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A characterization of dendroids by the n-connectedness of the Whitney levels

by

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Abstract. Let X be a continuum. Let $C(X)$ denote the hyperspace of all subcontinua of X. In this paper we prove that the following assertions are equivalent: (a) X is a dendroid, (b) each positive Whitney level in $C(X)$ is 2-connected, and (c) each positive Whitney level in $C(X)$ is ∞-connected (n-connected for each $n \geq 0$).

Introduction. Throughout this paper X will denote a *continuum* (i.e., a compact connected metric space) with metric d. Let $C(X)$ be the hyperspace of all subcontinua of X with the Hausdorff metric $\mathcal{H}$. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to \mathbb{R}$ satisfying: (a) $\mu(\{x\}) = 0$ for each $x \in X$, (b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$, and (c) $\mu(X) = 1$. A (*positive*) *Whitney level* is a set of the form $\mu^{-1}(t)$ where $0 \leq t \leq 1$ (resp. $0 < t \leq 1$). S^n denotes the n-sphere. A space Y is *n-connected* if, for every $0 \leq i \leq n$, each map $f : S^i \to Y$ is null homotopic; Y is *∞-connected* if it is n-connected for each n. A topological property P is a *Whitney property* provided whenever a continuum X has property P, so does every positive Whitney level in $C(X)$. A *map* is a continuous function. The unit closed interval is denoted by I, and the set of positive integers by $\mathbb{N}$.

Positive Whitney levels are continua [1]. Answering questions by J. Krasinkiewicz and S. B. Nadler, Jr., in [9] A. Petrus showed that if D is a 2-cell, then there exists a Whitney level $\mathcal{A}$ in $C(D)$ which is not contractible, in fact $\mathcal{A}$ has non-trivial fundamental group and non-trivial first singular homology group.

The main theorem in this paper is:

THEOREM. *The following assertions are equivalent*:

(i) *X is a dendroid*,
(ii) *Each positive Whitney level in $C(X)$ is 2-connected.*
(iii) *Each positive Whitney level in $C(X)$ is ∞-connected.*

We divide the proof into two independent sections. In the first section we prove that (ii)⇒(i), and in the second one we prove that (i)⇒(iii).

1. 2-connectedness of Whitney levels implies that X is a dendroid. We will need the following lemma.

1.1. LEMMA. *Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map. Let $t_0 \in I$. Let Y be a continuum such that $C(Y)$ is contractible. Then every map $f : Y \to \mu^{-1}([0,t_0])$ is homotopic to a map $g : Y \to \mu^{-1}([0,t_0])$ such that* $\operatorname{Im} g \subset \mu^{-1}(t_0)$.

Proof. Take a map $f : Y \to \mu^{-1}([0,t_0])$. Since $C(Y)$ is contractible, by [12, Thm. 16.7] there exists a map $F : Y \times I \to C(Y)$ such that, for every $y \in Y$, $F(y,0) = \{y\}$, $F(y,1) = Y$ and $s \le t$ implies that $F(y,s) \subset F(y,t)$.

We distinguish two cases:

(a) $\mu(\bigcup f(Y)) = \mu(\bigcup\{f(y) \in C(X) : y \in Y\}) \ge t_0$. Define $G : Y \times I \to C(X)$ by $G(y,t) = \bigcup f(F(y,t)) = \bigcup\{f(v) \in C(X) : v \in F(y,t)\}$ Then G is a map such that $G(y,0) = f(y)$ and $G(y,1) = \bigcup f(Y)$ for every $y \in Y$. Define $K : Y \times I \to \mu^{-1}([0,t_0])$ by

$$K(y,t) = \begin{cases} G(y,t) & \text{if } \mu(G(y,t)) \le t_0, \\ G(y,s) & \text{if } \mu(G(y,t)) \ge t_0, \end{cases}$$

where $s \in [0,t_0]$ is chosen in such a way that $\mu(G(y,s)) = t_0$.

Then $K(y,0) = f(y)$ and $K(y,1) \in \mu^{-1}(t_0)$, and we define $g : Y \to \mu^{-1}([0,t_0])$ by $g(y) = K(y,1)$ for every $y \in Y$.

(b) $\mu(\bigcup f(Y)) \le t_0$. Defining G as in (a), we see that f is homotopic (within $\mu^{-1}([0,t_0])$) to the constant map $y \to \bigcup f(Y)$. Since $\bigcup f(Y) \in \mu^{-1}([0,t_0])$, there exists an ordered arc ([12, Thm. 1.8]) joining $\bigcup f(Y)$ to an element $A_0 \in \mu^{-1}(t_0)$ (within $\mu^{-1}([0,t_0])$). Then we complete the proof of the lemma by defining $g(y) = A_0$ for every $y \in Y$.

We will use the following notions related to Whitney levels:

The *space of Whitney levels*, $N(X)$, of X is defined by $N(X) = \{\mathcal{A} \in C(C(X)) : \mathcal{A}$ is a Whitney level in $C(X)\}$. This space was introduced in [5]–[7]. In [7, Lemma 2.2] it was proved that an equivalent metric for $N(X)$ is $\mathcal{H}^*(\mathcal{A},\mathcal{B}) = \max\{\mathcal{H}(A,B) : A \in \mathcal{A},\ B \in \mathcal{B}$ and $A \subset B\}$. A partial order for $N(X)$ is defined in [5] by $\mathcal{A} \le \mathcal{B}$ if and only if for each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $A \subset B$. If $\mathfrak{A} \subset N(X)$ is compact and γ is an ordered arc in $C(X)$ beginning with a singleton and ending with X, then ([5]) $A_\gamma = \bigcap\{A \in \gamma :$ there exists $\mathcal{A} \in \mathfrak{A}$ such that $A \in \mathcal{A}\} \in \gamma \cap \mathcal{B}$ for some $\mathcal{B} \in \mathfrak{A}$. Finally, in [5] it is shown that $\inf(\mathfrak{A}) = \{\mathfrak{A}_\gamma \in C(X) : \gamma$ is an ordered arc in $C(X)$ beginning with a singleton and ending with $X\}$ is a Whitney level which is the infimum, in $(N(X),\le)$, of the set $\mathfrak{A}$.

CONVENTIONS. $\mathbb{R}^n$ denotes the Euclidean n-dimensional space. $\mathsf{e}:\mathbb{R}\to S^1$ denotes the exponential map defined by $\mathsf{e}(t)=(\cos t,\sin t)$. D^2 is the unit disk in $\mathbb{R}^2$. If Y is a topological space, a map $f:Y\to S^1$ *can be lifted* ($f\simeq 1$) if there exists a map $g:Y\to\mathbb{R}$ such that $\mathsf{e}\circ g=f$ (equivalently, if f is null homotopic, see [10, Lemma 5]). If $A\in C(X)$ and $\varepsilon>0$ then $N(\varepsilon,A)$ denotes the set $\{x\in X:$ there exists $y\in A$ such that $\mathsf{d}(x,y)<\varepsilon\}$ and $B(A,\varepsilon)$ denotes the set $\{B\in C(X):\mathcal{H}(A,B)<\varepsilon\}$. 2^X denotes the hyperspace of all closed nonempty connected subsets of X.

From now on, in this section, we will suppose that if $\mathcal{A}$ is a positive Whitney level in $C(X)$, then every map $f:S^i\to\mathcal{A}$ is null homotopic for $i=1,2$ (we are not supposing yet that $\mathcal{A}$ is pathwise connected).

1.2. THEOREM. *X is hereditarily unicoherent.*

Proof. Suppose, on the contrary, that there exist $A_1,B_1\in C(X)$ such that $A_1\cap B_1$ is not connected. Let $H,K\in 2^X$ be such that $H\cap K=\emptyset$ and $A_1\cap B_1=H\cup K$. We will construct:

(a) A Whitney map ω for $C(X)$,
(b) A number $t_0\in(0,1]$,
(c) Two open subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\omega^{-1}([0,t_0])$,
(d) A map $\lambda:S^1\to\mathcal{V}_1\cap\mathcal{V}_2$ and
(e) A map $h_1:\mathcal{V}_1\cap\mathcal{V}_2\to S^1$

such that $\omega^{-1}([0,t_0])=\mathcal{V}_1\cup\mathcal{V}_2$, $h_1\circ\lambda$ is not homotopic to a constant and, for $i=1,2$, $\lambda:S^1\to\mathcal{V}_i$ can be extended to the disk D^2. Then, using Lemma 1.1 and a Mayer–Vietoris type sequence we will obtain a contradiction. The construction of these elements is divided into a sequence of steps.

A. There exists $A_0\in C(X)$ such that $A_0\subset A_1$, $A_0\cap H\neq\emptyset$, $A_0\cap K\neq\emptyset$ and A_0 is minimal with these properties.

To construct A_0, choose a Whitney map μ for $C(X)$. Let $t_1=\min\{\mu(A)\in I:A\subset A_1,\ A\cap H\neq\emptyset$ and $A\cap K\neq\emptyset\}$. Take $A_0\in C(X)$ such that $\mu(A_0)=t_1$.

B. Let $H_1=A_0\cap H$ and $K_1=A_0\cap K$. Then there exists $B_0\in C(X)$ such that $B_0\subset B_1$, $B_0\cap H_1\neq\emptyset$, $B_0\cap K_1\neq\emptyset$ and B_0 is minimal with these properties. Define $H_0=H_1\cap B_0$ and $K_0=K_1\cap B_0$. Then $A_0\cap B_0=H_0\cup K_0$, $H_0\cap K_0=\emptyset$ and $H_0,K_0\in 2^X$. Furthermore, if A (resp. B) is a proper subcontinuum of A_0 (resp. B_0), then $A\cap H_0=\emptyset$ (resp. $B\cap H_0=\emptyset$) or $A\cap K_0=\emptyset$ (resp. $B\cap K_0=\emptyset$).

