FUNDAMENTA
MATHEMATICAE
140 (1992)

On the computation of the Nielsen numbers and the converse of the Lefschetz coincidence theorem

by

Peter Wong (Lewiston, Me.)

Abstract. Let $f, g : M_1 \to M_2$ be maps where M_1 and M_2 are connected triangulable oriented n-manifolds so that the set of coincidences $C_{f,g} = \{x \in M_1 \mid f(x) = g(x)\}$ is compact in M_1. We define a Nielsen equivalence relation on $C_{f,g}$ and assign the coincidence index to each Nielsen coincidence class. In this note, we show that, for $n \geq 3$, if $M_2 = \widetilde{M}_2/K$ where $\widetilde{M}_2$ is a connected simply connected topological group and K is a discrete subgroup then all the Nielsen coincidence classes of f and g have the same coincidence index. In particular, when M_1 and M_2 are compact, f and g are deformable to be coincidence free if the Lefschetz coincidence number $L(f, g)$ vanishes.

0. Introduction. In topological fixed point theory, the Lefschetz fixed point theorem states that the vanishing of the Lefschetz number $L(f)$ of a self-map $f : X \to X$ on a compact manifold is necessary to deforming f to be fixed point free. The converse, however, is false in general. If X satisfies the so-called Jiang condition, $L(f) = 0$ is also sufficient. The essence of the Jiang condition is that all fixed point classes have the same index. Thus the Nielsen number $N(f)$ divides $L(f)$ which is easier to compute. A theorem of Anosov asserts that $|L(f)| = N(f)$ for any self-map f on a nilmanifold. This result is recently generalized to coincidences independently by Jezierski [7], McCord [10], Brooks and Wong [3]. For nilmanifolds, coincidence classes have the same index.

The technique used in [3] is a local version of the root theory developed by R. Brooks [1] (see also [9,V]). It is shown in [3] that if $\varphi : N \to M$ is a map from an oriented n-manifold N to an oriented triangulable n-manifold M such that the set of roots $R_\varphi = \{x \in N \mid \varphi(x) = e\}$ is compact, then all the root classes have the same index. The purpose of this note is to use this fact about roots to show that (local) coincidence and hence (local) fixed point

---

1991 *Mathematics Subject Classification*: Primary 55M20.
Key words and phrases: fixed points, coincidences, roots, Lefschetz number, Nielsen number.

classes have the same index if the range is an oriented triangulable manifold of the form $\widetilde{M}/K$ where $\widetilde{M}$ is a connected simply connected topological group and K is a discrete subgroup. In particular, we obtain a local version of the converse of the Lefschetz fixed point theorem (Cor. 2.3).

In §1, we outline the coincidence and root theories necessary for §2. We also study how coincidences can be changed to roots. In §2, we prove our main results concerning the Nielsen numbers for coincidences as well as for fixed points.

Throughout H_* and H^* will denote singular homology and cohomology with integer coefficients.

The author would like to thank the referee for a number of helpful suggestions.

1. Preliminaries. Throughout this section, M_2 is a connected triangulable oriented n-manifold ($n \geq 3$) and M_1 is an oriented n-manifold. Suppose that $f, g : M_1 \to M_2$ are maps such that $C_{f,g} = \{x \in M_1 \mid f(x) = g(x)\}$ is compact in M_1. Following [12, Ch. 6], we define the *coincidence index* of f and g in M_1 to be the integer

$$I(f, g; M_1) = \langle (f, g)^* \mu_2, o_C \rangle$$

where $o_C \in H_n(M_1, M_1 - C_{f,g})$ is the fundamental homology class around $C_{f,g}$, $\mu_2 \in H^n(M_2 \times M_2, M_2 \times M_2 - \Delta(M_2))$ is the Thom class and $(f, g) : M_1 \to M_2 \times M_2$ is the map given by $(f, g)(x) = (f(x), g(x))$. The index $I(f, g; M_1)$ satisfies the homotopy invariance and the additivity properties (see [12, 6.1, 6.4]).

