FUNDAMENTA
MATHEMATICAE
140 (1992)

Some refinements of a selection theorem with 0-dimensional domain

by

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Abstract. The following known selection theorem is sharpened, primarily, by weakening the hypothesis that all the sets $\varphi(x)$ are closed in Y: Let X be paracompact with $\dim X = 0$, let Y be completely metrizable and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. Then φ has a selection.

1. Introduction. The purpose of this note is to sharpen the following selection theorem, primarily by weakening the assumption that the sets $\varphi(x)$ are all closed in the complete metric space Y. Our study grew out of some valuable conversations with Roman Pol while he was a visitor at the University of Washington, and Corollary 1.6 answers a question which he asked.

THEOREM 1.1 [M_2, Theorem 2]. *Let X be paracompact with* $\dim X = 0$, *let Y be completely metrizable*, *and let $\varphi : X \to \mathcal{F}(Y)$ be l.s.c. Then φ has a selection.*

In the above result, $\mathcal{F}(Y)$ denotes $\{E \in 2^Y : E \text{ closed in } Y\}$, where $2^Y = \{E \subset Y : E \neq \emptyset\}$. A function $\varphi : X \to 2^Y$ is *l.s.c.* (= *lower semi-continuous*) if $\{x \in X : \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y, and a *selection* for φ is a continuous $f : X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

The requirement in Theorem 1.1 that each $\varphi(x)$ be closed in Y cannot be omitted (see [M_6, Example 9.1]). Before examining how it can be relaxed, we pause to record a generalization of Theorem 1.1 in a different direction which will form the basis for our subsequent results. If $A \subset X$ is closed, we say that $\varphi : X \to 2^Y$ has the *SEP* (= *selection extension property*) at A if every selection for $\varphi|A$ extends to a selection for φ; if φ has the SEP at every closed $A \subset X$, then we simply say that φ has the *SEP* (see [M_6, p. 1]).

The following result reduces to Theorem 1.1 when $A = \emptyset$.

THEOREM 1.2 [M_4, Theorem 1.2]. *Let X be paracompact, Y completely metrizable, $A \subset X$ closed with* $\dim_X X \backslash A = 0$ (1), *and $\varphi : X \to \mathcal{F}(Y)$ l.s.c. Then φ has the SEP at A.*

We now turn to refinements of Theorem 1.2 which do not require that every $\varphi(x)$ be closed in Y. The simplest of these is the following result, which reduces to Theorem 1.2 when $C = \emptyset$.

THEOREM 1.3 [M_6, Theorem 1.3]. *Let X, Y and $A \subset X$ be as in Theorem 1.2. Let $C \subset X$ be countable, and let $\varphi : X \to 2^Y$ be l.s.c. with $\varphi(x) \in \mathcal{F}(Y)$ for $x \notin C$. Then φ has the SEP at A.*

It was observed in [M_6, Section 8] that Theorem 1.3 remains valid if C is only assumed to be a countable union of closed, discrete subsets of X. Proceeding further in this direction, we obtain the following generalization of Theorem 1.3. We say that $\varphi : X \to 2^Y$ has the *SNEP* (= *selection neighborhood extension property*) at a closed set $A \subset X$ if every selection for $\varphi|A$ extends to a selection for $\varphi|U$ for some open $U \supset A$; if φ has the SNEP at every closed (resp. singleton) $A \subset X$, then we say that φ has the *SNEP* (resp. *pointwise SNEP*).

THEOREM 1.4. *Let X, Y and $A \subset X$ be as in Theorem 1.2. Let $C = \bigcup_n C_n$ with each C_n closed in X, and let $\varphi : X \to 2^Y$ be l.s.c. with $\varphi(x) \in \mathcal{F}(Y)$ for $x \notin C$ and with $\varphi|C_n$ having the SNEP for all n. Then φ has the SEP at A.*

Remark. Theorem 1.4 is of some interest even when $C = X$. In that case, Y need only be metrizable (rather than completely metrizable), since we can then simply replace Y by a completely metrizable space containing it.