C. Let $E=A_0\cup B_0$. Let $S^+=\{(x,y)\in S^1:y\geq 0\}$ and $S^-=\{(x,y)\in S^1:y\leq 0\}$. Since X is metric, Tietze's Theorem implies that there exists a map $f_0:E\to S^1$ such that $H_0=f_0^{-1}((-1,0))$, $K_0=f_0^{-1}((1,0))$, $f_0(A_0)\subset S^+$ and $f_0(B_0)\subset S^-$. Since S^1 is an ANR (metric), there exists an open subset U in X and a map $f : U \to S^1$ such that $E \subset U$ and $f|E = f_0$. Then the Unique Lifting Theorem implies that $f|E$ cannot be lifted.

D. If A is a proper subcontinuum of E, then $f|A \simeq 1$.

To see this, suppose, for example, that A_0 is not contained in A. Let $A_H = \bigcup\{L \in C(X) : L$ is a component of $A \cap A_0$ and $L \cap H_0 \neq \emptyset\}$ and let $A_K = \bigcup\{L \in C(X) : L$ is a component of $A \cap A_0$ and $L \cap H_0 = \emptyset\}$. Then A_H is closed in X. We will prove that A_K is closed. If $A \subset A_0$, then either $A_K = A$ or $A_K = \emptyset$. Suppose then that A is not contained in A_0. If L is a component of $A \cap A_0$, then ([12, Thm. 20.2]) L intersects either H_0 or K_0 but not both of them. If $x \in \mathrm{Cl}(A_K)$ then $x = \lim x_n$ where $(x_n)_n$ is a sequence such that, for each n, $x_n \in L_n$ for some component L_n of $A_0 \cap A$ such that $L_n \cap H_0 = \emptyset$ (then $L_n \cap K_0 \neq \emptyset$). Therefore the component L of $A_0 \cap A$ which contains x intersects K_0. Hence $L \cap H_0 = \emptyset$ and $x \in A_K$. The minimality of A_0 implies that $A_H \cap K_0 = \emptyset$. Notice that $A_H \cap A_K = \emptyset$ and $A_K \cap H_0 = \emptyset$.

Thus $A = A_H \cup A_K \cup (A \cap B_0)$. Since $A_H, A_K \subset A_0 = f^{-1}(S^+)$ and $A \cap B_0 \subset B_0 = f^{-1}(S^-)$, we find that $f|A_H, f|A_K$ and $f|(A \cap B_0)$ can be lifted. Since $A_H \cap A \cap B_0 \subset H_0 = f^{-1}((-1,0))$, $A_K \cap A \cap B_0 \subset K_0 = f^{-1}((1,0))$ and $A_H \cap A_K = \emptyset$, it follows that $f|A$ can be lifted.

E. There exists an open subset $\mathcal{V}$ of $C(X)$ such that $C(E) - \{E\} \subset \mathcal{V}$ and for each $A \in \mathcal{V}$, $A \subset U$ and $f|A \simeq 1$.

Indeed, let $A \in C(E) - \{E\}, f|A \simeq 1$. Then ([2]) there exists an open subset U_A of U containing A such that $f|U_A \simeq 1$. Therefore there exists $\varepsilon_A > 0$ such that if $\mathcal{H}(A, B) < \varepsilon_A$, then $f|B \simeq 1$. Define $\mathcal{V} = \{B \in C(X) : \mathcal{H}(A, B) < \varepsilon_A$ for some $A \in C(E) - \{E\}\}$.

F. Fix a Whitney map $\nu_0 : 2^X \to I$. Let $\nu = \nu_0|C(X)$. Define $t^* = \nu(E) > 0$ and define $h : C(X) \times I \times (0, t^*) \to \mathbb{R}$ by $h(A, t, s) = \min\{\nu(A)t^*/s, \nu_0(A \cup E) + t(\nu(A) - \nu(E))\}$. Then h is continuous and $h(E, t, s) = t^*$ for every $t \in I$ and $s \in (0, t^*)$. Fix $t \in (0, 1]$ and $s \in (0, t^*)$. Then the map $A \to h(A, t, s)/h(X, t, s)$ from $C(X)$ to I is a Whitney map.

G. If $0 < s_1 < s_2 < t^*$, then there exists $r \in (0, 1]$ such that if $0 < t \leq r$, $A \in \nu^{-1}([s_1, s_2])$ and $h(A, t, s_1) < t^*$, then $A \in \mathcal{V}$.

Indeed, otherwise we can choose sequences $(t_n)_n \subset (0, 1]$ and $(D_n)_n \subset \nu^{-1}([s_1, s_2])$ such that $t_n \to 0$ and $h(D_n, t_n, s_1) < t^*$ and $D_n \notin \mathcal{V}$ for all n. We may suppose that $D_n \to A$ for some $A \in \nu^{-1}([s_1, s_2])$. Then $A \notin \mathcal{V}$ and $\nu(A) \leq s_2 < \nu(E)$. Thus A is not contained in E and $\nu_0(A \cup E) > t^*$. Since $t_n(\nu(D_n) - \nu(E)) + \nu_0(D_n \cup E) \to \nu_0(A \cup E)$ and $\nu(D_n)t^*/s_1 \geq t^*$, we conclude that there exists $n \in \mathbb{N}$ such that $h(D_n, t_n, s_1) \geq t^*$. This contradiction completes the proof of G.

H. Choose a sequence $(s_n)_n \subset (0, t^*)$ such that $s_n \to t^*$ and $0 < s_1 < s_2 < \ldots$ Let $(t_n)_n \subset (0,1]$ be a sequence such that $t_n \to 0$, $t_1 > t_2 > \ldots$ and, for each n, if $A \in \nu^{-1}([s_n, s_{n+1}])$ and $h(A, t_n, s_n) < t^*$, then $A \in \mathcal{V}$.

I. Let $\mathcal{A} = \nu^{-1}(t^*)$. For each n, define $\mathcal{A}_n = \{A \in C(X) : h(A, t_n, s_n) = t^*\}$. Then $E \in \mathcal{A}_n$, $\mathcal{A}_n$ is a positive Whitney level, $\nu^{-1}(s_n) \le \mathcal{A}_n \le \mathcal{A}$ and $\mathcal{A}_n \to \mathcal{A}$.

To see this, let $A \in \mathcal{A}_n$; then $t^* \le \nu(A)t^*/s_n$. Thus $s_n \le \nu(A)$. Then there exists $B \in \nu^{-1}(s_n)$ such that $B \subset A$. Hence $\nu^{-1}(s_n) \le \mathcal{A}_n$. Now, let $A \in \mathcal{A}$. Then $h(A, t_n, s_n) = \min\{\nu_0(A \cup E), (t^*)^2/s_n\}$. Therefore $h(A, t_n, s_n) \ge t^*$, so that there exists $B \in C(X)$ such that $B \subset A$ and $h(B, t_n, s_n) = t^*$. Thus $\mathcal{A}_n \le \mathcal{A}$.

By [7, Lemma 2.2(b)], $\mathcal{H}^*(\mathcal{A}_n, \mathcal{A}) \le \mathcal{H}^*(\nu^{-1}(s_n), \nu^{-1}(t^*)) \to 0$. Hence $\mathcal{A}_n \to \mathcal{A}$.

J. Define $\mathcal{B} = \inf(\{\mathcal{A}\} \cup \{\mathcal{A}_n : n \ge 1\})$. Then $\mathcal{B}$ is a Whitney level. Thus there exists $t_0 \in I$ and a Whitney map μ for $C(X)$ such that $\mathcal{B} = \mu^{-1}(t_0)$. Since $E \in \mathcal{A}$ and $E \in \mathcal{A}_n$ for all n, it follows that $E \in \mathcal{B}$ and $t_0 > 0$.

K. The set $\mathcal{W} = \nu^{-1}((s_1, t^*)) \cap \mu^{-1}([0, t_0))$ is contained in $\mathcal{V}$.

Indeed, let $A \in \mathcal{W}$. Then there exists N such that $A \in \nu^{-1}([s_N, s_{N+1}])$. By H, we must show that $h(A, t_N, s_N) < t^*$. Suppose, on the contrary, that $h(A, t_N, s_N) \ge t^*$. Then there exists a subcontinuum A^* of A such that $h(A^*, t_N, s_N) = t^*$. Choose a point $a \in A^*$. Let γ be an ordered arc in $C(X)$ joining $\{a\}$ to X such that $A^*, A \in \gamma$. Let A_2 be the unique element in $\gamma \cap \mathcal{B}$. Since $\mu(A) < t_0 = \mu(A_2)$, we find that $A \subsetneq A_2$. Thus $A \subsetneq A_2 = \bigcap\{B \in C(X) : B \in \gamma \cap (\{\mathcal{A}\} \cap \{\mathcal{A}_n : n \in \mathbb{N}\})\} \subset A^*$. This contradiction proves that $A \in \mathcal{V}$.

L. Choose a Whitney map $\overline{\mu} : 2^X \to I$ which extends μ (see [14, Cor. 3.3]). Define $\omega : C(X) \to I$ by $\omega(A) = (\overline{\mu}(A \cup E)\overline{\mu}(A))^{1/2}$. Then ω is a Whitney map such that $\omega(E) = \mu(E) = t_0$, $\omega^{-1}(t_0) - \{E\} \subset \mu^{-1}([0, t_0))$ and $\nu^{-1}((s_1, 1]) \cap \omega^{-1}(t_0) \subset \mathcal{V} \cup \{E\}$.

To prove this, let $A \in (\nu^{-1}((s_1, 1]) \cap \omega^{-1}(t_0)) - \{E\}$. By K, to show that $A \in \mathcal{V}$, it is enough to prove that $\nu(A) < t^*$. Suppose that $\nu(A) \ge t^*$. Then there exists $A^* \in \nu^{-1}(t^*)$ such that $A^* \subset A$. Since $\mathcal{B} \le \nu^{-1}(t^*)$, there exists $B \in \mathcal{B}$ such that $B \subset A^*$. Since E is not contained in A, we have $t_0 = \omega(A) \ge \omega(B) > \mu(B) = t_0$. This contradiction proves that $A \in \mathcal{V}$.

M. There exists $\varepsilon > 0$ such that $B(E, \varepsilon) \subset \nu^{-1}((s_1, 1])$ and if $\mathcal{H}(A, E) < \varepsilon$, $A \subset B$ and $B \in \omega^{-1}(t_0)$, then $B \in \mathcal{V} \cup \{E\}$.

