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On weakly infinite-dimensional subspaces

by

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Abstract. We will construct weakly infinite-dimensional (in the sense of Y. Smirnov) spaces X and Y such that Y contains X topologically and $\dim Y = \omega_0$ and $\dim X = \omega_0+1$. Consequently, the subspace theorem does not hold for the transfinite dimension dim for weakly infinite-dimensional spaces.

Preliminaries. All the spaces considered are separable and metrizable. Let us first establish some notational conventions. As far as standard notions from general topology and dimension theory are concerned we mostly follow [E1] and [E2].

In particular, we note that the boundary of a subset A of a space X is denoted by $\operatorname{Fr} A$. We denote by C the Cantor set, and by ω the collection of natural numbers. The first infinite ordinal is denoted by ω_0.

1. Definitions. Let us start with some fundamental definitions.

1.1. Definition. A sequence $\{(A_i, B_i)\}_{i=1}^n$ of pairs of disjoint closed sets in a space X is called *inessential* (resp. *inessential on a subspace* F) if we can find open sets O_i (resp. O_i open in F), $i = 1, \ldots, n$, such that

$$A_i \subset O_i \subset \overline{O}_i \subset X - B_i \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Fr} O_i = \emptyset$$

(resp. $(\bigcap_{i=1}^n \operatorname{Fr} O_i) \cap F = \emptyset$). Otherwise it is called *essential* (resp. *essential on* F).

1.2. Definition. A space X is called *weakly infinite-dimensional in the sense of Smirnov*, abbreviated *S-w.i.d.*, if for every sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of pairs of disjoint closed sets in X there exist open sets O_i, $i = 1, 2, \ldots,$

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such that

$$A_i \subset O_i \subset \overline{O}_i \subset X - B_i \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Fr} O_i = \emptyset \text{ for some } n\,.$$

In [B1] and [B2] we developed a transfinite extension of the covering dimension, dim. This dimension function classifies all weakly infinite-dimensional spaces. We also saw that the classification resulting from R. Pol's index [P] is equivalent and that the essential mappings defined by D. W. Henderson [He] give a classification which differs by at most 1 class from the one resulting from dim. We will see that the subspace theorem does not hold for the dimension dim. We prove this by constructing spaces X and Y such that Y contains X topologically and moreover $\dim X = \omega_0 + 1$ and $\dim Y = \omega_0$. We also see that this result is relevant to the characterization theorem.

To define the transfinite dimension function dim, we need the following notions:

Let L be an arbitrary set. $\operatorname{Fin} L$ denotes the collection of all finite, non-empty subsets of L. Let M be a subset of $\operatorname{Fin} L$.

For each $\sigma \in \{\emptyset\} \cup \operatorname{Fin} L$ we put

$$M^\sigma = \{\tau \in \operatorname{Fin} L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}\,.$$

$M^{\{a\}}$ is abbreviated as M^a.

1.3. Definition. Define the ordinal $\operatorname{Ord} M$ as follows:

$\operatorname{Ord} M = 0$ iff $M = \emptyset$,

$\operatorname{Ord} M \le \alpha$ iff for every $a \in L, \operatorname{Ord} M^a < \alpha$,

$\operatorname{Ord} M = \alpha$ iff $\operatorname{Ord} M \le \alpha$ and $\operatorname{Ord} M < \alpha$ is not true, and

$\operatorname{Ord} M = \infty$ iff $\operatorname{Ord} M > \alpha$ for every ordinal α.

Let X be a normal space. Then we define

$$L(X) = \{(A, B) : A \cap B = \emptyset,\ A \text{ and } B \text{ are closed in } X\}\,.$$

Moreover, let

$$M_{L(X)} = \{\sigma \in \operatorname{Fin} L(X) : \sigma \text{ is essential}\}\,.$$

We have the following equality:

1.4. Theorem. *Let X be a space. Then*

$$\operatorname{Ord} M_{L(X)} \le n \quad \textit{iff} \quad \dim X \le n\,. \ \blacksquare$$

This inspired our definition of transfinite dimension dim.