We now come to the principal results of this paper. Theorem 1.5 sharpens Theorem 1.4 in a new direction, under the mild restriction that A and the C_n's are G_δ-subsets of X (2). We adopt the notation that, if $\varphi : X \to 2^Y$, then $\overline{\varphi} : X \to \mathcal{F}(Y)$ is defined by $\overline{\varphi}(x) = \overline{\varphi(x)}$. As observed in [$M_1$, Proposition 2.3], $\overline{\varphi}$ is l.s.c. if and only φ is l.s.c.

THEOREM 1.5. *Let X, Y, $A \subset X$ and $C = \bigcup_n C_n$ be as in Theorem 1.4, and assume that A and the C_n's are G_δ-subsets of X. Let Z be a G_δ in Y, and let $\varphi : X \to 2^Y$ be l.s.c. with $\varphi(x) \in \mathcal{F}(Z)$ for $x \notin C$, with $\overline{\varphi}(x) \cap Z$ dense in $\overline{\varphi}(x)$ for $x \in C$ (3), and with $\varphi|C_n$ having the pointwise SNEP for all n. Then φ has the SEP at A.*

(1) This means that $X \backslash A$ is non-empty and $\dim S \le 0$ for all closed (in X) $S \subset X \backslash A$. If X is metrizable (more generally, if A is a G_δ in X), this is equivalent to $\dim X \backslash A = 0$.

(2) Examples 6.1 and 6.2 show that this restriction cannot omitted.

(3) Example 6.3 shows that this assumption, which is automatically satisfied when $Z = Y$, cannot be omitted.

While the statement of Theorem 1.5 is somewhat involved, it has the following easily stated corollary which answers a question raised by R. Pol.

COROLLARY 1.6. *Let X be metrizable with* $\dim X = 0$, *let Y be completely metrizable*, $Z \subset Y$ *a dense* G_δ, $C \subset X$ *countable*, *and* $\varphi : X \to 2^Y$ *with* $\varphi(x) = Z$ *for* $x \notin C$ *and with* $\overline{\varphi}(x) = Y$ *all* $x \in X$. *Then φ has the SEP.*

We conclude this introduction with one more result. It was proved in [M_7, Theorem 1.2(b)] that Theorem 1.2 remains valid if the assumption that $\varphi(x)$ is closed in Y for every $x \in X$ is weakened to assuming only that there exists a G_δ-subset Z of $X \times Y$ such that $\varphi(x)$ is relatively closed in $Z(x)$ for every $x \in X$ ([4]). Similarly, Theorem 1.5 can be generalized as follows.

THEOREM 1.7. *Let* X, Y, $A \subset X$ *and* $C = \bigcup_n C_n$ *be as in Theorem* 1.5. *Let Z be a G_δ in $X \times Y$, and let* $\varphi : X \to 2^Y$ *be l.s.c. with* $\varphi(x) \in \mathcal{F}(Z(x))$ *for* $x \notin C$, *with* $\overline{\varphi}(x) \cap Z(x)$ *dense in* $\overline{\varphi}(x)$ *for* $x \in C$, *and with* $\varphi|C_n$ *having the pointwise SNEP for all n. Then φ has the SEP at A.*

After establishing some lemmas in Section 2, Theorem 1.4 will be proved in Section 3, Theorem 1.5 in Section 4, and Theorem 1.7 in Section 5. Section 6 is devoted to examples.

2. Some lemmas.

2. Some lemmas. Lemmas 2.1 and 2.2 are simple observations, stated without proof. Lemma 2.3 will be applied in the proof of Theorem 1.4, and Lemma 2.5 and Corollary 2.6 in the proof of Theorem 1.5.

LEMMA 2.1. *If* $\varphi : X \to 2^Y$ *has the pointwise SNEP*, *so does* $\varphi|E$ *for every* $E \subset X$.

LEMMA 2.2. *Let* $\varphi : X \to 2^Y$, *let* $W \subset X \times Y$ *be open with* $\varphi(x) \cap W(x) \neq \emptyset$ *for all* $x \in X$, *and define* $\alpha : X \to 2^Y$ *by* $\alpha(x) = \varphi(x) \cap W(x)$. *If φ is l.s.c. or has the SNEP at* $A \subset X$, *then α has the same property.*

LEMMA 2.3. *Let X be paracompact*, $A \subset X$ *closed with* $\dim_X X \backslash A = 0$, *and let* $\varphi : X \to 2^Y$ *have the SNEP at A and at every* $x \in X \backslash A$. *Then φ has the SEP at A.*

Proof. Let g be a selection for $\varphi|A$. We must extend g to a selection for φ.