Indeed, let $\varepsilon_1 > 0$ be such that if $\mathcal{H}(E, A) < \varepsilon_1$ then $A \in \nu^{-1}((s_1, 1])$. Let $\delta > 0$ be such that $A \subset B$ and $|\omega(A) - \omega(B)| < \delta$ imply that $\mathcal{H}(A, B) < \varepsilon_1/2$ (see [12, Lemma 1.28]). Choose $r_0 \in [0, t_0)$ such that $t_0 - r_0 < \delta$. Finally, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon_1/2$ and $\mathcal{H}(A, E) < \varepsilon$ imply that

$A \in \omega^{-1}((r_0, 1])$.

N. Define $\mathcal{V}_1 = B(E, \varepsilon) \cap \omega^{-1}([0, t_0])$ and $\mathcal{V}_2 = \omega^{-1}([0, t_0]) - \{E\}$. Then $\mathcal{V}_1$ and $\mathcal{V}_2$ are open subsets of $\omega^{-1}([0, t_0])$ such that $\omega^{-1}([0, t_0]) = \mathcal{V}_1 \cup \mathcal{V}_2$ and if $A \in \mathcal{V}_1 \cap \mathcal{V}_2$, then $f|A \simeq 1$.

O. Define $h_1 : \mathcal{V}_1 \cap \mathcal{V}_2 \to S^1$ in the following way: Given $A \in \mathcal{V}_1 \cap \mathcal{V}_2$, take a map $g_A : A \to \mathbb{R}$ such that $\mathfrak{e} \circ g_A = f|A$. Define $h_1(A) = \mathfrak{e}(\min g_A(A))$. Then h_1 is well defined and continuous.

Indeed, it is easy to prove that h_1 is well defined. To prove that h_1 is continuous, take a sequence $(D_n)_n$ in $\mathcal{V}_1 \cap \mathcal{V}_2$ such that $D_n \to A \in \mathcal{V}_1 \cap \mathcal{V}_2$. Let $g_A : A \to \mathbb{R}$ be a map such that $\mathfrak{e} \circ g_A = f|A$. Let U_1 be an open subset of X such that $A \subset U_1 \subset U$ and $f|U_1 \simeq 1$. Let $g : U_1 \to \mathbb{R}$ be a map such that $\mathfrak{e} \circ g = f|U_1$. Since $D_n \to A$, there exists N such that $D_n \subset U_1$ for all $n \geq N$. Then, for all $n \geq N$, $h_1(D_n) = \mathfrak{e}(\min g(D_n)) \to \mathfrak{e}(\min g(A)) = h_1(A)$.

P. Choose $\delta > 0$ such that $A \subset B$ and $|\omega(A) - \omega(B)| < \delta$ imply that $\mathcal{H}(A, B) < \varepsilon$. Choose $s^* \in (0, t_0)$ such that $t_0 - s^* < \delta$ and $\omega(A_0), \omega(B_0) < s^*$. Choose $p_0 \in H_0$ and $q_0 \in K_0$. Finally, choose maps $\alpha_1, \alpha_2, \beta_1$ and β_2 from I to $C(X)$ such that $\alpha_1(0) = \{p_0\} = \beta_1(0), \alpha_2(0) = \{q_0\} = \beta_2(0), \alpha_1(1) = A_0 = \alpha_2(1), \beta_1(1) = B_0 = \beta_2(1)$ and, for $i = 1, 2$, $s < t$ implies that $\alpha_i(s)$ (resp. $\beta_i(s)$) is properly contained in $\alpha_i(t)$ (resp. $\beta_i(t)$) (see [12, Thm. 1.8]).

Q. Choose $r_1 \in I$ such that $\omega(B_0 \cup \alpha_2(r_1)) = s^*$. Define $\gamma : [0, 4] \to C(X)$ by

$$\gamma = \begin{cases} \alpha_2((1-t)r_1 + t) \cup \beta_2(w(t)) & \text{if } t \in [0, 1], \\ \beta_2((2-t)(w(1))) \cup A_0 \cup \beta_1(x(t)) & \text{if } t \in [1, 2], \\ \beta_1((3-t)(x(2)) + t - 2) \cup \alpha_1(y(t)) & \text{if } t \in [2, 3], \\ \alpha_1((4-t)y(3)) \cup B_0 \cup \alpha_2(z(t)) & \text{if } t \in [3, 4]. \end{cases}$$

Here $w(t), x(t), y(t), z(t) \in I$, for t in the respective intervals, are consecutively chosen in such a way that $\omega(\gamma(t)) = s^*$ for all $t \in [0, 4]$. Then γ is well defined, continuous, $\gamma(0) = \gamma(4)$ and $\gamma(t) \in \omega^{-1}(s^*) \cap C(E) \cap \mathcal{V}_1 \cap \mathcal{V}_2$ for every $t \in [0, 4]$.

R. Define $\lambda : S^1 \to \omega^{-1}(s^*) \cap \mathcal{V}_1 \cap \mathcal{V}_2$ by $\lambda(\cos t, \sin t) = \gamma(2(t + \pi)/\pi)$ if $t \in [-\pi, \pi]$. Then λ is well defined, continuous and $h_1 \circ \lambda$ is not homotopic to a constant.

To see that $h_1 \circ \lambda$ cannot be lifted, we first show that, for each $z \in S^-$, there exists a map $g_z : \lambda(z) \to [-\pi, 2\pi)$ such that $\mathfrak{e} \circ g_z = f|\lambda(z)$ and $0 \in \operatorname{Im} g_z$. Set $z = (\cos t, \sin t)$ with $t \in [-\pi, 0]$. If $t \in [-\pi, -\pi/2]$, then $s = 2(t + \pi)/\pi \in [0, 1]$ and $\lambda(z) = \gamma(s) = \alpha_2((1-s)r_1 + s) \cup \beta_2(w(s))$. If $\beta_2(w(s)) = B_0$, then $\alpha_2((1-s)r_1 + s)$ is a proper subset of A_0 since $s^* < t_0$. The minimality of A_0 implies that $\alpha_2((1-s)r_1 + s) \cap H_0 = \emptyset$. Thus $f(\alpha_2((1-s)r_1 + s))$ is a compact subset of $S^+ - \{(-1, 0)\}$ and, since

$f(\beta_2(w(s)))$ is contained in S^-, there exists a map $g_z : \lambda(z) \to [-\pi, \pi)$ such that $f|\lambda(z) = \mathsf{e} \circ g_z$. Since $(1, 0) = f(q_0) \in f(\lambda(z))$, we have $0 \in \operatorname{Im} g_z$. If $\beta_2(w(s))$ is a proper subset of B_0, the minimality of B_0 implies that $\beta_2(w(s)) \cap H_0 = \emptyset$, so that $f(\beta_2(w(s)))$ is a compact subset of $S^- - \{(-1, 0)\}$. Thus there exists a map $g_z : \lambda(z) \to (-\pi, \pi]$ such that $\mathsf{e} \circ g_z = f|\lambda(z)$. In the case that $t \in [-\pi/2, 0]$, similar considerations lead to the existence of g_z.

Similarly, for each $z \in S^+$, there exists a map $g_z : \lambda(z) \to [0, 3\pi)$ such that $\mathsf{e} \circ g_z = f|\lambda(z)$ and $\pi \in \operatorname{Im} g_z$.

If $z \in S^-$, then $h_1(\lambda(z)) = \mathsf{e}(\min g_z(\lambda(z))) \in \mathsf{e}([-\pi, 0]) = S^-$, so $h_1(\lambda(z)) \in S^-$ for each $z \in S^-$. Since $\lambda((-1, 0)) = \gamma(0) = \alpha_2(r_1) \cup \beta_2(w(0)) = \alpha_2(r_1) \cup B_0$ and $f(p_0) = (-1, 0)$, it follows that $-\pi$ is in the image of the map $g_{(-1,0)} : \lambda((-1, 0)) \to [-\pi, \pi)$. Then $h_1(\lambda((-1, 0))) = \mathsf{e}(-\pi) = (-1, 0)$. Similarly $h_1(\lambda((1, 0))) = (1, 0)$.

Thus $h_1 \circ \lambda$ is a map from S^1 to S^1 sending S^+ into S^+, S^- into $S^-, (-1, 0)$ into $(-1, 0)$ and $(1, 0)$ into $(1, 0)$. This implies that $h_1 \circ \lambda$ cannot be lifted.

S. $\lambda : S^1 \to \mathcal{V}_1$ can be extended to a map $\overline{\lambda} : D^2 \to \mathcal{V}_1$.

To see this, let $F : S^1 \times I \to C(S^1)$ $(= D^2)$ be a map such that, for each $x \in S^1$, $F(x, 0) = \{x\}$, $F(x, 1) = S^1$ and $s \le t$ implies that $F(x, s) \subset F(x, t)$. Define $\overline{\lambda} : S^1 \times I \to C(X)$ by $\overline{\lambda}(x, s) = \bigcup\{\lambda(z) \in C(X) : z \in F(x, s)\}$. Then $\overline{\lambda}$ is continuous, $\overline{\lambda}(x, 0) = \lambda(x)$ and $\overline{\lambda}(x, 1) = \bigcup\{\lambda(z) \in C(X) : z \in S^1\} = E$ for all $x \in S^1$. Identifying D^2 with $(S^1 \times I)/(S^1 \times \{1\})$, we deduce that $\overline{\lambda}$ is an extension of λ to D^2. If $x \in S^1$ and $s \in I$, $\lambda(x) = \overline{\lambda}(x, 0) \subset \overline{\lambda}(x, s) \subset E$, then $\mathcal{H}(\overline{\lambda}(x, s), E) \le \mathcal{H}(\lambda(x), E) < \varepsilon$ and so $\overline{\lambda}(x, s) \in \omega^{-1}([0, t_0])$. Thus $\overline{\lambda}(x, s) \in \mathcal{V}_1$ for every $x \in S^1$ and $s \in I$.

