A characterization of dendroids
by the \( n \)-connectedness of the Whitney levels

by

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Abstract. Let \( X \) be a continuum. Let \( C(X) \) denote the hyperspace of all subcontinua of \( X \). In this paper we prove that the following assertions are equivalent: (a) \( X \) is a dendroid, (b) each positive Whitney level in \( C(X) \) is 2-connected, and (c) each positive Whitney level in \( C(X) \) is \( \infty \)-connected (\( n \)-connected for each \( n \geq 0 \)).

Introduction. Throughout this paper \( X \) will denote a continuum (i.e., a compact connected metric space) with metric \( d \). Let \( C(X) \) be the hyperspace of all subcontinua of \( X \) with the Hausdorff metric \( \mathcal{H} \). A Whitney map for \( C(X) \) is a continuous function \( \mu : C(X) \to \mathbb{R} \) satisfying: (a) \( \mu(\{x\}) = 0 \) for each \( x \in X \), (b) if \( A, B \in C(X) \) and \( A \subsetneq B \), then \( \mu(A) < \mu(B) \), and (c) \( \mu(X) = 1 \). A (positive) Whitney level is a set of the form \( \mu^{-1}(t) \) where \( 0 \leq t \leq 1 \) (resp. \( 0 < t \leq 1 \)). \( S^n \) denotes the \( n \)-sphere. A space \( Y \) is \( n \)-connected if, for every \( 0 \leq i \leq n \), each map \( f : S^i \to Y \) is null homotopic; \( Y \) is \( \infty \)-connected if it is \( n \)-connected for each \( n \). A topological property \( P \) is a Whitney property provided whenever a continuum \( X \) has property \( P \), so does every positive Whitney level in \( C(X) \). A map is a continuous function. The unit closed interval is denoted by \( I \), and the set of positive integers by \( \mathbb{N} \).

Positive Whitney levels are continua [1]. Answering questions by J. Krasińskiwicz and S. B. Nadler, Jr., in [9] A. Petrus showed that if \( D \) is a 2-cell, then there exists a Whitney level \( \mathcal{A} \) in \( C(D) \) which is not contractible, in fact \( \mathcal{A} \) has non-trivial fundamental group and non-trivial first singular homology group.

The main theorem in this paper is:

**Theorem.** The following assertions are equivalent:

(i) \( X \) is a dendroid,

(ii) Each positive Whitney level in \( C(X) \) is 2-connected.

(iii) Each positive Whitney level in \( C(X) \) is \( \infty \)-connected.
We divide the proof into two independent sections. In the first section we prove that (ii)⇒(i), and in the second one we prove that (i)⇒(iii).

1. 2-connectedness of Whitney levels implies that $X$ is a dendroid. We will need the following lemma.

1.1. Lemma. Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map. Let $t_0 \in I$. Let $Y$ be a continuum such that $C(Y)$ is contractible. Then every map $f : Y \to \mu^{-1}([0, t_0])$ is homotopic to a map $g : Y \to \mu^{-1}([0, t_0])$ such that $\operatorname{Im}g \subset \mu^{-1}(t_0)$.

Proof. Take a map $f : Y \to \mu^{-1}([0, t_0])$. Since $C(Y)$ is contractible, by [12, Thm. 16.7] there exists a map $F : Y \times I \to C(Y)$ such that, for every $y \in Y$, $F(y, 0) = \{y\}$, $F(y, 1) = Y$ and $s \leq t$ implies that $F(y, s) \subset F(y, t)$.

We distinguish two cases:

(a) $\mu(\bigcup f(Y)) = \mu(\{f(y) \in C(Y) : y \in Y\}) \geq t_0$. Define $G : Y \times I \to C(X)$ by $G(y, t) = \bigcup f(F(y, t)) = \{f(v) \in C(X) : v \in F(y, t)\}$. Then $G$ is a map such that $G(y, 0) = f(y)$ and $G(y, 1) = \bigcup f(Y)$ for every $y \in Y$. Define $K : Y \times I \to \mu^{-1}([0, t_0])$ by

$$K(y, t) = \begin{cases} G(y, t) & \text{if } \mu(G(y, t)) \leq t_0, \\ G(y, s) & \text{if } \mu(G(y, t)) > t_0, \end{cases}$$

where $s \in [0, t_0]$ is chosen in such a way that $\mu(G(y, s)) = t_0$.

Then $K(y, 0) = f(y)$ and $K(y, 1) \in \mu^{-1}(t_0)$, and we define $g : Y \to \mu^{-1}([0, t_0])$ by $g(y) = K(y, 1)$ for every $y \in Y$.