Case 1: $\dim X = 0$. Extend g to a selection h for $\varphi|U$ for some open $U \supset A$. Choose an open cover (U_λ) of $X \backslash A$ such that $\varphi|U_\lambda$ has a selection f_λ for each λ. The open cover $\{U\} \cup (U_\lambda)$ of X has a disjoint open refinement $\{U'\} \cup (U'_\lambda)$. Define $f : X \to Y$ by $f|U' = h|U'$ and $f|U'_\lambda = f_\lambda|U'_\lambda$. This f is a selection for φ which extends g.

([4]) For $E \subset X \times Y$ and $x \in X$, we define $E(x) = \{y \in Y : (x, y) \in E\}$.

Case 2: $\dim_X X\backslash A = 0$. As in Case 1, extend g to selection h for $\varphi|U$ for some open $U \supset A$. Pick V open in X with $A \subset V \subset \overline{V} \subset U$, and let $X^* = X\backslash V$ and $A^* = \overline{V}\backslash V$. Now X^* is closed in X and $X^* \subset X\backslash A$, so $\dim X^* \le 0$. By Case 1 applied to X^* and A^*, the selection $h|A^*$ for $\varphi|A^*$ extends to a selection k for $\varphi|X^*$. Define $f : X \to Y$ by $f|\overline{V} = h|\overline{V}$ and $f|X^* = k$. This f is a selection for φ extending g. ■

Corollary 2.4. *If X is paracompact with* $\dim X = 0$, *then* $\varphi : X \to 2^Y$ *has the* (*pointwise*) *SEP if and only if it has the* (*pointwise*) *SNEP.*

Proof. This follows immediately from the special case of Lemma 2.3 where $\dim X = 0$. ■

Lemma 2.5. *Let X be paracompact*, *$A \subset X$ a closed G_δ with* $\dim_X X\backslash A = 0$, *Y metrizable*, *and let $\varphi : X\backslash A \to 2^Y$ have the pointwise SNEP. Suppose that $g : A \to Y$ and that g extends to a continuous $f : X \to Y$ such that $f(x) \in \overline{\varphi}(x)$ for $x \in X\backslash A$. Then g extends to a continuous $h : X \to Y$ such that $h(x) \in \varphi(x)$ for $x \in X\backslash A$.*

Proof. Since A is a closed G_δ in the normal space X, there is a continuous $u : X \to [0,1]$ such that $A = u^{-1}(0)$. Let ϱ be a compatible metric on Y, and let

$$V = \{(x,y) \in (X\backslash A) \times Y : \varrho(y, f(x)) < u(x)\}.$$

Then V is open in $(X\backslash A) \times Y$ and $\varphi(x) \cap V(x) \ne \emptyset$ for all $x \in X\backslash A$. Define $\alpha : X\backslash A \to 2^Y$ by $\alpha(x) = \varphi(x) \cap V(x)$. Then α has the pointwise SNEP by Lemmas 2.1 and 2.2.

Since $X\backslash A$ is an F_σ in X, we have $X\backslash A$ paracompact (by [M_1, Proposition 3]) and $\dim X\backslash A = 0$. Hence α has a selection k by Case 1 of Lemma 2.3 (with X replaced by $X\backslash A$ and φ by α). Define $h : X \to Y$ by $h|A = g$ and $h|(X\backslash A) = k$. This h satisfies all our requirements. ■

The following corollary should be compared to Lemma 2.3.

Corollary 2.6. *Let X be paracompact*, *$A \subset X$ a closed G_δ with* $\dim_X X\backslash A = 0$, *and let $\varphi : X \to 2^Y$ be l.s.c. and have the SNEP at every $x \in X\backslash A$. Then φ has the SEP at A. More generally*, *every selection g for $\overline{\varphi}|A$ extends to a continuous $k : X \to Y$ such that $k(x) \in \varphi(x)$ for every $x \in X\backslash A$.*

Proof. We may clearly suppose that Y is completely metrizable. Since $\overline{\varphi}$ is also l.s.c., g extends to a selection f for $\overline{\varphi}$ by Theorem 1.2. Hence, by Lemma 2.5, g extends to a continuous $k : X \to Y$ such that $k(x) \in \varphi(x)$ for every $x \in X\backslash A$. ■