1.5. Definition. Let X be a space. Then $\dim X = \operatorname{Ord} M_{L(X)}$.

We have seen in [B1] that

1.6. Theorem. *For a space X, $\dim X$ exists iff X is S-w.i.d.* $\blacksquare$

2. The subspace theorem. In [B1] we have proved the following subspace theorem on dim.

2.1. THEOREM [B1, 3.1.6]. *Let F be a closed subset of a space X. Then* $\dim F \le \dim X$. ∎

We also obtained the next result on open subspaces.

2.2. PROPOSITION [B1, 3.5.5]. *Let Y be a space and let X be an open subspace of Y such that* $\omega_0 \le \dim X < \infty$. *Moreover*, *assume* $\dim(Y - X)$ *is finite. Then* $\dim X = \dim Y$. ∎

2.3. EXAMPLE. Let us define Smirnov's spaces S_α for $\alpha < \omega_1$:

— $S_0 = \{0\}$,
— $S_{\alpha+1} = S_\alpha \times [-1, 1]$,
— if α is a limit then $S_\alpha = \omega(\bigoplus_{\beta<\alpha} S_\beta)$ (one-point compactification).

It is well known that each S_α is S-w.i.d. Moreover, $\dim S_\alpha = \alpha$. The proof is completely analogous to [B2; 4.1.11].

The discrete sum $Z = \bigoplus_{n<\omega_0} S_n$ is not S-w.i.d. but it is topologically contained in S_{ω_0}. Therefore we see that a subspace X of an S-w.i.d. space Y need not be S-w.i.d. itself.

So we have $Z \subset S_{\omega_0}$, $\dim Z = \infty$ and $\dim S_{\omega_0} = \omega_0$. ∎

The natural question arises whether for every S-w.i.d. space Y and every S-w.i.d. subspace X of Y we have $\dim X \le \dim Y$.

2.4. Remark. For the transfinite dimension function Ind, which has very similar properties to dim and classifies a similar category of infinite-dimensional spaces, L. A. Lyuksemburg proved

2.5. THEOREM [L]. *Let Y be a metric space and let X be a subspace of Y such that both* $\operatorname{Ind} X$ *and* $\operatorname{Ind} Y$ *exist. Then* $\operatorname{Ind} X \le \operatorname{Ind} Y$. ∎

Thus for Ind the subspace theorem holds.

We will answer our question in the negative by constructing a counterexample.

3. The counterexample. We will define S-w.i.d. spaces X and Y such that $X \subset Y \subset S_{\omega_0+1}$, $\dim X = \omega_0 + 1$ and $\dim Y = \omega_0$.

Put $T_n = S_n \times I$ for $n = 1, 2, \ldots$ and let

$$S_{\omega_0+1} = S_{\omega_0} \times I = \{p\} \times I \cup \bigoplus_{n<\omega_0} (S_n \times I) = \{p\} \times I \cup \bigoplus_{n<\omega_0} T_n .$$

For the construction of Y, for each $n = 1, 2, \ldots$ express the interval $I = [-1, 1]$ as $I = \bigcup_{m=-n+1}^{n} K_{nm}$ where $K_{nm} = [(m-1)/n, m/n]$. Fix some n and some $m \in \{-n+1, \ldots, n\}$. We may define O^i_{nm}, $i = 1, 2, \ldots$, as open intervals in I within K_{nm} such that $O^i_{nm} \cap O^j_{nm} = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots$ Let us also define

$$F^i_n = \bigtimes_{j=1}^{n} [-1/(i+1), 1/(i+1)] \subset S_n$$

for each $n = 1, 2, \ldots$ and $i = 1, 2, \ldots$ For each n let

$$\begin{aligned} Y_n = \{(x,y) \in S_n \times I : y \notin O^i_{nm} \\ \text{for every } m = -n+1, \ldots, n \text{ and every } i = 1, 2, \ldots\} \\ \cup \{(x,y) \in S_n \times I : y \in O^i_{nm} \text{ and } x \in F^i_n \\ \text{for some } m = -n+1, \ldots, n \text{ and some } i = 1, 2, \ldots\}\,. \end{aligned}$$

Clearly, $Y_n \subset T_n$.

We define $Y = \{p\} \times I \cup \bigoplus_{n<\omega_0} Y_n$. Then Y is closed subspace of S_{ω_0+1} and hence a compact metric space.

