname{Fix} f$.

Point (a) can be arranged because $\|X\|$ is positive, indeed bounded away from zero, on $U - \overline{U}_1$ and everywhere $< \frac{1}{4}$ on U. Point (b) can be arranged by the usual method of moving and amalgamating zeros (see e.g. [1]). Indeed, it can be supposed that $\widehat{X}|U_\lambda$ has at most a single zero, which lies in $U_\lambda \cap \operatorname{Fix} f$.

Now choose, for each component U_λ of U, a C^∞ function $h_\lambda : U_\lambda \to \mathbb{R}$ such that $0 \le h_\lambda \le 1$, $h_\lambda \equiv 1$ on $U_\lambda - U_1$, and $h_\lambda = 0$ precisely on $U_\lambda \cap \operatorname{Fix} f$. The existence of such a function h_λ for each U_λ is an easy consequence of the fact that every closed set in a manifold is the (exact) vanishing set of some nonnegative C^∞ function on the manifold.

Now define a mapping $f_1 : U \to M$ by

$$f_1(x) = \exp_x(h_\lambda(x)\widehat{X}) \quad \text{if } x \in U_\lambda .$$

Next choose a C^∞ mapping f_2 from $V = M - \overline{U}_1$ into M such that

$$\operatorname{dis}(f_2(x), f(x)) < \min(\tfrac{1}{4}\operatorname{dis}(x, f(x)), \tfrac{1}{8})$$

for all $x \in V$. This choice is possible because $\operatorname{dis}(x, f(x))$ is positive and bounded away from zero on V.

Now choose a partition of unity subordinate to $\{U, V\}$, say $\{\varrho,\ 1-\varrho\}$. Then for each $x \in U \cap V$,

$$\begin{aligned}\operatorname{dis}(f_1(x), f_2(x)) &\le \operatorname{dis}(f_1(x), f(x)) + \operatorname{dis}(f(x), f_2(x))\\ &\le \tfrac{1}{8}\|X\| + \tfrac{1}{4}\|X\| = \tfrac{3}{8}\|X\| < \tfrac{3}{8}\cdot\tfrac{1}{4} < 1\,.\end{aligned}$$

(Recall that $\|X\| = \operatorname{dis}(x, f(x))$ for $x \in U$ and that $\operatorname{dis}(x, f(x)) < \frac{1}{4}$ for $x \in U$.)

Thus the patching of f_1 and f_2 via the partition of unity $\{\varrho, 1-\varrho\}$ is well defined. Call the result $\widehat{f}$. The map $\widehat{f}$ is C^∞ (since there are only finitely many U_λ). Then, for $x \in V - U$, $\widehat{f}(x) = f_2(x) \ne x$ because

$$\begin{aligned}\operatorname{dis}(x, f_2(x)) &\ge \operatorname{dis}(x, f(x)) - \operatorname{dis}(f(x), f_2(x))\\ &> \operatorname{dis}(x, f(x)) - \tfrac{1}{4}\operatorname{dis}(x, f(x)) - \tfrac{1}{4}\operatorname{dis}(x, f(x))\\ &= \tfrac{3}{4}\operatorname{dis}(x, f(x)) > 0\,.\end{aligned}$$

For $x \in U \cap V$,

$$\begin{aligned}\operatorname{dis}(x, \widehat{f}(x)) &\ge \operatorname{dis}(x, f_1(x)) - \operatorname{dis}(f_1(x), f_2(x))\\ &\ge \|\widehat{X}\| - \operatorname{dis}(f_1(x), f_2(x))\,.\end{aligned}$$

To see that $\|\widehat{X}\| - \operatorname{dis}(f_1(x), f_2(x))$ is positive, first note that $\|\widehat{X} - X\| < \frac{1}{8}\|X\|$ at $x \in U$ by the triangle inequality and condition (a) on $\widehat{X}$. Hence $\|\widehat{X}\| \ge \frac{7}{8}\|X\|$. Also

$$\begin{aligned}\operatorname{dis}(f_1(x), f_2(x)) &\le \operatorname{dis}(f_1(x), f(x)) + \operatorname{dis}(f(x), f_2(x))\\ &\le \tfrac{1}{8}\|X\| + \tfrac{1}{4}\|X\| = \tfrac{3}{8}\|X\|\end{aligned}$$

by the choices of f_1 and f_2. Thus, for $x \in U \cap V$,

$$\operatorname{dis}(x, \widehat{f}(x)) \ge \tfrac{7}{8}\|X\| - \tfrac{3}{8}\|X\| = \tfrac{1}{2}\|X\| > 0\,.$$

For $x \in U - V$, if $x \in U_\lambda$, then

$$\operatorname{dis}(x, \widehat{f}(x)) = \operatorname{dis}(x, \exp_x h_\lambda \widehat{X})\,;$$

this is 0 if and only if $x \in \operatorname{Fix} f$ because $h_\lambda \widehat{X} = 0$ at x if and only if $x \in \operatorname{Fix} f \cap U_\lambda$.

Thus $\operatorname{Fix}\widehat{f} = \operatorname{Fix} f$ on M.