T. $\lambda : S^1 \to \mathcal{V}_2$ can be extended to a map $\lambda' : D^2 \to \mathcal{V}_2$.

This follows from the fact that $\operatorname{Im} \lambda \subset \omega^{-1}(s^*) \subset \mathcal{V}_2$ and every map from S^1 into $\omega^{-1}(t_1)$ is homotopic to a constant.

This completes the construction of $\omega, t_0, \mathcal{V}_1, \mathcal{V}_2, \lambda$ and h_1. Now we consider the Mayer–Vietoris sequences for the triads $(V_1 \cup V_2, V_1, V_2)$ and (S^2, S^2_+, S^2_-) where $S^2_+ = \{(x, y, z) \in S^2 : z \ge 0\}$ and $S^2_- = \{(x, y, z) \in S^2 : z \le 0\}$. Consider the diagram

$$\begin{array}{ccccccc}
0 = H_2(S^2_+) \oplus H_2(S^2_-) & \longrightarrow & H_2(S^2) & \xrightarrow{\partial_0} & H_1(S^1) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \Lambda_*} & & \big\downarrow{\scriptstyle \lambda_*} & & \\
H_2(V_1) \oplus H_2(V_2) & \longrightarrow & H_2(\mathcal{V}_1 \cup \mathcal{V}_2) & \xrightarrow{\partial} & H_1(\mathcal{V}_1 \cap \mathcal{V}_2) & &
\end{array}$$

where $\Lambda : S^2 \to \mathcal{V}_1 \cup \mathcal{V}_2 = \omega^{-1}([0, t_0])$ is defined in such a way that $\Lambda|S^1 = \lambda, \Lambda|S^2_+ = \overline{\lambda}$ and $\Lambda|S^2_- = \lambda'$.

By Lemma 1.1, Λ is homotopic to a map $\Lambda_0 : S^2 \to \omega^{-1}([0, t_0])$ such that $\operatorname{Im} \Lambda_0 \subset \omega^{-1}(t_0)$. Since $\omega^{-1}(t_0)$ is a positive Whitney level, Λ_0 is homotopic to a constant. Therefore Λ_* is the zero homomorphism. This implies that so is λ_*, and hence also the composition $h_{1*} \circ \lambda_* = (h_1 \circ \lambda)_*$. This is a contradiction since $h_1 \circ \lambda : S^1 \to S^1$ is not homotopic to a constant. Therefore X is hereditarily unicoherent.

Remark. If Y is a hereditarily indecomposable continuum then every Whitney level $\mathcal{A}$ in $C(Y)$ is hereditarily indecomposable (see [12, Thm. 14.1]); thus every map from S^n into $\mathcal{A}$ is constant for each $n \in \mathbb{N}$. Therefore it is not enough to suppose that the maps from n-spheres ($n \geq 1$) into positive Whitney levels in $C(X)$ are null homotopic to conclude that X is a dendroid. On the other hand [11, Example 3], it is not enough to suppose that every positive Whitney level $\mathcal{A}$ in $C(Z)$ is pathwise connected to conclude that Z is pathwise connected. However, as shown below, it suffices to add the assumption that Z is hereditarily unicoherent.

1.3. Lemma. *Suppose that Z is a hereditarily unicoherent continuum with the following property: If $p, q \in Z$ and $\varepsilon > 0$, then there exist $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in C(Z)$ such that $p \in A_1$, $q \in A_n$, $A_1 \cap A_2 \neq \emptyset, \ldots, A_{n-1} \cap A_n \neq \emptyset$ and $\operatorname{diam}(A_i) < \varepsilon$ for each i. Then Z is pathwise connected.*

Proof. Let p and q be two different points in Z and let $A = \bigcap\{B \in C(Z) : p, q \in B\}$. Since Z is hereditarily unicoherent, we have $A \in C(Z)$. We will prove that A is connected *im kleinen* at each point. Let $a \in A$ and let $\varepsilon > 0$. Take $A_1, \ldots, A_n \in C(Z)$ such that $p \in A_1$, $q \in A_n$, $A_1 \cap A_2 \neq \emptyset, \ldots, A_{n-1} \cap A_n \neq \emptyset$ and $\operatorname{diam}(A_i) < \varepsilon$ for each i. Let $D = \bigcup\{A_i : a \in A_i\}$ and let $W = A - \bigcup\{A_i : a \notin A_i\}$. Then $D \in C(Z)$, $A \subset A_1 \cup \ldots \cup A_n$, W is an open subset of A and $a \in W \subset D \subset B(\{a\}, \varepsilon)$. Hence A is connected *im kleinen* at a. Therefore A is a locally connected continuum. Thus A is pathwise connected (in fact, this implies that A is an arc). Hence Z is pathwise connected.

1.4. Theorem. *If Z is hereditarily unicoherent and all its positive Whitney levels are pathwise connected, then Z is pathwise connected.*

Proof. Let $p, q \in Z$ and let $\varepsilon > 0$. Fix a Whitney map μ for $C(Z)$. Let $0 < \delta < 1$ be such that if $A, B \in C(Z)$, $|\mu(A) - \mu(B)| < \delta$ and $A \subset B$, then $\mathcal{H}(A, B) < \varepsilon$. Let $0 < t \leq \delta/2$. Choose $A, B \in \mu^{-1}(t)$ such that $p \in A$ and $q \in B$. Let $\alpha : I \to \mu^{-1}(t)$ be a map such that $\alpha(0) = A$ and $\alpha(1) = B$. Let $\lambda > 0$ be such that $|t - s| < \lambda$ implies that $\mathcal{H}(\alpha(t), \alpha(s)) < \varepsilon/3$. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of I such that $t_i - t_{i-1} < \delta$ for all $i \geq 1$. For $i \geq 1$, define $A_i = \bigcup\{\alpha(t) : t_{i-1} \leq t \leq t_i\}$. Then $A_1, \ldots, A_n \in C(Z), \operatorname{diam}(A_i) < \varepsilon$ for all i, $p \in A_1$, $q \in A_n$ and $A_1 \cap A_2 \neq \emptyset, \ldots, A_{n-1} \cap A_n \neq \emptyset$. Therefore Z is pathwise connected.

1.5. THEOREM. *If each positive Whitney level in $C(X)$ is 2-connected, then X is a dendroid.*

1.6. COROLLARY. *If every positive Whitney level in $C(X)$ is contractibile, then X is a dendroid.*

2. If X is a dendroid then every positive Whitney level in $C(X)$ is ∞-connected. In [12, Thm. 14.8], it was shown that if X is pathwise connected then every Whitney level for $C(X)$ is also pathwise connected. So we concentrate our attention on the null homotopy of maps from n-spheres ($n \geq 1$) into positive Whitney levels.

Throughout this section we will suppose that X is a dendroid. Fix a Whitney map μ, a number $t_0 \in (0,1]$ and an integer $N \in \mathbb{N}$. We will show that every map $G : S^N \to \mu^{-1}(t_0)$ is null homotopic. To do this, we will need to define a strong form of convergence in $C(X)$.

2.1. DEFINITION. Given $x \neq y \in X$, the unique arc joining x and y in X will be denoted by $\overline{xy}$. The set $\{x\}$ will be denoted by $\overline{xx}$. Define $L : C(X) \times X \to C(X)$ by $L(A,x) = \overline{ax}$ where a is the unique element in A such that $\overline{ax} \cap A = \{a\}$. Given a sequence $(A_n)_n$ in $C(X)$ and an element $A \in C(X)$, we say that $(A_n)_n$ *strongly converges* to A ($A_n \xrightarrow{\text{s}} A$) if $A_n \to A$ and $L(A_n, a) \to \{a\}$ for each $a \in A$.

The following lemma is easy to prove.

2.2. LEMMA. (a) *If $A_n \xrightarrow{\text{s}} A$, $B_n \xrightarrow{\text{s}} B$ and $A_n \cap B_n \neq \emptyset$ for each n, then $A_n \cup B_n \xrightarrow{\text{s}} A \cup B$.*

(b) *Let $(A_n)_n \subset C(X)$ and $A \in C(X)$ be such that, for each infinite subset S of $\mathbb{N}$, there exists a subsequence $(A_{n_k})_k$ such that $n_k \in S$ for every k and $A_{n_k} \xrightarrow{\text{s}} A$. Then $A_n \xrightarrow{\text{s}} A$.*

Define $J : C(X) \times C(X) \to C(X)$ by

$$J(A,B) = \begin{cases} A \cap B & \text{if } A \cap B \neq \emptyset, \\ \{b\} & \text{if } A \cap B = \emptyset, \end{cases}$$

where b is the unique point in B such that $\overline{ab} \cap B = \{b\}$ for each $a \in A$.

2.3. LEMMA. *If $A_n \xrightarrow{\text{s}} A$ and $B_n \xrightarrow{\text{s}} B$, then $J(A_n, B_n) \xrightarrow{\text{s}} J(A,B)$.*

Proof. Case 1: $A \cap B = \emptyset$. Then there exists M such that $A_n \cap B_n = \emptyset$ for all $n \geq M$. Let $\{a\} = J(B,A)$ and $\{b\} = J(A,B)$. For each $n \geq M$, let $\{a_n\} = J(B_n, A_n)$, $\{b_n\} = J(A_n, B_n)$ and let $c_n \in A_n$ and $d_n \in B_n$ be such that $\overline{ac_n} = L(A_n, a)$ and $\overline{bd_n} = L(B_n, b)$. Since the set $\overline{c_n a} \cup \overline{ab} \cup \overline{bd_n}$ is connected and intersects A_n and B_n, it contains $\overline{a_n b_n}$. In particular, $b_n \in \overline{c_n a} \cup \overline{ab} \cup \overline{bd_n} \to \overline{ab}$. Thus the limit points of the sequence $(b_n)_n$ are in $\overline{ab} \cap B = \{b\}$. Therefore $b_n \to b$. Hence $J(A_n, B_n) \to J(A,B)$.