3. Proof of Theorem 1.4. Let g be a selection for $\varphi|A$. We must extend g to a selection f for φ.

Let $B_0 = A$ and let $B_n = A \cup \bigcup_{i=1}^n C_i$ for $n > 0$. Let ϱ be a compatible, complete metric on Y. By induction, we will construct selections f_n for $\overline{\varphi}$ such that:

(a) $f_0|A = g$,
(b) $f_{n+1}(x) = f_n(x) \in \varphi(x)$ for $x \in B_n$,
(c) $\varrho(f_{n+1}(x), f_n(x)) \le 2^{-n}$ for all $x \in X$.

Suppose that we had such functions f_n. Then (f_n) converges uniformly to a selection f for $\overline{\varphi}$ by (c). Clearly f extends g by (a) and (b). Also $f(x) \in \varphi(x)$ for $x \in \bigcup_n B_n$ by (b), and $f(x) \in \varphi(x)$ for $x \notin \bigcup_n B_n$ because $\varphi(x) = \overline{\varphi}(x)$ for such x by our hypotheses. Hence f is a selection for φ which extends g.

It remains to construct the sequence (f_n). Since φ is l.s.c., so is $\overline{\varphi}$, and hence g extends to a selection f_0 for $\overline{\varphi}$ by Theorem 1.2. Now suppose $f_0, \ldots, f_n$ have been chosen, and let us choose f_{n+1}. Let

$$W = \{(x, y) \in X \times Y : \varrho(y, f_n(x)) < 2^{-n}\}$$

Then W is open in $X \times Y$ and $\varphi(x) \cap W(x) \neq \emptyset$ for all $x \in X$. Define $\alpha : X \to 2^Y$ by $\alpha(x) = \varphi(x) \cap W(x)$. Since $\varphi|C_{n+1}$ has the SNEP, so does $\alpha|C_{n+1}$ by Lemma 2.2. Also $\dim_{C_{n+1}} C_{n+1} \backslash (C_{n+1} \cap B_n) \le 0$ because $\dim_X X \backslash A = 0$ and $B_n \supset A$. Hence, by Lemma 2.3, the selection $f_n|(C_{n+1} \cap B_n)$ for $\alpha|(C_{n+1} \cap B_n)$ extends to a selection h for $\alpha|C_{n+1}$. Define $k : B_{n+1} \to Y$ by $k|B_n = f_n|B_n$ and $k|C_{n+1} = h$. This k is a selection for $\alpha|B_{n+1}$. Now $\alpha : X \to 2^Y$ is l.s.c. by Lemma 2.2, and hence so is $\overline{\alpha} : X \to \mathcal{F}(Y)$. Since $\dim_X X \backslash B_{n+1} \le 0$ (because $B_{n+1} \supset A$), k extends to a selection f_{n+1} for $\overline{\alpha}$ by Theorem 1.2. This f_{n+1} has all required properties. ∎

4. Proof of Theorem 1.5. The proof proceeds along the same lines as the proof of Theorem 1.4, but the details are more complicated.

Let g be a selection for $\varphi|A$. We must extend g to a selection f for φ.

Let $B_0 = A$ and let $B_n = A \cup \bigcup_{i=1}^n C_i$ for $n > 0$. Then B_n is a G_δ in X for all n. Let ϱ be a compatible, complete metric on Y. Since Z is a G_δ in Y, it is completely metrizable; let d be a compatible, complete metric on Z. Finally, define $\psi : X \to \mathcal{F}(Z)$ by $\psi(x) = \overline{\varphi}(x) \cap Z$. Our assumptions imply that $\psi(x) = \varphi(x)$ for $x \notin C$ and that $\overline{\psi}(x) = \overline{\varphi}(x)$ for all $x \in X$. Thus $\overline{\psi}$ is l.s.c., and hence so is ψ.