Two coincidences $x, y \in C_{f,g}$ are said to be *Nielsen equivalent* if there exists a path $\alpha : [0, 1] \to M_1$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $f \circ \alpha \sim g \circ \alpha$ (rel. endpoints) in M_2. This is an equivalence relation on $C_{f,g}$ and there are a finite number of *Nielsen coincidence* (equivalence) *classes* each of which is open and closed in $C_{f,g}$ (see [2], [6]). Let N be a Nielsen coincidence class of f and g. Denote by $I(f, g; N)$ the *index* of N given by

$$I(f, g; N) = I(f, g; U_N) = \langle (f, g)^* \mu_2, o_N \rangle$$

where U_N is an open neighborhood of N in M_1 which does not contain any other coincidences of f and g. Note that $I(f, g; N)$ is independent of the choice of U_N.

A coincidence class N is *essential* if $I(f, g; N) \neq 0$. The *local* Nielsen coincidence number of f and g, denoted by $N(f, g; M_1)$, is the number of essential Nielsen coincidence classes of f and g in M_1 (compare [2]). When $M_1 \subset M_2$ and g is the inclusion, the notion of coincidence index and class reduces to that of fixed point index and class ([6]). When g is the constant map $g(x) = e$, we recover the local Nielsen root theory in [3] where the preferred generator in $H^n(M_2, M_2 - e)$ is given by $i^*(\mu_2)$, the image of the Thom class under the inclusion induced by $i : (M_2, M_2 - e) \to (M_2 \times M_2, M_2 \times M_2 - \Delta(M_2))$.

Suppose that M_1 is connected, $\widetilde{M_1} \xrightarrow{p_1} M_1$ and $\widetilde{M_2} \xrightarrow{p_2} M_2$ are universal covers. In addition, we assume that $C_{f,g} \neq \emptyset$ and $M_2 = \widetilde{M_2}/K$ where $\widetilde{M_2}$ is a topological group with unit $\widetilde{e}$ and K is a discrete subgroup. Let $\widetilde{f}, \widetilde{g} : \widetilde{M_1} \to \widetilde{M_2}$ be lifts of f and g, respectively. Define $\eta : \widetilde{M_1} \to M_2$ by

$$\eta(\widetilde{x}) = p_2[(\widetilde{f}(\widetilde{x}))^{-1}(\widetilde{g}(\widetilde{x}))].$$

It is easy to see that $\eta(\widetilde{x}) = e$ if, and only if, $f(p_1(\widetilde{x})) = g(p_1(\widetilde{x}))$ where $e = p_2(\widetilde{e})$. Let C be a coincidence class of f and g. Choose $\widetilde{x} \in M_1$ such that $p_1(\widetilde{x}) = x \in C$. Let $y \in C$ and $\alpha : [0,1] \to M_1$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $f \circ \alpha \sim g \circ \alpha$. Suppose that $H : [0,1] \times [0,1] \to M_2$ so that $H(s,0) = g \circ \alpha(s)$ and $H(s,1) = f \circ \alpha(s)$. Also, $H(0,t) = f(x) = g(x)$ and $H(1,t) = f(y) = g(y)$. Let $\widetilde{\alpha} : [0,1] \to \widetilde{M_1}$ be the lift of α with $\widetilde{\alpha}(0) = \widetilde{x}$ and $\widetilde{y} = \widetilde{\alpha}(1)$. Define $\overline{H} : [0,1] \times [0,1] \to M_2$ by

$$(*) \qquad \overline{H}(s,t) = p_2([\widetilde{f}(\widetilde{\alpha}(s))]^{-1}\widetilde{H}(s,t))$$

where $\widetilde{H}$ is the lift of H such that $\widetilde{H}(s,0) = \widetilde{g} \circ \widetilde{\alpha}(s)$. It follows that

$$\overline{H}(s,0) = \eta \circ \widetilde{\alpha}(s), \quad \overline{H}(s,1) = e, \quad \overline{H}(0,t) = e, \quad \overline{H}(1,t) = e\,.$$