Now let us construct the subspace X of Y. Set

$$I(n) = \Big(\bigtimes_{i=1}^{n} \{0\}\Big) \times I \subset T_n \quad \text{ for } n = 1, 2, \ldots$$

Note that also $I(n) \subset Y_n$ for each $n = 1, 2, \ldots$ Define $X_n = Y_n - I(n)$ and $X = \{p\} \times I \cup \bigoplus_{n<\omega_0} X_n$. Clearly we have $X \subset Y \subset S_{\omega_0+1}$.

We will prove that $\dim Y = \omega_0$ but its subspace X is S-w.i.d. and has dimension $\dim X > \omega_0$.

3.1. Claim. $\dim Y = \omega_0$.

Clearly, $\dim Y \geq \omega_0$ since Y contains closed subspaces of arbitrary large finite dimension. To show $\dim Y \leq \omega_0$, let (A_0, B_0) be a pair of disjoint closed subsets in Y (in other words, $(A_0, B_0) \in L(Y)$). We prove that there exists some finite n_1 such that

$$\operatorname{Ord} M^{(A_0,B_0)}_{L(Y)} < n_1\,. \tag{3.1.1}$$

Consider $\{p\} \times I$ in Y as the unit interval $[-1,1]$. There exists some k such that for all the subintervals $K_{km} = [(m-1)/k, m/k]$, $m = -k+1, \ldots, k$, we have

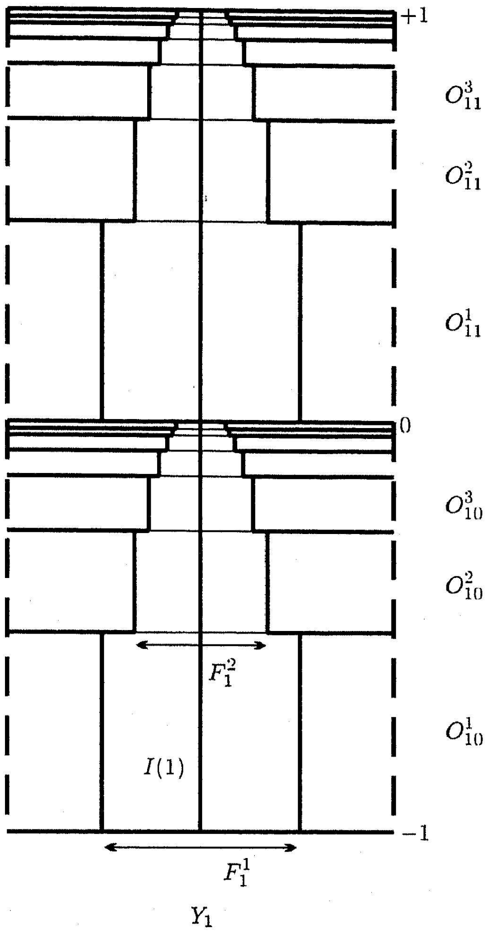
$$\text{if} \quad K_{km} \cap A_0 \neq \emptyset \quad \text{then} \quad (K_{k,m-1} \cup K_{km} \cup K_{k,m+1}) \cap B_0 = \emptyset\,.$$

For each m we can find some $n(m)$ such that for each $n \geq n(m)$ we have

$$\begin{aligned} &\text{if} \quad K_{km} \cap A_0 = \emptyset \quad \text{then} \quad (S_n \times K_{km}) \cap A_0 = \emptyset \quad \text{and} \\ &\text{if} \quad K_{km} \cap B_0 = \emptyset \quad \text{then} \quad (S_n \times K_{km}) \cap B_0 = \emptyset\,. \end{aligned} \tag{3.1.2}$$

Put $n_1 = \max\{2k, n(m) : m = -k+1, \ldots, k\}$. We prove that n_1 is as required. Take $\{(A_i, B_i)\}_{i=0}^{n_1} \in \operatorname{Fin} L(Y)$. If we prove

$$\{(A_i, B_i)\}_{i=0}^{n_1} \quad \text{is inessential}\,, \tag{3.1.3}$$



we are done. Let $G = \bigoplus_{n=0}^{n_1-1} Y_n$. Then $\dim G \le n_1$; hence by the theorem on partitions [E2; 3.2.6] we can find open sets U_i in G, $i = 0, \ldots, n_1$, such that

$$A_i \cap G \subset U_i \subset \overline{U}_i \subset G - B_i \quad \text{and} \quad \bigcap_{i=0}^{n_1} \operatorname{Fr} U_i = \emptyset\,. \tag{3.1.4}$$