By induction, we will construct selections f_n for $\overline{\varphi}$ such that:

(a) $f_0|A = g$,
(b) $f_{n+1}(x) = f_n(x) \in \varphi(x)$ for $x \in B_n$,
(c) $\varrho(f_{n+1}(x), f_n(x)) < 2^{-n}$ for all $x \in X$,
(d) $f_n(x) \in \psi(x)$ for $x \notin B_n$,
(e) $d(f_{n+1}(x), f_n(x)) < 2^{-n}$ for $x \notin B_{n+1}$.

Suppose that we had such functions f_n. Then (f_n) converges ϱ-uniformly to a continuous $f : X \to Y$ by (c), and f extends g by (a) and (b). Finally, if $x \notin \bigcup_n B_n$ then $f(x) \in \psi(x) = \varphi(x)$ by (d) and (e), and if $x \in \bigcup_n B_n$ then $f(x) \in \varphi(x)$ by (a) and (b). Thus f is a selection for φ which extends g.

It remains to construct the sequence (f_n). Since $\overline{\varphi}$ is l.s.c., g extends to a selection g' for $\overline{\varphi} = \overline{\psi}$ by Theorem 1.2. Also $X\backslash A$ is an F_σ in X, so $X\backslash A$ is paracompact (by [M$_1$, Proposition 3]) and $\dim X\backslash A = 0$. Hence $\psi|(X\backslash A) : X\backslash A \to \mathcal{F}(Z)$ has the SEP by Theorem 1.2. By Lemma 2.5, g therefore extends to a continuous $f_0 : X \to Y$ such that $f_0(x) \in \psi(x)$ for $x \in X\backslash A$.

Suppose now that we have $f_0, \ldots, f_n$. To construct f_{n+1}, we will first extend $f_n|B_n$ to a suitable selection h for $\varphi|B_{n+1}$ and will then extend h to obtain f_{n+1}.

Since $f_n(x) \in \psi(x) \subset Z$ for all $x \in X\backslash B_n$, we can define

$$V = \{(x,y) \in (X\backslash B_n) \times Z : d(y, f_n(x)) < 2^{-n}\}\,.$$

Then V is relatively open in $X \times Z$, so $V = \widetilde{V} \cap (X \times Z)$ for some open $\widetilde{V}$ in $X \times Y$. Define

$$W = \widetilde{V} \cap \{(x,y) \in X \times Y : \varrho(y, f_n(x)) < 2^{-n}\}\,,$$

so W is open in $X \times Y$.

For all $x \in X\backslash B_n$ we have $f_n(x) \in \psi(x) \subset \overline{\varphi}(x)$ and $f_n(x) \in W(x)$; we can thus define $\alpha : X\backslash B_n \to 2^Y$ by $\alpha(x) = \varphi(x) \cap W(x)$, and we note that $f_n(x) \in \overline{\alpha}(x)$ for all $x \in X\backslash B_n$. Since $\varphi|C_{n+1}$ has the pointwise SNEP, so does $\varphi|(B_{n+1}\backslash B_n)$ by Lemma 2.1, and hence so does $\alpha|(B_{n+1}\backslash B_n)$ by Lemma 2.2. Also B_n is a G_δ in X, and $\dim_{B_{n+1}} B_{n+1}\backslash B_n \le 0$ because $\dim_X X\backslash A = 0$ and $B_n \supset A$. Thus, by Lemma 2.5, $f_n|B_n$ extends to a continuous $h : B_{n+1} \to Y$ such that $h(x) \in \alpha(x)$ for every $x \in B_{n+1}\backslash B_n$.

Define $\beta : X \to 2^Y$ by

$$\beta(x) = \begin{cases} \{f_n(x)\} & \text{if } x \in B_n\,, \\ \psi(x) \cap W(x) & \text{if } x \in X\backslash B_n\,. \end{cases}$$

Using Corollary 2.6, we will extend h to a continuous $f_{n+1} : X \to Y$ such that $f_{n+1}(x) \in \beta(x)$ for every $x \in X\backslash B_{n+1}$.