Since $\overline{c_n a} \to \{a\}$, there exists $M_1 \geq M$ such that $b_n \notin \overline{c_n a}$ for every $n \geq M_1$. Thus $b_n \in \overline{ab} \cup \overline{bd_n}$ for all $n \geq M_1$. It follows that $\overline{b_n b} \to \{b\}$. So $L(J(A_n, B_n), b) \to \{b\}$. Thus $J(A_n, B_n) \xrightarrow{\text{s}} J(A, B)$.

Case 2: $A \cap B \neq \emptyset$. First we will prove that $\limsup J(A_n, B_n) \subset J(A, B)$. Let $x \in \limsup J(A_n, B_n)$. Then there exists a subsequence $(n_k)_k$ of $(n)_n$ and, for each k, there exists $x_k \in J(A_{n_k}, B_{n_k})$ such that $x_k \to x$. If $A_{n_k} \cap B_{n_k} \neq \emptyset$ for an infinite number of k's, then $x \in A \cap B = J(A, B)$ (in this case). Thus we may suppose that $A_{n_k} \cap B_{n_k} = \emptyset$ for every k.

If there exist $z, y \in A \cap B$ such that $z \neq y$, choose $p \in \overline{zy} - \{z, y\}$. For each $k \in \mathbb{N}$, let $a_k, c_k \in A_{n_k}$ be such that $L(A_{n_k}, z) = \overline{a_k z}$ and $L(A_{n_k}, y) = \overline{c_k y}$. Since $\overline{a_k z} \to \{z\}$ and $\overline{c_k y} \to \{y\}$, there exists $K \in \mathbb{N}$ such that, for all $k \geq K$, $\overline{a_k z} \cap \overline{c_k y} = \emptyset$, $\overline{a_k z} \cap \overline{py} = \emptyset$ and $\overline{pz} \cap \overline{c_k y} = \emptyset$. Given $k \geq K$, $\overline{a_k c_k} \subset A_{n_k} \cap (\overline{a_k z} \cup \overline{zp} \cup \overline{py} \cup \overline{yc_k})$ and $(\overline{a_k z} \cup \overline{zp}) \cap (\overline{py} \cap \overline{yc_k}) = \{p\}$. Therefore $p \in \overline{a_k c_k}$. Hence $p \in A_{n_k}$ for all $k \geq K$. Similarly, there exists K_1 such that $p \in B_{n_k}$ for all $k \geq K_1$. This contradicts our assumption. Therefore $A \cap B$ consists of a single point a_0.

For each $k \in \mathbb{N}$, let $a_k \in A_{n_k}$ and $b_k \in B_{n_k}$ be such that $\overline{a_k b_k} \cap A_{n_k} = \{a_k\}$ and $\overline{a_k b_k} \cap B_{n_k} = \{b_k\}$. Then $\{b_k\} = J(A_{n_k}, B_{n_k})$. So $x_k = b_k$. Suppose that $L(A_{n_k}, a_0) = \overline{c_k a_0}$ and $L(B_{n_k}, a_0) = \overline{d_k a_0}$ with $c_k \in A_{n_k}$ and $d_k \in B_{n_k}$. Then $x_k \in \overline{a_k b_k} \subset \overline{c_k a_0} \cup \overline{a_0 d_k} \to \{a_0\}$. Therefore $x = a_0 \in A \cap B = J(A, B)$. Hence $\limsup J(A_n, B_n) \subset J(A, B)$.

Now take a point $x \in J(A, B) = A \cap B$. For each n, let $a_n \in A_n$ and $b_n \in B_n$ be such that $L(A_n, x) = \overline{a_n x}$ and $L(B_n, x) = \overline{b_n x}$. If $A_n \cap B_n \neq \emptyset$, then $\overline{a_n b_n} \subset A_n \cup B_n$. Thus $\overline{a_n b_n} \cap A_n \cap B_n \neq \emptyset$. Hence $(\overline{a_n x} \cup \overline{x b_n}) \cap A_n \cap B_n \neq \emptyset$. This implies that $L(A_n \cap B_n, x) \subset \overline{a_n x} \cup \overline{x b_n}$. If $A_n \cap B_n = \emptyset$, let $\{d_n\} = J(A_n, B_n)$. Then $d_n \in \overline{a_n x} \cup \overline{x b_n}$ and $L(J(A_n, B_n), x) \subset \overline{a_n x} \cup \overline{x b_n}$. Therefore $L(J(A_n, B_n), x) \subset \overline{a_n x} \cup \overline{x b_n}$ for all n. Since $\overline{a_n x} \cup \overline{x b_n} \to \{x\}$, we have $L(J(A_n, B_n), x) \to \{x\}$. Thus $x \in \liminf J(A_n, B_n)$ and we conclude that $J(A_n, B_n) \xrightarrow{\text{s}} J(A, B)$.

In order to give a "uniform" parametrization of the arcs in X, we define, for $a, b \in X$, the function $\gamma(a, b) : I \to \overline{ab}$ by $\gamma(a, b)(t) = x$ if $\mu(\overline{ax}) = t\mu(\overline{ab})$ and $x \in \overline{ab}$. Then we have:

2.4. Lemma. *For each $a, b \in X$, $\gamma(a, b)$ is a map, $\gamma(a, b)(0) = a$, $\gamma(a, b)(1) = b$ and, if $a \neq b$, then $\gamma(a, b)$ is injective.*

2.5. Lemma. *If $\{a_n\} \xrightarrow{\text{s}} \{a\}$, $\{b_n\} \xrightarrow{\text{s}} \{b\}$, $r_n \to r$ and $t_n \to t$, then $\overline{\gamma(a_n, b_n)(r_n)\gamma(a_n, b_n)(t_n)} \xrightarrow{\text{s}} \overline{\gamma(a, b)(r)\gamma(a, b)(t)}$ and $\{\gamma(a_n, b_n)(r_n)\} \xrightarrow{\text{s}} \{\gamma(a, b)(r)\}$.*

Proof. Let $\gamma_n = \gamma(a_n, b_n)$ and $\gamma = \gamma(a, b)$. Since $\overline{a_n b_n} \subset \overline{a_n a} \cup \overline{ab} \cup \overline{bb_n}$ and $\overline{ab} \subset \overline{aa_n} \cup \overline{a_n b_n} \cup \overline{b_n b}$, we have $\overline{a_n b_n} \to \overline{ab}$. First, we will show that $\{\gamma_n(r_n)\} \xrightarrow{\text{s}} \{\gamma(r)\}$.

If $r=0$ or $a=b$, then $\overline{a_n\gamma_n(r_n)}\to a$, since $\mu(\overline{a_n\gamma_n(r_n)})=r_n\mu(\overline{a_nb_n})\to 0$ and $a_n\to a$. Since $L(\{\gamma_n(r_n)\},\gamma(r))=\overline{a\gamma_n(r_n)}\subset\overline{aa_n}\cup\overline{a_n\gamma_n(r_n)}\to\{a\}$, we have $\{\gamma_n(r_n)\}\xrightarrow{\rm s}\{\gamma(r)\}$.

If $r=1$ and $a\ne b$, then for $p\in\overline{ab}-\{a,b\}$, $\overline{a_n\gamma_n(r_n)}\subset\overline{a_na}\cup\overline{ap}\cup\overline{pb}\cup\overline{bb_n}$. Since $\mu(\overline{a_na}\cup\overline{ap})\to\mu(\overline{ap})<\mu(\overline{ab})$ and $\mu(\overline{a_n\gamma_n(r_n)})=r_n\mu(\overline{a_nb_n})\to\mu(\overline{ab})$, there exists M such that $\gamma_n(r_n)\notin\overline{a_na}\cup\overline{ap}$ for all $n\ge M$. Thus $\gamma_n(r_n)\in\overline{pb}\cup\overline{pb_n}$ for all $n\ge M$. This implies that $\{\gamma_n(r_n)\}\xrightarrow{\rm s}\{\gamma(r)\}$.

If $0<r<1$ and $a\ne b$, then for $p\in\overline{a\gamma(r)}-\{\gamma(r)\}$ and $q\in\overline{\gamma(r)b}-\{\gamma(r)\}$, $\overline{a_n\gamma_n(r_n)}\subset\overline{a_na}\cup\overline{ap}\cup\overline{pq}\cup\overline{qb}\cup\overline{bb_n}$. Proceeding as above, there exists M such that $\gamma_n(r_n)\notin\overline{a_na}\cup\overline{ap}$ for all $n\ge M$. If there exists a subsequence $(\gamma_{n_k}(r_{n_k}))_k$ of $(\gamma_n(r_n))_n$ such that $\gamma_{n_k}(r_{n_k})\in\overline{qb}\cup\overline{bb_{n_k}}$, we may suppose that $\gamma_{n_k}(r_{n_k})\to x$ for some $x\in\overline{qb}$ and $\overline{a_{n_k}\gamma_{n_k}(r_{n_k})}\to A$ for some $A\in C(X)$. Then $a,x\in A$, $\mu(\overline{a_{n_k}\gamma_{n_k}(r_{n_k})})\to r\mu(\overline{ab})=\mu(\overline{a\gamma(r)})<\mu(\overline{aq})\le\mu(\overline{ax})\le\mu(A)=\lim\mu(\overline{a_{n_k}\gamma_{n_k}(r_{n_k})})$. This contradiction proves that there exists $M\in\mathbb{N}$ such that $\gamma_n(r_n)\in\overline{pq}$ for all $n\ge M$. It follows that $\{\gamma_n(r_n)\}\xrightarrow{\rm s}\{\gamma(r)\}$.

Now we will prove that $\overline{\gamma_n(r_n)\gamma_n(t_n)}\xrightarrow{\rm s}\overline{\gamma(r)\gamma(t)}$. Notice that $\overline{\gamma_n(r_n)\gamma_n(t_n)}\to\overline{\gamma(r)\gamma(t)}$. Given $p=\gamma(s)\in\overline{\gamma(r)\gamma(t)}$, there exists a sequence $(s_n)_n\subset I$ such that $s_n\to s$ and s_n is between r_n and t_n. Then $\gamma(s_n)\xrightarrow{\rm s}\gamma(s)$. Since $L(\overline{\gamma_n(r_n)\gamma_n(t_n)},\gamma(s))\subset\overline{\gamma_n(s_n)\gamma_n(s)}\to\{\gamma(s)\}$, we obtain $\overline{\gamma_n(r_n)\gamma_n(t_n)}\xrightarrow{\rm s}\overline{\gamma(r)\gamma(t)}$.