(b) $\mu(\bigcup f(Y)) \leq t_0$. Defining $G$ as in (a), we see that $f$ is homotopic (within $\mu^{-1}([0, t_0])$) to the constant map $y \to \bigcup f(Y)$. Since $\bigcup f(Y) \in \mu^{-1}([0, t_0])$, there exists an ordered arc ([12, Thm. 1.8]) joining $\bigcup f(Y)$ to an element $A_0 \in \mu^{-1}(t_0)$ (within $\mu^{-1}([0, t_0])$). Then we complete the proof of the lemma by defining $g(y) = A_0$ for every $y \in Y$.

We will use the following notions related to Whitney levels:

The space of Whitney levels, $N(X)$, of $X$ is defined by $N(X) = \{A \in C(C(X)) : A$ is a Whitney level in $C(X)\}$. This space was introduced in [5]–[7]. In [7, Lemma 2.2] it was proved that an equivalent metric for $N(X)$ is $H^*(A, B) = \max\{H(A, B) : A \in A, B \in B \text{ and } A \subset B\}$. A partial order for $N(X)$ is defined in [5] by $A \leq B$ if and only if for each $B \in B$, there exists $A \in A$ such that $A \subset B$. If $A \subset N(X)$ is compact and $\gamma$ is an ordered arc in $C(X)$ beginning with a singleton and ending with $X$, then ([5]) $A_\gamma = \bigcap\{A \in \gamma : \text{ there exists } A \in A \subset A \in A \subset B \text{ for some } B \in B\}$. Finally, in [5] it is shown that $\inf(A) = \{A, A \in C(X) : \gamma \text{ is an ordered arc in } C(X) \begin{array}{l} \text{ beginning with a singleton and ending with } X \end{array}\}$ is a Whitney level which is the infimum, in $(N(X), \leq)$, of the set $A$. 

A. Illanes
CONVENTIONS. \( \mathbb{R}^n \) denotes the Euclidean \( n \)-dimensional space. \( e : \mathbb{R} \to S^1 \) denotes the exponential map defined by \( e(t) = (\cos t, \sin t) \). \( D^2 \) is the unit disk in \( \mathbb{R}^2 \). If \( Y \) is a topological space, a map \( f : Y \to S^1 \) can be lifted (\( f \simeq 1 \)) if there exists a map \( g : Y \to \mathbb{R} \) such that \( e \circ g = f \) (equivalently, if \( f \) is null homotopic, see [10, Lemma 5]). If \( A \in C(X) \) and \( \varepsilon > 0 \) then \( N(\varepsilon, A) \) denotes the set \( \{ x \in X : \text{there exists } y \in A \text{ such that } d(x, y) < \varepsilon \} \) and \( B(A, \varepsilon) \) denotes the set \( \{ B \in C(X) : \mathcal{H}(A, B) < \varepsilon \} \). \( 2^X \) denotes the hyperspace of all closed nonempty connected subsets of \( X \).

From now on, in this section, we will suppose that if \( A \) is a positive Whitney level in \( C(X) \), then every map \( f : S^1 \to A \) is null homotopic for \( i = 1, 2 \) (we are not supposing yet that \( A \) is pathwise connected).

1.2. THEOREM. \( X \) is hereditarily unicoherent.

Proof. Suppose, on the contrary, that there exist \( A_1, B_1 \in C(X) \) such that \( A_1 \cap B_1 \) is not connected. Let \( H, K \in 2^X \) be such that \( H \cap K = \emptyset \) and \( A_1 \cap B_1 = H \cup K \). We will construct:

(a) A Whitney map \( \omega \) for \( C(X) \),
(b) A number \( t_0 \in (0, 1] \),
(c) Two open subsets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) in \( \omega^{-1}([0, t_0]) \),
(d) A map \( \lambda : S^1 \to \mathcal{V}_1 \cap \mathcal{V}_2 \) and
(e) A map \( h_1 : \mathcal{V}_1 \cap \mathcal{V}_2 \to S^1 \)

such that \( \omega^{-1}([0, t_0]) = \mathcal{V}_1 \cup \mathcal{V}_2 \), \( h_1 \circ \lambda \) is not homotopic to a constant and, for \( i = 1, 2 \), \( \lambda : S^1 \to \mathcal{V}_i \) can be extended to the disk \( D^2 \). Then, using Lemma 1.1 and a Mayer–Vietoris type sequence we will obtain a contradiction. The construction of these elements is divided into a sequence of steps.

A. There exists \( A_0 \in C(X) \) such that \( A_0 \subsetneq A_1 \), \( A_0 \cap H \neq \emptyset \), \( A_0 \cap K \neq \emptyset \) and \( A_0 \) is minimal with these properties.