Now consider $F = Y - G = \{p\} \times I \cup \bigoplus_{n_1 \le n < \omega_0} Y_n$. Since $\dim\{p\} \times I = 1$ we can find open sets V'_i in F, $i = 0, \ldots, n_1$, such that

$$A_i \cap F \subset V'_i \subset \overline{V}'_i \subset F - B_i \quad \text{and} \quad \bigcap_{i=0}^{n_1} \operatorname{Fr} V'_i \cap \{p\} \times I = \emptyset\,.$$

There is some $n_2 \ge n_1$ such that

$$\bigcap_{i=0}^{n_1} \operatorname{Fr} V'_i \cap \Big(\{p\} \times I \cup \bigoplus_{n_2 < n < \omega_0} Y_n\Big) = \emptyset\,.$$

For $i=0,\ldots,n$, let $V_i = V_i' \cap (\{p\} \times I \cup \bigoplus_{n_2<n<\omega_0} Y_n)$. Then

$$\bigcap_{i=0}^{n_1} \operatorname{Fr} V_i = \emptyset\,. \tag{3.1.5}$$

For each $n = n_1,\ldots,n_2$ consider the set Y_n and $\{(A_i,B_i)\}_{i=0}^{n_1}$ restricted to the subspace Y_n. The space Y_n is compact, (A_1,B_1) is a pair of disjoint closed sets and for every $m=-n+1,\ldots,n$ we have

$$\lim_{i\to\infty} \operatorname{diam}(F_n^i \times O_{nm}^i) = 0\,.$$

Since $n \geq n_1 \geq 2k$, for every $l = -k+1,\ldots,k$ there exists an $m(l) \in \{-n+1,\ldots,n\}$ such that $K_{nm(l)} \subset K_{kl}$.

We can find some i_0 such that for each $m(l)$, where $l=-k+1,\ldots,k$, we have

$$(F_n^{i_0} \times \overline{O}_{nm(l)}^{i_0}) \cap A_1 = \emptyset \quad \text{or} \quad (F_n^{i_0} \times \overline{O}_{nm(l)}^{i_0}) \cap B_1 = \emptyset\,. \tag{3.1.6}$$

Let $l_1,\ldots,l_p$ be all l, $-k+1 \leq l \leq k$, such that $K_{kl} \cap A_0 = \emptyset$ and $K_{kl} \cap B_0 = \emptyset$. Then by our choice of n_1 we have for $n=n_1,\ldots,n_2$, $j=1,\ldots,p$

$$(S_n \times K_{kl_j}) \cap A_0 = \emptyset \quad \text{and} \quad (S_n \times K_{kl_j}) \cap B_0 = \emptyset\,.$$

Since $\overline{O}_{nm(l_j)}^{i_0} \subset K_{nm(l_j)} \subset K_{kl}$, (3.1.2) and the construction of Y_n show that the set

$$P_n = \bigcup_{j=1}^{p} (F_n^{i_0} \times \overline{O}_{nm(l_j)}^{i_0})$$

forms a partition of Y_n between $A_0 \cap Y_n$ and $B_0 \cap Y_n$. Consequently, we can find some open set V_0^n in Y_n such that

$$A_0 \cap Y_n \subset V_0^n \subset \overline{V}_0^n \subset Y_n - B_0 \quad \text{and} \quad \operatorname{Fr} V_0^n \subset P_n\,. \tag{3.1.7}$$

By virtue of (3.1.6) and the normality of Y we can find an open set V_1^n such that

$$A_1 \cap Y_n \subset V_1^n \subset \overline{V}_1^n \subset Y_n - B_1 \quad \text{and} \quad \operatorname{Fr} V_1^n \cap P_n = \emptyset\,. \tag{3.1.8}$$

Then, by (3.1.7), also $\operatorname{Fr} V_0^n \cap \operatorname{Fr} V_1^n = \emptyset$.