Thus $\widetilde{x}$ and $\widetilde{y}$ belong to the same root class. Conversely, if $\widetilde{x}$ and $\widetilde{y}$ are Nielsen equivalent as roots of η then by $(*)$, the coincidences $x = p_1(\widetilde{x})$ and $y = p_1(\widetilde{y})$ are Nielsen equivalent as coincidences of f and g.

Let $C_1, \ldots, C_k$ be the distinct coincidence classes of f and g. Choose $\widetilde{x}_1, \ldots, \widetilde{x}_k \in \widetilde{M_1}$ such that $p_1(\widetilde{x}_i) \in C_i$ for $i = 1, \ldots, k$. For any path α_i in M_1 from x_i to $y_i \in C_i$ such that $f \circ \alpha_i \sim g \circ \alpha_i$, we let $\widetilde{\alpha}_i$ be the path in $\widetilde{M_1}$ covering α_i starting at $\widetilde{x}_i$. We may assume that α_i contains no other coincidences of f and g and hence $\widetilde{\alpha}_i$ contains no other coincidences of η. Let $\widetilde{C}_i = \{\widetilde{y} \in \widetilde{M_1} \mid \widetilde{y}$ is an endpoint of some $\widetilde{\alpha}_i\}$. There is a bijection between the sets $\widetilde{C}_i$ and C_i for each $i = 1, \ldots, k$. Let W be a connected open neighborhood of the union of the paths $\widetilde{\alpha}$ so that $\mathrm{cl}(W) \cap R_\eta = \bigcup_i \widetilde{C}_i$ and let $\varphi : W \to M_2$ be the restriction $\varphi = \eta|W$.

We have just shown the following

Proposition 1.1. *There is a* 1-1 *correspondence between the roots* R_φ *of* φ *and the coincidences* $C_{f,g}$ *of* f *and* g. *Furthermore*, *there is a* 1-1 *correspondence between the root classes* $\widetilde{C}_i$ *and the coincidence classes* C_i *given by* $C_i = p_1(\widetilde{C}_i)$ *for* $i = 1, \ldots, k$.

Remark. This 1-1 correspondence holds for $n = 2$ and is independent of the choice of $\alpha_i, \widetilde{\alpha}_i$ and W.

Let us state the following result which is essential for §2.

LEMMA 1.2 [3, 2.3]. *Let W be a connected open subset of an oriented n-manifold X. Let $\varphi : W \to Y$ be a map where Y is an oriented triangulable n-manifold with $e \in Y$. Suppose that R_φ is compact and α is an essential root class of φ. Then for any root class β, $\omega(\varphi;\alpha) = \omega(\varphi;\beta)$, i.e., α and β have the same root index.*

2. Main results. Here is our main theorem.

THEOREM 2.1. *Let $f, g : M_1 \to M_2$ be maps from an oriented n-manifold $(n \geq 3)$ M_1 to a connected triangulable oriented n-manifold $M_2 = \widetilde{M}_2/K$ where $\widetilde{M}_2$ is a connected simply connected topological group and K is a discrete subgroup. Suppose that $C_{f,g}$ is compact in M_1. If $I(f,g;M_1) = 0$ then $N(f,g;M_1) = 0$. Otherwise, $N(f,g;M_1) > 0$ and $N(f,g;M_1)$ divides $I(f,g;M_1)$.*

Proof. By additivity,

$$I(f,g;M_1) = \sum_{N \in \mathcal{N}(f,g;M_1)} I(f,g;N)$$

where $\mathcal{N}(f,g;M_1)$ is the set of essential Nielsen coincidence classes of f and g. It suffices to show that if there exists an essential Nielsen class then all the Nielsen classes are essential of the same index. Thus, we need to show that the coincidence index $I(f,g;C_i)$ coincides with the root index $\omega(\varphi;\widetilde{C}_i)$ in 1.1 and then apply 1.2.