To construct \( A_0 \), choose a Whitney map \( \mu \) for \( C(X) \). Let \( t_1 = \min \{ \mu(A) \in I : A \subsetneq A_1, A \cap H \neq \emptyset \text{ and } A \cap K \neq \emptyset \} \). Take \( A_0 \in C(X) \) such that \( \mu(A_0) = t_1 \).

B. Let \( H_1 = A_0 \cap H \) and \( K_1 = A_0 \cap K \). Then there exists \( B_0 \in C(X) \) such that \( B_0 \subsetneq B_1 \), \( B_0 \cap H_1 \neq \emptyset \), \( B_0 \cap K_1 \neq \emptyset \) and \( B_0 \) is minimal with these properties. Define \( H_0 = H_1 \cap B_0 \) and \( K_0 = K_1 \cap B_0 \). Then \( A_0 \cap B_0 = H_0 \cup K_0 \), \( H_0 \cap K_0 = \emptyset \) and \( H_0, K_0 \in 2^X \). Furthermore, if \( A \) (resp. \( B \)) is a proper subcontinuum of \( A_0 \) (resp. \( B_0 \)), then \( A \cap H_0 = \emptyset \) (resp. \( B \cap H_0 = \emptyset \)) or \( A \cap K_0 = \emptyset \) (resp. \( B \cap K_0 = \emptyset \)).

C. Let \( E = A_0 \cup B_0 \). Let \( S^+ = \{(x, y) \in S^1 : y \geq 0 \} \) and \( S^- = \{(x, y) \in S^1 : y \leq 0 \} \). Since \( X \) is metric, Tietze’s Theorem implies that there exists a map \( f_0 : E \to S^1 \) such that \( H_0 = f_0^{-1}((-1, 0)) \), \( K_0 = f_0^{-1}((1, 0)) \), \( f_0(A_0) \subset S^+ \) and \( f_0(B_0) \subset S^- \). Since \( S^1 \) is an ANR (metric), there exists
an open subset $U$ in $X$ and a map $f : U \to S^1$ such that $E \subseteq U$ and $f|E = f_0$. Then the Unique Lifting Theorem implies that $f|E$ cannot be lifted.

D. If $A$ is a proper subcontinuum of $E$, then $f|A \simeq 1$.

To see this, suppose, for example, that $A_0$ is not contained in $A$. Let $A_H = \bigcup \{ L \in C(X) : L$ is a component of $A \cap A_0$ and $L \cap H_0 \neq \emptyset \}$ and let $A_K = \bigcup \{ L \in C(X) : L$ is a component of $A \cap A_0$ and $L \cap H_0 = \emptyset \}$. Then $A_H$ is closed in $X$. We will prove that $A_K$ is closed. If $A \subseteq A_0$, then either $A_K = A$ or $A_K = \emptyset$. Suppose then that $A$ is not contained in $A_0$. If $L$ is a component of $A \cap A_0$, then $L_0$ intersects either $H_0$ or $K_0$ but not both of them. If $x \in \text{Cl}(A_K)$ then $x = \lim n x_n$ where $(x_n)_n$ is a sequence such that, for each $n$, $x_n \in L_n$ for some component $L_n$ of $A_0 \cap A$ such that $L_n \cap H_0 = \emptyset$ (then $L_n \cap K_0 \neq \emptyset$). Therefore the component $L$ of $A_0 \cap A$ which contains $x$ intersects $K_0$. Hence $L \cap H_0 = \emptyset$ and $x \in A_K$. The minimality of $A_0$ implies that $A_H \cap K_0 = \emptyset$. Notice that $A_H \cap A_K = \emptyset$ and $A_K \cap H_0 = \emptyset$.

Thus $A = A_H \cup A_K \cup (A \cap B_0)$. Since $A_H, A_K \subseteq A = f^{-1}(S^\ast)$ and $A \cap B_0 \subseteq B_0 = f^{-1}((S^\ast)^c)$, we find that $f|A_H, f|A_K$ and $f|(A \cap B_0)$ can be lifted. Since $A_H \cap A \cap B_0 \subseteq H_0 = f^{-1}((-1, 0))$, $A_K \cap A \cap B_0 \subseteq K_0 = f^{-1}((1, 0))$ and $A_H \cap A_K = \emptyset$, it follows that $f|A$ can be lifted.

E. There exists an open subset $\mathcal{V}$ of $C(X)$ such that $C(E) - \{ E \} \subseteq \mathcal{V}$ and for each $A \in \mathcal{V}$, $A \subseteq U$ and $f|A \simeq 1$.