Now for $i=2,\ldots,n_1$ let V_i^n be an open subset of Y_n such that

$$A_i \cap Y_n \subset V_i^n \subset \overline{V}_i^n \subset Y_n - B_i\,. \tag{3.1.9}$$

Then also $\bigcap_{i=0}^{n_1} \operatorname{Fr} V_i^n = \emptyset$. For $i=0,\ldots,n_1$ let

$$O_i = U_i \cup V_i \cup \Bigl(\bigcup_{n=n_1}^{n_2} V_i^n\Bigr)\,.$$

Then each O_i, $i=0,\ldots,n_1$, is an open set in Y such that according to (3.1.4), (3.1.5) and (3.1.9) we have

$$A_i \subset O_i \subset \overline{O}_i \subset Y - B_i\,. \tag{3.1.10}$$

Then also $\bigcap_{i=0}^{n_1} \operatorname{Fr} O_i = \emptyset$. Consequently, we have proven (3.1.3) and we are done. ■

3.2. Claim. $\dim X \geq \omega_0 + 1$.

For this, consider $A = S_{\omega_0} \times [-1, -1/2]$ and $B = S_{\omega_0} \times [1/2, 1]$. Clearly, (A, B) forms a pair of disjoint closed sets in $S_{\omega_0+1} = S_{\omega_0} \times I$. Let $A_0 = A \cap X$ and $B_0 = B \cap X$. We will prove our claim by showing that

$$\operatorname{Ord} M_{L(X)}^{(A_0,B_0)} \geq n \quad \text{for each } n < \omega_0 .$$

For this, fix some finite n and consider the subspace $T_{n+1} = S_{n+1} \times I$ within S_{ω_0+1}. Observe that $C = S_{n+1} \times \{-1\} \subset A$ and $D = S_{n+1} \times \{1\} \subset B$ can be considered as opposite faces of the cube T_{n+1}. We have

$$S_{n+1} = \{x = (x_1, \ldots, x_{n+1}) : x_i \in [-1, 1],\ i = 1, \ldots, n+1\} .$$

For $i = 1, \ldots, n+1$ put $C_i = \{x \in S_{n+1} : x_i = -1\}$ and $D_i = \{x \in S_{n+1} : x_i = 1\}$. The pairs $(C_i \times I, D_i \times I)$ also form pairs of opposite faces of the cube T_{n+1}.

In addition, define

$$F_i = \{x \in S_{n+1} : x_i \leq 0 \text{ and } x_j \geq x_i \text{ for } j \neq i\} ,$$
$$G_i = \{x \in S_{n+1} : x_i \geq 0 \text{ and } x_j \leq x_i \text{ for } j \neq i\} .$$

Clearly, $C_i \subset F_i$ and $D_i \subset G_i$ for $i = 1, \ldots, n+1$. Observe that $F_i \cap G_i = \{(0, \ldots, 0)\}$. Consequently, $F_i \times I \cap G_i \times I = \{(0, \ldots, 0)\} \times I = I(n+1)$, so that by the construction of X we have $F_i \times I \cap G_i \times I \cap X = \emptyset$.

Put $A_i = F_i \times I \cap X$ and $B_i = G_i \times I \cap X$ for $i = 1, \ldots, n+1$. It is sufficient to prove that

(3.2.1) $\qquad \{(A_i, B_i)\}_{i=0}^{n}$ is essential on the subspace X_{n+1}.

Assume the contrary. Then there are open sets U_i, $i = 0, \ldots, n$, in X_{n+1} such that

(3.2.2) $\qquad A_i \cap X_{n+1} \subset U_i \subset \overline{U}_i \subset X_{n+1} - B_i \quad \text{for } i = 0, \ldots, n$

and $\bigcap_{i=0}^{n} \operatorname{Fr} U_i = \emptyset$. According to [E2; 1.2.9], we can extend the U_i to open sets V_i in T_{n+1}, for $i = 0, \ldots, n$, such that

(3.2.3) $$[(F_i - \{(0, \ldots, 0)\}) \times I] \cap T_{n+1} \subset V_i \subset \overline{V}_i \subset T_{n+1} - [(G_i - \{(0, \ldots, 0)\}) \times I] \quad \text{for } i = 1, \ldots, n$$

and

$$C \subset V_0 \subset \overline{V}_0 \subset T_{n+1} - D , \qquad \operatorname{Fr} V_i \cap X_{n+1} = \operatorname{Fr} U_i \quad \text{for } i = 0, \ldots, n$$

so that $\bigcap_{i=0}^{n} \operatorname{Fr} V_i \cap X_{n+1} = \emptyset$. For brevity we put $E = \bigcap_{i=0}^{n} \operatorname{Fr} V_i$ so that