Suppose that $C = C_i$ is an essential coincidence class of f and g and $\widetilde{C}$ is the corresponding root class of φ in W. Since p_1 is a covering map and $p_1|\widetilde{C} : \widetilde{C} \to C$ is a homeomorphism, we can choose an open neighborhood $\widetilde{V}$ of $\widetilde{C}$ in W such that $\mathrm{cl}(\widetilde{V})$ does not contain any other roots of φ and $p_1|\widetilde{V}$ is a homeomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{V}, \widetilde{V} - \widetilde{C} & \xrightarrow{(\tilde{f},\tilde{g})} & \widetilde{M}_2 \times \widetilde{M}_2, \widetilde{M}_2 \times \widetilde{M}_2 - (p_2 \times p_2)^{-1}(\Delta M_2) & \xleftarrow{l} & \widetilde{M}_2, \widetilde{M}_2 - p_2^{-1}(e) \\
{\scriptstyle p_1|\tilde{V}}\downarrow & & {\scriptstyle p_2 \times p_2}\downarrow & & {\scriptstyle p_2}\downarrow \\
p_1(\widetilde{V}), p_1(\widetilde{V}) - C & \xrightarrow{(f,g)} & M_2 \times M_2, M_2 \times M_2 - \Delta M_2 & \xleftarrow{k} & M_2, M_2 - e
\end{array}$$

where $l(\widetilde{x}) = (\widetilde{e}, \widetilde{x})$ and $k(x) = (e,x)$. Let $\sigma : \widetilde{M}_2 \times \widetilde{M}_2, \widetilde{M}_2 \times \widetilde{M}_2 - (p_2 \times p_2)^{-1}(\Delta M_2) \to \widetilde{M}_2, \widetilde{M}_2 - p_2^{-1}(e)$ be given by $\sigma(\widetilde{y}_1, \widetilde{y}_2) = \widetilde{y}_1^{-1}\widetilde{y}_2$.

Then we have

$$k \circ p_2 \circ \sigma \circ (\widetilde{f}, \widetilde{g}) = k \circ \varphi\,,$$

which implies

$$\varphi^* \circ k^* = (\widetilde{f}, \widetilde{g})^* \circ \sigma^* \circ p_2^* \circ k^*\,.$$

A straightforward argument using the relative Hurewicz Isomorphism Theorem shows that l^* is an isomorphism and hence $\sigma^* = (l^*)^{-1}$. Therefore,

$$\varphi^* \circ k^* = (\widetilde{f}, \widetilde{g})^* \circ \sigma^* \circ l^* \circ (p_2 \times p_2)^* = (\widetilde{f}, \widetilde{g})^* \circ (p_2 \times p_2)^* = (p_1|\widetilde{V})^* \circ (f, g)^*$$

It follows that

$$\langle \varphi^* \circ k^*(\mu_2), o_{\tilde{C}} \rangle = \langle (p_1|\widetilde{V})^* \circ (f, g)^*(\mu_2), o_{\tilde{C}} \rangle ,$$

which yields

$$\omega(\varphi; \widetilde{C}) = \langle \varphi^* \circ k^* \mu_2, o_{\tilde{C}} \rangle = \langle (f, g)^* \mu_2, o_C \rangle = I(f, g; C)$$

where $\mu_2 \in H^n(M_2 \times M_2, M_2 \times M_2 - \Delta(M_2))$ is the Thom class, $o_{\tilde{C}}$ and o_C are the fundamental homology classes around $\widetilde{C}$ and C, respectively. ∎