Define $\mathfrak{A}=\{(A,B)\in C(X)\times C(X):A\subset B\}$ and $F:\mathfrak{A}\times I\to C(X)$ by $F(A,B,t)=\bigcup\{\overline{ax}\in C(X):a\in A,\ x\in B\text{ and }\mu(\overline{ax})\le t\}$.

2.6. Lemma. (a) *F is well defined.*

(b) *$F|\{(A,B)\}\times I$ is continuous for every $(A,B)\in\mathfrak{A}$.*

(c) *$F(A,B,0)=A$ and $F(A,B,1)=B$.*

(d) *If $s\le t$, then $F(A,B,s)\subset F(A,B,t)$.*

Proof. We only prove (b). Let $(A,B)\in\mathfrak{A}$ and let $\varepsilon>0$. Let $\delta>0$ be such that if $A_1\subset B_1$ and $|\mu(A_1)-\mu(B_1)|<\delta$, then $\mathcal{H}(A_1,B_1)<\varepsilon$. It is easy to check that if $|s-t|<\delta$, then $\mathcal{H}(F(A,B,t),F(A,B,s))<\varepsilon$. Thus $F|\{(A,B)\}\times I$ is continuous.

2.7. Lemma. *If $A_n\xrightarrow{\rm s}A$, $B_n\xrightarrow{\rm s}B$ and $t_n\to t$ with $(A_n,B_n)\in\mathfrak{A}$ for each n, then $F(A_n,B_n,t_n)\xrightarrow{\rm s}F(A,B,t)$.*

Proof. Take $x\in\limsup F(A_n,B_n,t_n)$. Then $x=\lim x_k$ where $x_k\in F(A_{n_k},B_{n_k},t_{n_k})$ and $(n_k)_k$ is a subsequence of $(n)_n$. For each k, there exists $a_k\in A_{n_k}$ and $b_k\in B_{n_k}$ such that $x_k\in\overline{a_kb_k}$ and $\mu(\overline{a_kb_k})\le t_{n_k}$. We may suppose that $a_k\to a$ for some $a\in A$ and $\overline{a_kb_k}\to C$ for some $C\in C(X)$. Then $\overline{ax}\subset C\subset B$ and $\mu(\overline{ax})\le\mu(C)\le t$. Hence $x\in F(A,B,t)$. Therefore $\limsup F(A_n,B_n,t_n)\subset F(A,B,t)$.

Now take $x \in F(A,B,t)$. Then $x \in B$ and there exists $a \in A$ such that $\mu(\overline{ax}) \le t$. Let $s = \mu(\overline{ax})$. Then there exists a sequence $(s_n)_n$ with $0 \le s_n \le t_n$ for all n and $s_n \to s$. For each $n \in \mathbb{N}$, let $a_n \in A_n$ and $x_n \in B_n$ be such that $L(A_n, a) = \overline{a_n a}$ and $L(B_n, x) = \overline{x_n x}$. Let $y_n \in F(A_n, B_n, t_n)$ be such that $L(F(A_n, B_n, t_n), x) = \overline{y_n x}$. If $\mu(\overline{a_n x_n}) \le s_n$, define $z_n = x_n$. If $\mu(\overline{a_n x_n}) \ge s_n$, let z_n be the unique element in $\overline{a_n x_n}$ such that $\mu(\overline{a_n z_n}) = s_n$. Then $z_n \in F(A_n, B_n, t_n)$.

If $x = a$, then $L(F(A_n, B_n, t_n), x) = \overline{y_n a} \subset \overline{a_n a} \to \{a\}$. Therefore $L(F(A_n, B_n, t_n), x) \to \{x\}$. Now suppose that $x \ne a$. Given $p \in \overline{ax} - \{a, x\}$, $z_n \in \overline{a_n x_n} \subset \overline{a_n a} \cup \overline{ap} \cup \overline{px} \cup \overline{x x_n}$. Since $\mu(\overline{a_n a} \cup \overline{ap}) \to \mu(\overline{ap}) < s$, there exists M such that $z_n \in \overline{px} \cup \overline{x x_n}$ for all $n \ge M$. This implies that $\overline{z_n x} \to \{x\}$. Since $\overline{y_n x} \subset \overline{z_n x}$, we have $L(F(A_n, B_n, t_n), x) \to \{z\}$. It follows that $F(A_n, B_n, t_n) \xrightarrow{\mathrm{s}} F(A, B, t)$.

Now we "uniformize" the map F. Define $G : \mathfrak{A} \times I \to C(X)$ by $G(A,B,t) = F(A,B,s)$ where s is chosen in such a way that $\mu(G(A,B,t)) = \mu(A) + t(\mu(B) - \mu(A))$.

2.8. Lemma. (a) *$G(A,B,0) = A$ and $G(A,B,1) = B$.*

(b) *If $s \le t$, then $G(A,B,s) \subset G(A,B,t)$.*

(c) *If $A_n \xrightarrow{\mathrm{s}} A$, $B_n \xrightarrow{\mathrm{s}} B$ and $t_n \to t$ with $(A_n, B_n) \in \mathfrak{A}$ for each n, then $G(A_n, B_n, t_n) \xrightarrow{\mathrm{s}} G(A,B,t)$.*

(d) *$G|\{(A,B)\} \times I$ is continuous for every $(A,B) \in \mathfrak{A}$.*

Proof. We only prove (c). We will use Lemma 2.2(b). Let S be an infinite subset of $\mathbb{N}$. For each $n \in S$, let $G(A_n, B_n, t_n) = F(A_n, B_n, s_n)$ with $s_n \in I$. Let $G(A,B,t) = F(A,B,s)$. Take a subsequence $(n_k)_k$ of $(n)_n$ such that $n_k \in S$ for all k and $s_{n_k} \to s^*$ for some $s^* \in I$. Then $G(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\mathrm{s}} F(A,B,s^*)$. This yields $\mu(F(A,B,s^*)) = \lim(\mu(A_{n_k}) + t_{n_k}(\mu(B_{n_k}) - \mu(A_{n_k}))) = \mu(G(A,B,t)) = \mu(F(A,B,s))$. It follows that $F(A,B,s^*) = F(A,B,s)$. Hence $G(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\mathrm{s}} G(A,B,t)$. Therefore $G(A_n, B_n, t_n) \xrightarrow{\mathrm{s}} G(A,B,t)$.

Now we define "standard" arcs joining elements in $\mu^{-1}(t_0)$. Define $\alpha : \mu^{-1}(t_0) \times \mu^{-1}(t_0) \times I \to \mu^{-1}(t_0)$ in the following way:

A. If $A \cap B = \emptyset$, let $\{a\} = J(B,A)$, $\{b\} = J(A,B)$ and $\gamma = \gamma(a,b)$.

A.1. If $\mu(\overline{ab}) \le t_0$, let s_0 be the unique number in I such that $\mu(\overline{ab} \cup G(\{a\}, A, s_0)) = t_0$ then define

$$\alpha(A,B,t) = \begin{cases} \overline{a\gamma(3t)} \cup G(\{a\}, A, s) & \text{if } 0 \le t \le 1/3, \\ G(\{a\}, A, (2-3t)s_0) \cup \overline{ab} \cup G(\{b\}, B, s) & \text{if } 1/3 \le t \le 2/3, \\ \overline{\gamma(3t-2)b} \cup G(\{b\}, B, s) & \text{if } 2/3 \le t \le 1. \end{cases}$$

In the three cases the element $s \in I$ is chosen in such a way that $\mu(\alpha(A,B,t)) = t_0$.

A.2. If $\mu(\overline{ab}) \geq t_0$, let s_0 and r_0 be the unique elements in I such that $\mu(\overline{a\gamma(s_0)}) = t_0 = \mu(\overline{\gamma(r_0)b})$. Then define

$$\alpha(A,B,t) = \begin{cases} \overline{a\gamma(3ts_0)} \cup G(\{a\},A,s) & \text{if } 0 \leq t \leq 1/3, \\ \overline{\gamma(s)\gamma((2-3t)s_0+3t-1)} & \\ \qquad \text{where } s \in [0,(2-3t)s_0+3t-1] & \text{if } 1/3 \leq t \leq 2/3, \\ \overline{\gamma(3t-2+(3-3t)r_0)b} \cup G(\{b\},B,s) & \text{if } 2/3 \leq t \leq 1, \end{cases}$$

with s chosen as above.

B. If $A \cap B \neq \emptyset$, define

$$\alpha(A,B,t) = \begin{cases} A & \text{if } 0 \leq t \leq 1/3, \\ G(A\cap B, A, 2-3t) \cup G(A\cap B, B, s) & \text{if } 1/3 \leq t \leq 2/3, \\ B & \text{if } 2/3 \leq t \leq 1, \end{cases}$$

with s chosen in the same way.

It is easy to check that α is well defined, $\alpha(A,B,0) = A$ and $\alpha(A,B,1) = B$ for all $(A,B) \in \mu^{-1}(t_0) \times \mu^{-1}(t_0)$ and if $A, B \subset A_0 \in C(X)$, then $\alpha(A,B,t) \subset A_0$ for each $t \in I$.

2.9. Lemma. *If $A_n \xrightarrow{s} A$, $B_n \xrightarrow{s} B$ and $t_n \to t$, then $\alpha(A_n,B_n,t_n) \xrightarrow{s} \alpha(A,B,t)$ $(A_n, B_n, A$ and B in $\mu^{-1}(t_0))$.*

Proof. We will use Lemma 2.2(b). Let S be an infinite subset of $\mathbb{N}$. We need to analyze several cases.

1. $A \cap B \neq \emptyset$.