(3.2.4) $$E \cap X_{n+1} = \emptyset .$$

Let $O = \{x \in S_{n+1} : x_{n+1} < -3/4\}$. Then

$$C_{n+1} \subset O \subset \overline{O} \subset S_{n+1} - D_{n+1}\,.$$

Thus $O \times I$ is an open set in $S_{n+1} \times I = T_{n+1}$ such that

$$C_{n+1} \times I \subset O \times I \subset \overline{O} \times I \subset T_{n+1} - D_{n+1} \times I\,.$$

According to the construction of Y and X and the definition of O we have

$$(\overline{O} \times I) - X = (\overline{O} \times I) - Y = \overline{O} \times \bigcup_{\substack{i=1,2,\ldots \\ m=-n+1,\ldots,n}} O^i_{nm}\,.$$

Statement (3.2.4) then yields

$$(\overline{O} \times I) \cap E \subset \overline{O} \times \bigcup_{\substack{i=1,2,\ldots \\ m=-n+1,\ldots,n}} O^i_{nm}\,.$$

By the compactness of E we may assume

$$(\overline{O} \times I) \cap E \subset \overline{O} \times \bigcup_{j=1}^{k} O^{i_j}_{nm_j}\,. \tag{3.2.5}$$

Moreover, by (3.2.3) we have

$$E \subset \Big[\Big(S_{n+1} - \bigcup_{i=1}^{n} (F_i \cup G_i)\Big) \cup \{(0,\ldots,0)\}\Big] \times I\,. \tag{3.2.6}$$

For $j = 1,\ldots,k$ define

$$W_j = S_{n+1} - \bigg(\Big(\bigcup_{i=1}^{n} F_i \cup G_i\Big) \cup G_{n+1} \cup F^{i_j}_n\bigg)\,. \tag{3.2.7}$$

Then (3.2.6), $\overline{O} \cap G_{n+1} = \emptyset$, $\overline{O} \cap F^{i_j}_n = \emptyset$ and $(0,\ldots,0) \notin \overline{O}$ yield

$$E \cap (\overline{O} \times I) \subset W_j \times I \quad \text{for each } j = 1,\ldots,k\,.$$

Combining this and (3.2.5) we obtain

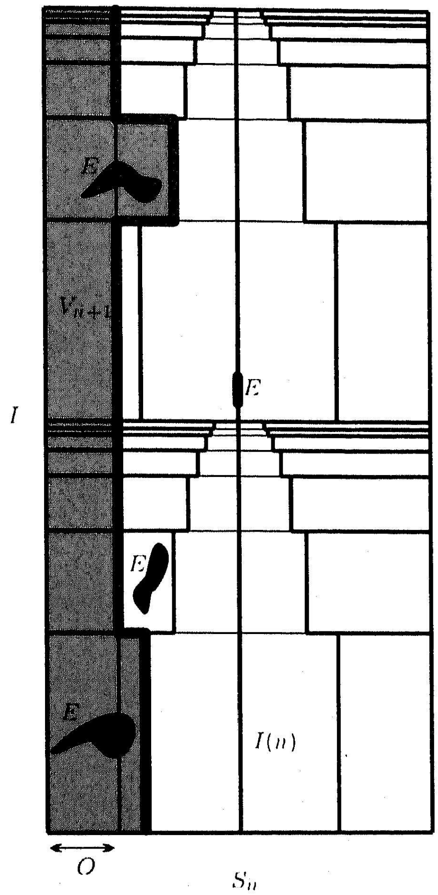
$$E \cap (\overline{O} \times I) \subset \bigcup_{j=1}^{k} (W_j \times O^{i_j}_{nm_j})\,. \tag{3.2.8}$$