Indeed, let $A \in C(E) - \{ E \}$, $f|A \simeq 1$. Then $\{ 0 \}$ there exists an open subset $U_A$ of $U$ containing $A$ such that $f|U_A \simeq 1$. Therefore there exists $\varepsilon_A > 0$ such that if $\mathcal{H}(A, B) < \varepsilon_A$, then $f|B \simeq 1$. Define $\mathcal{V} = \{ B \in C(X) : \mathcal{H}(A, B) < \varepsilon_A$ for some $A \in C(E) - \{ E \} \}$.

F. Fix a Whitney map $\nu_0 : 2^X \to I$. Let $\nu = \nu_0|C(X)$. Define $t^* = \nu(E) > 0$ and define $h : C(X) \times I \times (0, t^*) \to \mathbb{R}$ by $h(A, t, s) = \min \{ \nu(A) t^*/s, \nu_0(A \cup E) + t(\nu(A) - \nu(E)) \}$. Then $h$ is continuous and $h(E, t, s) = t^*$ for every $t \in I$ and $s \in (0, t^*)$. Fix $t \in (0, 1]$ and $s \in (0, t^*)$. Then the map $A \to h(A, t, s)/h(E, t, s)$ from $C(X)$ to $I$ is a Whitney map.

G. If $0 < s_1 < s_2 < t^*$, then there exists $r \in (0, 1]$ such that if $0 < t \leq r$, $A \in \nu^{-1}([s_1, s_2])$ and $h(A, t, s_1) < t^*$, then $A \in \mathcal{V}$.

Indeed, otherwise we can choose sequences $(t_n)_n \in (0, 1]$ and $(D_n)_n \in \nu^{-1}([s_1, s_2])$ such that $t_n \to 0$ and $h(D_n, t_n, s_1) < t^*$ and $D_n \not\subseteq \mathcal{V}$ for all $n$. We may suppose that $D_n \to A$ for some $A \in \nu^{-1}([s_1, s_2])$. Then $A \not\in \mathcal{V}$ and $\nu(A) \leq s_2 < \nu(E)$. Thus $A$ is not contained in $E$ and $\nu_0(A \cup E) > t^*$. Since $t_n(\nu(D_n) - \nu(E)) + \nu_0(D_n \cup E) \to \nu_0(A \cup E)$ and $\nu(D_n) t^*/s_1 \geq t^*$, we conclude that there exists $n \in \mathbb{N}$ such that $h(D_n, t_n, s_1) \geq t^*$. This contradiction completes the proof of G.
H. Choose a sequence \((s_n)_n \subset (0,t^*)\) such that \(s_n \to t^*\) and \(0 < s_1 < s_2 < \ldots\). Let \((t_n)_n \subset (0,1] \) be a sequence such that \(t_n \to 0, \ t_1 > t_2 > \ldots\) and, for each \(n\), if \(A \in \nu^{-1}([s_n, s_{n+1}])\) and \(h(A,t_n,s_n) < t^*\), then \(A \in \mathcal{V}\).

I. Let \(\mathcal{A} = s^{-1}(t^*)\). For each \(n\), define \(\mathcal{A}_n = \{ A \in C(X) : h(A,t_n,s_n) = t^*\}\). Then \(E \in \mathcal{A}_n, \mathcal{A}_n\) is a positive Whitney level, \(\nu^{-1}(s_n) \leq \mathcal{A}_n \leq \mathcal{A}\) and \(\mathcal{A}_n \to \mathcal{A}\).

To see this, let \(A \in \mathcal{A}_n\); then \(t^* \leq \nu(A)t^*/s_n\). Thus \(s_n \leq \nu(A)\). Then there exists \(B \in \nu^{-1}(s_n)\) such that \(B \subset A\). Hence \(\nu^{-1}(s_n) \leq \mathcal{A}_n\).

Now, let \(A \in \mathcal{A}\). Then \(h(A,t_n,s_n) = \min\{\nu_0(A \cup E), (t^*)^2/s_n\}\). Therefore \(h(A,t_n,s_n) \geq t^*\), so that there exists \(B \in C(X)\) such that \(B \subset A\) and \(h(B,t_n,s_n) = t^*\). Thus \(\mathcal{A}_n \leq \mathcal{A}\).

By [7, Lemma 2.2(b)], \(\mathcal{H}^s(\mathcal{A}_n, \mathcal{A}) \leq \mathcal{H}^s(\nu^{-1}(n), \nu^{-1}(t^*)) \to 0\). Hence \(\mathcal{A}_n \to \mathcal{A}\).