1.1. $A_{n_k} \cap B_{n_k} = \emptyset$ for infinitely many elements $n_1 < n_2 < \ldots$ in S. For each k, let $\{a_{n_k}\} = J(B_{n_k}, A_{n_k})$ and $\{b_{n_k}\} = J(A_{n_k}, B_{n_k})$. Since $\{b_{n_k}\} = J(A_{n_k}, B_{n_k}) \xrightarrow{s} J(A,B) = A \cap B$, $A \cap B$ consists of a single point a_0. Then $\{a_{n_k}\} = J(B_{n_k}, A_{n_k}) \xrightarrow{s} \{a_0\}$. For each k, let $\gamma_k = \gamma(a_{n_k}, b_{n_k})$. It follows that, for all sequences $(r_k)_k$ and $(m_k)_k$ in I, $\overline{\gamma_k(r_k)\gamma_k(m_k)} \xrightarrow{s} \{a_0\}$.

1.1.1. $t_0 = 0$. Then $\mu(\overline{a_{n_k}b_{n_k}}) \geq t_0$, so $\alpha(A_{n_k}, B_{n_k}, t_{n_k})$ is equal to either $\{a_{n_k}\}$, a point in $\overline{\gamma_k(0)\gamma_k(1)} = \overline{a_{n_k}b_{n_k}}$ or $\{b_{n_k}\}$. Thus $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{s} \{a_0\} = A = B = \alpha(A,B,t)$.

1.1.2. $t_0 > 0$. We may suppose that $\mu(\overline{a_{n_k}b_{n_k}}) < t_0$ for every k. For each k, let $s_0^k \in I$ be such that $\mu(\overline{a_{n_k}b_{n_k}} \cup G(\{a_{n_k}\}, A_{n_k}, s_0^k)) = t_0$ and let s_k be the number chosen so that $\mu(\alpha(A_{n_k}, B_{n_k}, t_{n_k})) = t_0$. We may suppose that $s_k \to s^*$ for some $s^* \in I$ and $s_0^k \to s'$ for some $s' \in I$. Then $\overline{a_0a_0} \cup G(\{a_0\}, A, s^*)$ is an element of $\mu^{-1}(t_0)$ which is contained in A. This implies that $G(\{a_0\}, A, s^*) = A$. But $\mu(G(\{a_0\}, A, s^*)) = \mu(\{a_0\}) + s^*(\mu(A) - \mu(\{a_0\}))$, and so $s^* = 1$. We may suppose that one of the following three cases holds:

1.1.2.1. $t_{n_k} \in [0,1/3]$ for every k. Then $t \in [0,1/3]$ and $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\text{s}} G(\{a_0\}, A, s') = A = \alpha(A, B, t)$.

1.1.2.2. $t_{n_k} \in [1/3,2/3]$ for every k. Then $t \in [1/3,2/3]$ and we have $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\text{s}} G(\{a_0\}, A, (2-3t)s^*) \cup \overline{a_0a_0} \cup G(\{a_0\}, B, s') = \alpha(A, B, t)$.

1.1.2.3. $t_{n_k} \in [2/3,1]$ for every k. Then $t \in [2/3,1]$ and $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\text{s}} G(\{a_0\}, B, s') = B = \alpha(A, B, t)$.

This completes Subcase 1.1.

1.2. $A_{n_k} \cap B_{n_k} \neq \emptyset$ for infinitely many elements $n_1 < n_2 < \ldots$ in S. Then we may suppose that one of the following three cases holds:

1.2.1. $t_{n_k} \in [0,1/3]$ for all k. Then $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) = A_{n_k} \xrightarrow{\text{s}} A = \alpha(A, B, t)$.

1.2.2. $t_{n_k} \in [1/3,2/3]$ for all k. So $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) = B_{n_k} \xrightarrow{\text{s}} B = \alpha(A, B, t)$.

1.2.3. $t_{n_k} \in [2/3,1]$ for every k. Then $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) = G(A_{n_k} \cap B_{n_k}, A_{n_k}, 2-3t_{n_k}) \cup G(A_{n_k} \cap B_{n_k}, B_{n_k}, s_k)$, where $s_k \in I$, and we may suppose that $s_k \to s'$ for some $s' \in I$. Then $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\text{s}} G(J(A,B), A, 2-3t) \cup G(J(A,B), B, s') = \alpha(A, B, t)$.

This completes the proof of Case 1.

2. $A \cap B = \emptyset$. Then we may suppose that $A_n \cap B_n = \emptyset$ for every $n \in S$. Here it is necessary to consider the following cases:

2.1. $\mu(\overline{a_{n_k} b_{n_k}}) \geq t_0$ for infinitely many elements $n_1 < n_2 < \ldots$ in S.
2.1.1. $t_{n_k} \in [0,1/3]$ for every k.
2.1.2. $t_{n_k} \in [1/3,2/3]$ for every k.
2.1.3. $t_{n_k} \in [2/3,1]$ for every k.

2.2. $\mu(\overline{a_{n_k} b_{n_k}}) \leq t_0$ for infinitely many elements $n_1 < n_2 < \ldots$ in S.
2.2.1. $t_{n_k} \in [0,1/3]$ for every k.
2.2.2. $t_{n_k} \in [1/3,2/3]$ for every k.
2.2.3. $t_{n_k} \in [2/3,1]$ for every k.

All of them can be treated similarly to Case 1.

Hence, in each one of the cases, infinitely many elements $n_1 < n_2 < \ldots$ of S can be obtained such that $\alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\text{s}} \alpha(A, B, t)$.

Therefore $\alpha(A_n, B_n, t_n) \xrightarrow{\text{s}} \alpha(A, B, t)$.

2.10. Construction. For each $r \in \mathbb{N}$, let $S_r = (\{0,1\})^r$. For each set $E = \{A_\sigma \in \mu^{-1}(t_0) : \sigma \in S_N\}$ define $f_E : I^N \to \mu^{-1}(t_0)$ through the following steps:

$f_E(a_1, \sigma_1) = \alpha(A_{(0,\sigma_1)}, A_{(1,\sigma_1)}, a_1)$ if $a_1 \in I$ and $\sigma_1 \in S_{N-1}$.

$f_E(a_1, a_2, \sigma_2) = \alpha(f_E(a_1, 0, \sigma_2), f_E(a_1, 1, \sigma_2), a_2)$ if $a_1, a_2 \in I$ and $\sigma_2 \in S_{N-2}$.

If $2 \le r < N$, then $f_E(a_1,\dots,a_r,\sigma_r) = \alpha(f_E(a_1,\dots,a_{r-1},0,\sigma_r), f_E(a_1,\dots,a_{r-1},1,\sigma_r),a_r)$ for $a_1,\dots,a_r \in I$ and $\sigma_r \in S_{N-r}$.

If $r = N$, then we set $f_E(a_1,\dots,a_N) = \alpha(f_E(a_1,\dots,a_{N-1},0), f_E(a_1,\dots,a_{N-1},1),a_N)$ for $a_1,\dots,a_N \in I$.

The following lemma is easy to prove.

2.11. LEMMA. (a) *f_E is well defined.*

(b) *If $(a_n)_n \subset I^N$ and $a \in I^N$ are such that $a_n \to a$ then $f_E(a_n) \xrightarrow{s} f_E(a)$.*

(c) *If $A_\sigma \subset A \in C(X)$ for each $\sigma \in S_N$, then $f_E(a) \subset A$ for every $a \in I^N$.*

2.12. LEMMA. *Let $p,q \in \{0,1\}$. Let $E = \{A_\sigma : \sigma \in S_N\}$ and $D = \{B_\sigma : \sigma \in S_N\}$ and let $r \in \{1,\dots,N\}$ be such that $A_{(\sigma_1,p,\sigma_2)} = B_{(\sigma_1,q,\sigma_2)}$ for each $\sigma_1 \in S_{r-1}$ and $\sigma_2 \in S_{N-r}$. Then $f_E(a_1,p,a_2) = f_D(a_1,q,a_2)$ for every $a_1 \in I^{r-1}$ and $a_2 \in I^{N-r}$.*

Proof. Let $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N) \in I^N$ be such that $x_r = p$, $y_r = q$ and $x_i = y_i$ for all $i \ne r$. We will show, by induction on k, that if $x_{k+1},\dots,x_N,y_{k+1},\dots,y_N \in \{0,1\}$ then $f_E(x) = f_D(y)$.

Suppose that $k = 1$. Let $\sigma = (x_2,\dots,x_N)$ and $\varrho = (y_2,\dots,y_N) \in S_{N-1}$. If $r > 1$, then $A_{(0,\sigma)} = B_{(0,\varrho)}$, $A_{(1,\sigma)} = B_{(1,\varrho)}$ and $x_1 = y_1$. Then $f_E(x) = \alpha(A_{(0,\sigma)},A_{(1,\sigma)},x_1) = \alpha(B_{(0,\varrho)},B_{(1,\varrho)},y_1) = f_D(y)$. If $r = 1$, then $\sigma = \varrho$. Notice that $f_E(x) = A_{(p,\sigma)}$ and $f_D(y) = B_{(q,\sigma)}$. Thus $f_E(x) = f_D(y)$.

Suppose that the assertion holds for $k < n$. Suppose that $x_{k+2},\dots,x_N, y_{k+2},\dots,y_N \in \{0,1\}$. Then $f_E(x) = \alpha(f_E(x_1,\dots,x_k,0,x_{k+2},\dots,x_N), f_E(x_1,\dots,x_k,1,x_{k+2},\dots,x_N),x_{k+1}) = (*)$. If $k+1 \ne r$, the induction hypothesis implies that $(*) = f_D(y)$, and if $k+1 = r$, then $f_E(x) = f_E(x_1,\dots,x_k,p,x_{k+2},\dots,x_N)$, which, by the induction hypothesis, is equal to $f_D(y_1,\dots,y_k,q,y_{k+2},\dots,y_N) = f_D(y)$.

This completes the induction. Then the theorem follows by taking $k = N$.