From the definition (3.2.7) it is clear that for $j = 1,\ldots,k$

$$\mathrm{Fr}\, W_j \subset \bigcup_{i=1}^{n} (F_i \cup G_i) \cup F^{i_j}_n\,.$$

Together with (3.2.4) and (3.2.6) this gives us

$$E \cap (\mathrm{Fr}\, W_j \times O^{i_j}_{nm_j}) = \emptyset\,. \tag{3.2.9}$$



Now put

$$V_{n+1} = \bigcup_{j=1}^{k} (W_j \times O_{nm_j}^{i_j}) \cup (O \times I)\,.$$

Then according to (3.2.8) and (3.2.9)

$$\emptyset = \operatorname{Fr} V_{n+1} \cap E = \operatorname{Fr} V_{n+1} \cap \bigcap_{i=0}^{n} \operatorname{Fr} V_i = \bigcap_{i=0}^{n+1} \operatorname{Fr} V_i\,.$$

By the definition of O and since $W_j \cap D_{n+1} = \emptyset$ (since $D_{n+1} \subset G_{n+1}$) we have

$$C_{n+1} \times I \subset V_{n+1} \subset \overline{V}_{n+1} \subset T_{n+1} - D_{n+1} \times I\,.$$

We have already seen that $C \subset V_0 \subset \overline{V}_0 \subset T_{n+1} - D$. However, the sequence $\{(C, D), (C_1 \times I, D_1 \times I), \dots, (C_{n+1} \times I, D_{n+1} \times I)\}$ contains the pairs of opposite faces of the $(n+2)$-dimensional cube T_{n+1} and hence is essential

[E2; 1.8.1]. We reach a contradiction. Therefore assertion (3.2.1) is proven and we are done. ■

For the proof of our next claim we need the following proposition [B1; 5.3.2].

3.3. PROPOSITION. *Suppose that for a space* X *there exists a closed subspace* G *such that* $\dim G$ *is finite and for each* F *closed in* X *such that* $G \cap F = \emptyset$, $\dim F$ *is finite. Then* $\dim X \leq \omega_0 + \dim G$. ■

3.4. CLAIM. $\dim X \leq \omega_0 + 1$ *and consequently*, X *is S-w.i.d.*

Indeed, we apply Proposition 3.3. Let $G = \{p\} \times I$. Then $\dim G = 1$. Moreover, for each F closed in X and disjoint from G we have $F \subset \bigoplus_{m=0}^{n} X_m$ for some n so that $\dim F \leq n + 1 < \omega_0$. Hence $\dim X \leq \omega_0 + \dim G = \omega_0 + 1$. By Theorem 1.6 we see that X is S-w.i.d. ■

We conclude that $X \subset Y$ and $\dim Y = \omega_0$, but for X we have $\dim X = \omega_0 + 1$.

4. Relation with the characterization theorem. In [B2] we proved the following theorem:

4.1. THEOREM [B2; 4.2.1]. *Let* X *be a locally compact space and* $\alpha < \omega_1$. *Then* $\dim X \geq \alpha$ *iff* $X \times C$ *admits an essential map onto* J^α. ■

The transfinite cubes J^α and the concept of essential mappings to J^α are defined by D. W. Henderson [He]. The local compactness restriction follows from the use of the following product theorem in the proof.

4.2. THEOREM [B1; 3.5.7]. *Let* X *be a locally compact space. Then* $\dim X = \dim X \times C$. ■

In [Ch] V. A. Chatyrko proved the following compactification theorem.

4.3. THEOREM [Ch]. *Let* X *be an S-w.i.d. space. Then*:

(1) $\dim X = \dim \beta X$.

(2) *We can find a compact metric space* Y *such that* Y *contains* X *topologically and* $\dim Y \leq \dim X$. ■

Combining results 2.1, 4.2 and 4.3 he observes that using the compactification Y of X we can almost prove $\dim X = \dim X \times C$ without the requirement of local compactness:

$$\dim X \times C \overset{(2.1)}{\geq} \dim X \overset{(4.3)}{\geq} \dim Y \overset{(4.2)}{=} \dim Y \times C \overset{?}{\geq} \dim X \times C\,.$$

We only need the subspace theorem and we are done.

In this regard, but also considering the general requirements for a dimension function, it is a pity the subspace theorem does not generally hold.

The following question remains unanswered.

4.4. QUESTION. Can Theorems 4.1 and 4.2 be extended beyond the class of locally compact spaces?

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