To apply Corollary 2.6, recall that B_{n+1} is a G_δ in X, and note that β is l.s.c. because $\beta|(X\backslash B_{n+1})$ is l.s.c. (by Lemma 2.2) and $f_n(x) \in \beta(x)$ for all $x \in X$. Next, every closed (in X) $E \subset X\backslash A$ is paracompact and 0-dimensional, so $\psi|E : E \to \mathcal{F}(Z)$ has the SEP by Theorem 1.2; it follows that ψ has the SNEP at every $x \in X\backslash A$, and hence so does β. Finally, h is a selection for $\overline{\beta}|B_{n+1}$, because if $x \in B_n$ then $h(x) = f_n(x) \in \beta(x)$, and if $x \in B_{n+1}\backslash B_n$ then $h(x) \in \varphi(x) \cap W(x) \subset \overline{\psi}(x) \cap W(x) \subset \overline{\beta}(x)$. By Corollary 2.6, we can therefore extend h to a continuous $f_{n+1} : X \to Y$ such that $f_{n+1}(x) \in \beta(x)$ for every $x \in X \backslash B_{n+1}$, and this f_{n+1} satisfies all our requirements. ∎

5. Proof of Theorem 1.7. The proof of Theorem 1.7 is similar to the proof of Theorem 1.5, with two principal modifications. The first of these, which is easy, replaces references to Theorem 1.2 by references to the following generalization of Theorem 1.2.

THEOREM 5.1 [M_7, Theorem 1.2(b)]. *Let X, Y and $A \subset X$ be as in Theorem 1.2, let $Z \subset X \times Y$ be a G_δ, and let $\varphi : X \to 2^Y$ be l.s.c. with $\varphi(x) \in \mathcal{F}(Z(x))$ for every $x \in X$. Then φ has the SEP at A.*

The second modification in the proof of Theorem 1.5 is more technical, and concerns condition (e) on the sequence (f_n). That condition depends on the existence of a compatible, complete metric d on $Z \subset Y$, and no analogue of such a metric appears to be available for $Z \subset X \times Y$ in Theorem 1.7. We now indicate how that difficulty can be circumvented.

Since Theorem 1.7 assumes that Z is a G_δ-subset of $X \times Y$, we can write $Z = \bigcap_n G_n$ with each G_n open in $X \times Y$. Now inductively construct *two* sequences, a sequence (f_n) of selections for $\overline{\varphi}$ and a sequence (H_n) of open subsets of $X \times Y$, satisfying conditions (a)–(d) for (f_n) in the proof of Theorem 1.5 as well as the following replacement for condition (e):

(e′) $f_{n+1}(x) \in H_{n+1}(x) \subset \overline{H_{n+1}(x)} \subset H_n(x) \cap G_{n+1}$ for $x \notin B_n$.

This guarantees that $f = \lim_n f_n$ satisfies all our requirements.

To construct the sequences (f_n) and (H_n), we begin by setting $G_0 = H_0 = X \times Y$ and choosing f_0 as in the proof of Theorem 1.5 (but invoking Theorem 5.1 instead of Theorem 1.2). Once we have $f_0, \ldots, f_n$ and $H_0, \ldots, H_n$, we note that $f_n(x) \in H_n(x) \cap G_{n+1}$ for $x \notin B_n$, and we choose an open H_{n+1} in $(X \backslash B_n) \times Y$ such that

$$f_n(x) \in H_{n+1}(x) \subset \overline{H_{n+1}(x)} \subset H_n(x) \cap G_{n+1} \quad \text{for } x \notin B_n .$$

(The existence of such an H_{n+1} follows easily from the fact that $X \backslash B_n$, being an F_σ in X, is paracompact.) Having chosen H_{n+1}, we now construct f_{n+1} as in the proof of Theorem 1.5, with $\widetilde{V}$ replaced by H_{n+1} in the definition of W and with the reference to Theorem 1.2 replaced by a reference to Theorem 5.1. ∎

6. Examples. Our first example shows that the requirement in Theorem 1.5 that the C_n's are all G_δ-subsets of X cannot be omitted. As usual, $\mathbb{R}$ denotes the reals, $\mathbb{Q}$ the rationals, and $\mathbb{P}$ the irrationals.

EXAMPLE 6.1. Let X be the one-point compactification of an uncountable discrete space, and let $x^* \in X$ be the point at infinity. Let $Y = \mathbb{R}$, and define $\varphi : X \to 2^Y$ by $\varphi(x^*) = \mathbb{Q}$ and $\varphi(x) = \mathbb{P}$ if $x \neq x^*$. Then all assumptions of Theorem 1.5 are satisfied, with $C_1 = \{x^*\}$ and with A and the other C_n's all empty, except that C_1 is not a G_δ in X. Nevertheless, φ has no selection.