2.13. CONSTRUCTION. Let $g : I^N \to \mu^{-1}(t_0)$ be a map. Given $m \in \mathbb{N} \cup \{0\}$ and $x = (x_1,\dots,x_N) \in (\{0,1,\dots,10^m-1\})^N$, define $Q(x) = [x_1/10^m,(x_1+1)/10^m] \times \ldots \times [x_N/10^m,(x_N+1)/10^m]$ and $E(x) = \{A_\sigma : \sigma \in S_N\}$ where $A_\sigma = g((x+\sigma)/10^m)$ for every $\sigma \in S_N$. Next, define $h_x : Q(x) \to \mu^{-1}(t_0)$ by $h_x(a) = f_{E(x)}(10^m(a - x/10^m))$. Then h_x is well defined. Now define $h_m : I^N \to \mu^{-1}(t_0)$ by $h_m(a) = h_x(a)$ if $a \in Q(x)$. Finally, define $h : I^{N+1} \to \mu^{-1}(t_0)$ by

$$h(a,t) = \begin{cases} g(a) & \text{if } t = 0,\\ \alpha(h_{m+1}(a),h_m(a),2^{m+1}(t-1/2^{m+1})) & \text{if } t \in [1/2^{m+1},1/2^m]. \end{cases}$$

2.14. Lemma. *For each m, h_m is well defined and, if $a_n \to a$, then $h_m(a_n) \xrightarrow{s} h_m(a)$.*

Proof. To see that h_m is well defined take a point $a \in Q(x) \cap Q(y)$. First suppose that x and y differ just in one coordinate r. Suppose that $x_r < y_r$. Then $a_r 10^m = y_r = x_r + 1$. Then $h_m(a)$ can be defined as $f_{E(x)}(10^m(a - x/10^m))$ and $f_{E(y)}(10^m(a - y/10^m))$ where $E(x) = \{g((x + \sigma)/10^m) : \sigma \in S_N\}$ and $E(y) = \{g((y + \sigma)/10^m) : \sigma \in S_N\}$.

We will apply Lemma 2.12. Let $c = 10^m(a - x/10^m)$ and $d = 10^m(a - y/10^m)$. Then $c_r = 1$ and $d_r = 0$. Let $p = 1$ and $q = 0$. For $\sigma_1 \in S_{r-1}$ and $\sigma_2 \in S_{N-r}$ we have $g((x + (\sigma_1, p, \sigma_2))/10^m) = g((y + (\sigma_1, q, \sigma_2))/10^m)$. Hence, by Lemma 2.12, $f_{E(x)}(c) = f_{F(y)}(d)$. Thus $f_{E(x)}(10^m(a - x/10^m)) = f_{E(y)}(10^m(a - y/10^m))$.

If x and y differ in more that one coordinate, considering the vectors $(x_1, y_2, \ldots, y_N)$, $(x_1, x_2, y_3, \ldots, y_N)$, … ,$(x_1, \ldots, x_{N-1}, y_N)$, we conclude that h_m is well defined.

The second part of the lemma follows from Lemma 2.11(b).

2.15. Lemma. *h is well defined and continuous.*

Proof. It is easy to check that h is well defined. From Lemma 2.13 it follows that if $(a_n, t_n) \to (a, t)$ and $t > 0$ then $h(a_n, t_n) \xrightarrow{s} h(a, t)$. Thus h is continuous at (a, t) if $t > 0$.

Now take a point $(a, 0) \in I^{N+1}$; we will check that h is continuous at this point. Let $\varepsilon > 0$. Consider the metric $\mathtt{d}_0$ in I^N defined by $\mathtt{d}_0(b, c) = \max\{|b_i - c_i| : 1 \le i \le N\}$. Let $\delta > 0$ be such that $\mathtt{d}_0(a, b) \le \delta$ implies that $\mathcal{H}(g(a), g(b)) < \varepsilon$. Let $A_0 = [a_1 - \delta, a_1 + \delta] \times \ldots \times [a_N - \delta, a_N + \delta]$ and let $A = \bigcup\{g(b) : b \in A_0 \cap I^N\}$. Then A is a subcontinuum of X and $A \subset N(\varepsilon, g(a))$. Fix $M \in \mathbb{N}$ such that $3/10^M < \delta$.

We will prove that $h(b, t) \subset N(\varepsilon, h(a, 0))$ for $(b, t) \in I^{N+1}$ such that $\mathtt{d}_0(a, b) \le 1/10^M$ and $t < 1/2^M$.

Given $m \ge M$, let $x \in (\{0, 1, \ldots, 10^m - 1\})^N$ be such that $b \in Q(x)$. If $\sigma \in S_N$, then $\mathtt{d}_0(a, (x + \sigma)/10^m) = \max\{|a_i - (x_i + \sigma_i)/10^m| : 1 \le i \le N\} \le \delta$. Thus $g((x + \sigma)/10^m) \subset A$ for each $\sigma \in S_N$. By Lemma 2.11(c), $f_{E(x)}(10^m(b - x/10^m)) \subset A$. Therefore $h_m(b) \subset A$ for each $m \ge M$. It follows that $h(b, t) \subset A \subset N(\varepsilon, h(a, 0))$.

Now suppose that h is not continuous at $(a, 0)$. Then there exists $B \in \mu^{-1}(t_0) - \{h(a, 0)\}$ and a sequence $((a_n, t_n))_n$ such that $(a_n, t_n) \to (a, 0)$ and $h(a_n, t_n) \to B$. By the paragraph above, for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $h(a_n, t_n) \subset N(\varepsilon, h(a, 0))$ for every $n \ge K$. This implies that $B \subset h(a, 0)$, so $B = h(a, 0)$. This contradiction completes the proof of the continuity of h.

2.16. LEMMA. *Let $g, g^* : I^N \to \mu^{-1}(t_0)$ be maps such that $g|\operatorname{Fr}(I^N) = g^*|\operatorname{Fr}(I^N)$. Let $h, h^* : I^{N+1} \to \mu^{-1}(t_0)$ be the maps constructed as in* 2.13 *for the maps g and g^* respectively. Then $h|\operatorname{Fr}(I^N) \times I = h^*|\operatorname{Fr}(I^N) \times I$ and $h|I^N \times \{1\} = h^*|I^N \times \{1\}$.*

Proof. Consider h^*_m, $E^*(x)$ and A^*_σ constructed as in 2.13 for the map g^*. Let $(a,t) \in \operatorname{Fr}(I^N) \times I$. If $t = 0$, then $h(a,t) = g(a) = g^*(a) = h^*(a,t)$. Now suppose that $t > 0$. To prove that $h(a,t) = h^*(a,t)$, it is enough to prove that $h_m(a) = h^*_m(a)$ for every $m \geq 0$. Let $x = (x_1, \ldots, x_N) \in (\{0, 1, \ldots, 10^m - 1\})^N$ be such that $a \in Q(x)$. We have to prove that $f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m))$. Since $a \in \operatorname{Fr}(I^N)$, there exists $r \in \{1, \ldots, N\}$ such that $a_r = 0$ or 1.

If $a_r = 0$, then $x_r = 0$. We will apply Lemma 2.13 to $p = q = 0$. Given $\sigma_1 \in S_{r-1}$ and $\sigma_2 \in S_{N-r}$, $A_{(\sigma_1,0,\sigma_2)} = g((x + (\sigma_1, 0, \sigma_2))/10^m) = g^*((x + (\sigma_1, 0, \sigma_2))/10^m) = A^*_{(\sigma_1,0,\sigma_2)}$. Thus Lemma 2.13 implies that $f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m))$.

If $a_r = 1$, then $x_r + 1 = 10^m$ and $a_r - x_r/10^m = 1/10^m$. Set $p = q = 1$. Given $\sigma_1 \in S_{r-1}$ and $\sigma_2 \in S_{N-r}$, $A_{(\sigma_1,1,\sigma_2)} = g((x + (\sigma_1, 1, \sigma_2))/10^m) = g^*((x + (\sigma_1, 1, \sigma_2))/10^m) = A^*_{(\sigma_1,1,\sigma_2)}$. Thus Lemma 2.13 implies that $f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m))$. Hence $h(a,t) = h^*(a,t)$.

Now take $a \in I^N$. We will prove that $h(a,1) = h^*(a,1)$. Notice that $h(a,1) = h_0(a) = f_{E(0)}(a)$ and $h^*(a,1) = f_{E^*(0)}(a)$. Given $\sigma \in S_N \subset \operatorname{Fr}(I^N)$, we have $A_\sigma = g(\sigma) = g^*(\sigma) = A^*_\sigma$. Thus $f_{E(0)} = f_{E^*(0)}$. Therefore $h(a,1) = h^*(a,1)$.

2.17. THEOREM. *Every map $G : S^N \to \mu^{-1}(t_0)$ is null homotopic.*

Proof. Let $G : S^N \to \mu^{-1}(t_0)$ be a map. Let $(S^N)^+$ and $(S^N)^-$ be the north and south hemispheres of S^N respectively. Let $g = G|(S^N)^+$ and $g^* = G|(S^N)^-$. Then $g|\operatorname{Fr}((S^N)^+) = g^*|\operatorname{Fr}((S^N)^-)$. Identifying $(S^N)^+$ and $(S^N)^-$ with I^N, we consider h and h^* as in Lemma 2.16. Then $h|(\operatorname{Fr}((S^N)^+) \times I) \cup ((S^N)^+ \times \{1\}) = h^*|(\operatorname{Fr}((S^N)^-) \times I) \cup ((S^N)^- \times \{1\})$. We consider the $(N+1)$-ball B^{N+1} as the space obtained by identifying, in the disjoint union $((S^N)^+ \times I) \mathring{\cup} ((S^N)^- \times I)$, the points of the set $(\operatorname{Fr}((S^N)^+) \times I) \cup ((S^N)^+ \times \{1\})$ with the points of the set $h^*|(\operatorname{Fr}((S^N)^-) \times I) \cup ((S^N)^- \times \{1\})$ in the natural way. Then there exists a map $\overline{h} : B^{N+1} \to \mu^{-1}(t_0)$ which extends both h and h^*. Thus $\overline{h}$ is an extension of G. Hence G is null homotopic.

Remark. Related with this topic, the following question by A. Petrus ([13]) remains open: If X is a contractible dendroid, is then every Whitney level for $C(X)$ contractible?

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