To see that $\widehat{f}$ is homotopic to f it suffices to see that $\operatorname{dis}(f(x), \widehat{f}(x)) < 1$ for all $x \in M$. In that case, a homotopy can be obtained by deformation along shortest geodesic connections. Note first that if $x \in V - U$, then

$\operatorname{dis}(f(x), \widehat{f}(x)) = \operatorname{dis}(f(x), f_2(x)) < \frac{1}{8}$. If $x \in U - V$, then $\widehat{f}(x) = f_1(x)$ and hence

$$\begin{aligned}\operatorname{dis}(f(x), \widehat{f}(x)) &\leq \operatorname{dis}(f(x), x) + \operatorname{dis}(x, f_1(x)) \\ &\leq \|X\| + \|\widehat{X}\| \leq \tfrac{1}{4} + \tfrac{9}{32} < 1\,.\end{aligned}$$

If $x \in U \cap V$, then

$$\begin{aligned}\operatorname{dis}(f(x), \widehat{f}(x)) &\leq \operatorname{dis}(f(x), f_1(x)) + \operatorname{dis}(f_1(x), \widehat{f}(x)) \\ &\leq \operatorname{dis}(f(x), f_1(x)) + \operatorname{dis}(f_1(x), f_2(x)) \\ &\leq \tfrac{1}{8}\|X\| + \tfrac{3}{8}\|X\| \leq \tfrac{1}{2}\|X\| < \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{1}{8}\,.\end{aligned}$$

(Recall that the estimate $\operatorname{dis}(f_1(x), f_2(x)) \leq \frac{3}{8}\|X\|$ was obtained earlier.)

Thus in all cases, $\operatorname{dis}(f(x), \widehat{f}(x)) < 1$ as required.

The modifications for the case of a noncompact (but paracompact) manifold M without boundary of the proof just given for a compact M involve only: (1) noting that M can be given a suitably controlled geometry, i.e. a metric with injectivity radius at least 1, as already discussed, and (2) noting that the finiteness of the cover U_λ (by components of U) could be replaced by local finiteness. The details are left to the reader.

The proof technique already presented can be extended to apply to manifolds with boundary, compact or noncompact, to yield the same result. One new feature arises, however, not in the result, which stays the same, but in the proof. A crucial ingredient in the proof is the fact that, given a point p in a Riemannian manifold, every point q sufficiently close to p can be written in the form $\exp_p X$, for some tangent vector X at p. Subsequently, we need to know that $\exp_p tX$, $t \in [0,1]$, was defined (and in M). For manifolds with boundary with general metrics, difficulties arise with this. However, these difficulties can be easily eliminated by choosing a metric on the manifold with boundary that has the form $g \otimes dt^2$ on a neighborhood of ∂M, where g is a Riemannian metric on ∂M and some "collared neighborhood" of ∂M has been chosen in the form $\partial M \times [0,1)$. Using this approach, a metric can be chosen on the manifold M with boundary ∂M which has injectivity radius at least 1 in the following extended sense: If we write dis for the distance with respect to the metric thus chosen on the manifold with boundary, then every pair of points p, q in M with $\operatorname{dis}(p,q) < 1$ are connected by an arc-length-parameter geodesic in M with length $= \operatorname{dis}(p,q)$, and this geodesic is unique and depends smoothly on p and q. With this metric in hand, the proof as given can be applied. Only one other additional feature appears: the smooth vector field $\widehat{X}$ approximating X must be chosen so that $\exp_p t\widehat{X} \in M$ for all $t \in [0,1]$. In particular, for $p \in \partial M$, it must be that $\widehat{X}$ points into M or is tangent to ∂M, in the obvious senses. The fact that the approximation $\widehat{X}$ can be so chosen is easily established by standard approximation techniques.

It is important to note that the upshot of this construction will be a map $\widehat{f}$ that need not be equal to f on ∂M, *even if* f is already smooth on ∂M. The question of choosing $\widehat{f} = f$ on the boundary of M, supposing $f|\partial M$ to be C^∞, is subtle. Such a choice is not always possible if, for example, f has no fixed points in $M - \partial M$ so that $\widehat{f}$ is also required to have no fixed points in $M - \partial M$. The obstructions to such extensions arise even at the level of seeking an $\widehat{f}$ that is only C^1. This question has been investigated in [1] and further studied in the paper to which this is an appendix.

The result proved here for C^∞ mappings can be generalized to the case of real-analytic mappings, but some new features arise from the fact that not every closed subset of a real-analytic manifold is the zero set of a real-analytic function. For convenience, we call a (closed) set C in a real-analytic manifold M a *real-analytic variety* if there is a real-analytic function $f : M \to \mathbb{R}$ with $C = \{x \in M : f(x) = 0\}$. The author has proved the following general result, the proof of which will appear elsewhere.

Let M be a real-analytic manifold. Then:

(1) if $F : M \to M$ is a real-analytic mapping, then $\operatorname{Fix} F$ is a real-analytic variety,

(2) if $C \subset M$ is a real-analytic variety and if $G : M \to M$ is a continuous mapping with $\operatorname{Fix} G = C$, then there is a real-analytic mapping $F : M \to M$ such that

(a) $\operatorname{Fix} F = C$ and

(b) F is homotopic to G.

References

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