Proof. Suppose f were a selection for φ, and let $y^* = f(x^*)$. Then $\{x^*\} = f^{-1}(y^*)$, so $\{x^*\}$ is a G_δ in X, which is impossible. ■

Our next example shows that the requirement in Theorem 1.5 that A is a G_δ in X cannot be omitted.

Example 6.2. Let X be a first-countable, paracompact space with $\dim X = 0$ which has a countable, closed, non-G_δ subset $A = \{x_1, x_2, \ldots\}$(5). Let $Y = \mathbb{R}$, and define $\varphi : X \to 2^Y$ by $\varphi(x) = \mathbb{Q}$ if $x \in A$ and $\varphi(x) = \mathbb{P}$ if $x \notin A$. Then all assumptions of Theorem 1.5 are satisfied with $C_n = \{x_n\}$, except that A is not a G_δ in X. However, φ does not have the SEP at A.

Proof. Let g be the selection for $\varphi|A$ defined by $g(x) = 0$ for $x \in A$. If g extended to a selection f for φ, then $A = f^{-1}(0)$ would be a G_δ in X, a contradiction. ■

Our last example, which was kindly provided by Roman Pol, shows that Theorem 1.5 becomes false if the requirement that $\overline{\varphi}(x) \cap Z$ is dense in $\overline{\varphi}(x)$ for $x \in C$ is omitted.

Example 6.3. Let X be a non-empty, completely metrizable space without isolated points, let $Y = [0, 1]$ and let $Z = (0, 1]$. Then there exists a countable $C \subset X$, and a l.s.c. $\varphi : X \to 2^Y$ with $\varphi(x) = Z$ for $x \notin C$ and with $\varphi(x) \in \mathcal{F}(Y)$ for $x \in C$, such that φ has no selection.

Proof. Let $x_0 \in X$. Since no $x \in X$ has a countable neighborhood, we can choose disjoint countable sets $C_0, C_1, \ldots$ in X, with $C_0 = \{x_0\}$, such that each $C_0 \cup \ldots \cup C_n$ is closed in X and is the set of accumulation points of C_{n+1}. Let $C = C_0 \cup C_1 \cup \ldots$, and define $\varphi : X \to 2^Y$ by

$$\varphi(x) = \begin{cases} \{0\} & \text{if } x = x_0\,, \\ \{0\} \cup [1/n, 1] & \text{if } x \in C_n\,, \\ (0, 1] & \text{if } x \notin C\,. \end{cases}$$

It is easily checked that φ is l.s.c., so it remains to show that φ has no selection.

Suppose f were a selection for φ. Let $D = f^{-1}(0)$. Then $x_0 \in D \subset C$ and D is closed in X. Also D has no isolated points, for if $x \in D \cap C_n$, there is a sequence $x_i \to x$ with $x_i \in C_{n+1}$; since $f(x) = 0$ and $f(x_i) \in$

(5) One can, for example, take X to be the set $\mathbb{R}$, topologized by $\{U \cup S : U \text{ open in } \mathbb{R},\ S \subset \mathbb{P}\}$ (this X is sometimes called the Michael line), and take A to be the rationals in X. See [M5] or [B, pp. 390–391].

$\{0\} \cup [1/(n+1), 1]$ for all i, it follows that $f(x_i) = 0$ (and hence $x \in D$) for almost all i. Thus D is a non-empty, closed, countable subset of X without isolated points, and that is impossible. ∎

References

[B] D. Burke, *Covering properties*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam 1984, 347–422.

[M$_1$] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. 4 (1953), 831–838.

[M$_2$] —, *Selected selection theorems*, Amer. Math. Monthly 63 (1956), 233–238.

[M$_3$] —, *Continuous selections I*, Ann. of Math. 63 (1956), 361–382.

[M$_4$] —, *Continuous selections II*, ibid. 64 (1956), 562–580.

[M$_5$] —, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), 375–376.

[M$_6$] —, *Continuous selections and countable sets*, Fund. Math. 111 (1981), 1–10.

[M$_7$] —, *A generalization of a theorem on continuous selections*, Proc. Amer. Math. Soc. 105 (1989), 236–243.

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Received 7 August 1991