As a consequence of Theorem 2.1, we obtain

COROLLARY 2.2. *Let M_1, M_2 be closed connected orientable triangulable n-manifolds ($n \geq 3$). Suppose that $M_2 = \widetilde{M}_2/K$ where $\widetilde{M}_2$ is a connected simply connected topological group and K is a discrete subgroup. Then for any maps $f, g : M_1 \to M_2$, the Lefschetz coincidence number $L(f, g)$ vanishes if, and only if, f and g are deformable to be coincidence free.*

P r o o f. By the Lefschetz coincidence theorem ([12, Ch. 6]), if $f \sim f'$ and $g \sim g'$ such that $C_{f',g'} = \emptyset$ then $L(f', g') = L(f, g) = 0$. Conversely, if $L(f, g) = 0$ then by Thm. 2.1, the Nielsen coincidence number $N(f, g) = 0$. By a result of Schirmer [11], f and g are deformable to be coincidence free. ∎

R e m a r k. For the converse in Cor. 2.2, we may homotope only one of the two maps to remove coincidences using a result of R. Brooks.

When we apply 2.1 to the fixed point situation, we obtain

COROLLARY 2.3. *Let $f : U \subset M \to M$ be a compactly fixed map* (*i.e.*, $\mathrm{Fix}\, f$ *is compact*) *defined on an open subset U of a connected oriented triangulable n-manifold ($n \geq 3$) $M = \widetilde{M}/K$ where $\widetilde{M}$ is a connected simply connected topological group and K is a discrete subgroup. If $I(f; U) \neq 0$ then all the local Nielsen fixed point classes are essential of the same index. If $I(f; U) = 0$ then f is homotopic via a compactly fixed homotopy to a fixed point free map.*

P r o o f. It follows from Thm. 2.1 and [6, 5.13]. ∎

COROLLARY 2.4. *Let M_1, M_2 be as in* 2.2 *and $M_1 = M_2 = M$. For any self-map $f : M \to M$, either* (i) *if $L(f) = 0$ then $N(f) = 0$ or* (ii) *if $L(f) \neq 0$, then $N(f) > 0$ and $L(f) = I \cdot N(f)$ for some nonzero integer I.*

R e m a r k. The class of orientable manifolds of the form $M = \widetilde{M}/K$ studied in this note is not contained in the class of Jiang spaces. All nilmanifolds and orientable infranilmanifolds, for instance, are not necessarily Jiang spaces.

References

[1] R. Brooks, *Certain subgroups of the fundamental group and the number of roots of* $f(x) = a$, Amer. J. Math. 95 (1973), 720–728.

[2] R. Brooks and R. Brown, *A lower bound for the* Δ*-Nielsen number*, Trans. Amer. Math. Soc. 143 (1969), 555–564.

[3] R. Brooks and P. Wong, *On changing fixed points and coincidences to roots*, Proc. Amer. Math. Soc., to appear.

[4] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott & Foresman, Glenview, Ill., 1971.

[5] A. Dold, *Lectures on Algebraic Topology*, Springer, Berlin 1972.

[6] E. Fadell and S. Husseini, *Local fixed point index theory for non simply connected manifolds*, Illinois J. Math. 25 (1981), 673–699.

[7] J. Jezierski, *The Nielsen number product formula for coincidences*, Fund. Math. 134 (1989), 183–212.

[8] B. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math. 14, Amer. Math. Soc., 1982.

[9] T. Kiang, *The Theory of Fixed Point Classes*, Springer, Berlin 1989.

[10] C. McCord, *Lefschetz and Nielsen coincidence numbers on nilmanifolds and solvmanifolds*, Topology Appl., to appear.

[11] H. Schirmer, *Mindestzahlen von Koinzidenzpunkten*, J. Reine Angew. Math. 194 (1955), 21–39.

[12] J. Vick, *Homology Theory* , Academic Press, New York 1973.

DEPARTMENT OF MATHEMATICS
BATES COLLEGE
LEWISTON, MAINE 04240
U.S.A.

Received 24 July 1991;
in revised form 8 October 1991