J. Define \(\mathcal{B} = \inf(\{\mathcal{A}_n : n \geq 1\})\). Then \(\mathcal{B}\) is a Whitney level. Thus there exists \(t_0 \in I\) and a Whitney map \(\mu\) for \(C(X)\) such that \(\mathcal{B} = \mu^{-1}(t_0)\). Since \(E \in \mathcal{A}\) and \(E \in \mathcal{A}_n\) for all \(n\), it follows that \(E \in \mathcal{B}\) and \(t_0 > 0\).

K. The set \(\mathcal{W} = \nu^{-1}((s_1,t^*)) \cap \mu^{-1}([0,t_0])\) is contained in \(\mathcal{V}\).

Indeed, let \(A \in \mathcal{W}\). Then there exists \(N\) such that \(A \in \nu^{-1}([s_N, s_{N+1}])\).

By H, we must show that \(h(A,t_N,s_N) < t^*\). Suppose, on the contrary, that \(h(A,t_N,s_N) \geq t^*\). Then there exists a subcontinuum \(A^*\) of \(A\) such that \(h(A^*,t_N,s_N) = t^*\). Choose a point \(a \in A^*\). Let \(\gamma\) be an ordered arc in \(C(X)\) joining \(\{a\}\) to \(X\) such that \(A^*, A \in \gamma\). Let \(A_2\) be the unique element in \(\gamma \cap \mathcal{B}\). Since \(\mu(A) < t_0 = \mu(A_2)\), we find that \(A \not\subseteq A_2\). Thus \(A \not\subseteq A_2 = \bigcap\{B \in C(X) : B \in \gamma \cap \{\mathcal{A} \cap \{\mathcal{A}_n : n \in \mathbb{N}\}\}\} \subset A^*\). This contradiction proves that \(A \in \mathcal{V}\).

L. Choose a Whitney map \(\overline{\pi} : 2^X \to I\) which extends \(\mu\) (see [14, Cor. 3.3]). Define \(\omega : C(X) \to I\) by \(\omega(A) = (\overline{\pi}(A \cup E)\overline{\pi}(A))^{1/2}\). Then \(\omega\) is a Whitney map such that \(\omega(E) = \mu(E) = t_0, \ \omega^{-1}(t_0) - \{E\} \subset \mu^{-1}([0,t_0])\) and \(\nu^{-1}((s_1,1]) \cap \omega^{-1}(t_0) \subset \mathcal{V} \cup \{E\}\).

To prove this, let \(A \in \omega^{-1}((s_1,1]) \cap \omega^{-1}(t_0) - \{E\}\). By K, to show that \(A \in \mathcal{V}\), it is enough to prove that \(\nu(A) < t^*\). Suppose that \(\nu(A) \geq t^*\). Then there exists \(A^* \in \nu^{-1}(t^*)\) such that \(A^* \subset A\). Since \(B \leq \nu^{-1}(t^*)\), there exists \(E \in \mathcal{B}\) such that \(B \subset A^*\). Since \(E\) is not contained in \(A\), we have \(t_0 = \omega(A) \geq \omega(B) > \mu(B) = t_0\). This contradiction proves that \(A \in \mathcal{V}\).

M. There exists \(\varepsilon > 0\) such that \(B(E, \varepsilon) \subset \nu^{-1}((s_1,1])\) and if \(\mathcal{H}(A, E) < \varepsilon, \ A \subset B\) and \(B \in \omega^{-1}(t_0)\), then \(B \in \mathcal{V} \cup \{E\}\).

Indeed, let \(\varepsilon_1 > 0\) be such that if \(\mathcal{H}(E, A) < \varepsilon_1\) then \(A \in \nu^{-1}((s_1,1])\).

Let \(\delta > 0\) be such that \(A \subset B\) and \(|\omega(A) - \omega(B)| < \delta\) imply that \(\mathcal{H}(A, B) < \varepsilon_1/2\) (see [12, Lemma 1.28]). Choose \(r_0 \in [0,t_0]\) such that \(t_0 - r_0 < \delta\). Finally, choose \(\varepsilon > 0\) such that \(\varepsilon < \varepsilon_1/2\) and \(\mathcal{H}(A, E) < \varepsilon\) imply that
A ∈ \omega^{-1}([r_0, 1]).

N. Define \( V_1 = B(E, \varepsilon) \cap \omega^{-1}([0, t_0]) \) and \( V_2 = \omega^{-1}([0, t_0]) \setminus \{E\} \). Then \( V_1 \) and \( V_2 \) are open subsets of \( \omega^{-1}([0, t_0]) \) such that \( \omega^{-1}([0, t_0]) = V_1 \cup V_2 \) and if \( A \in V_1 \cap V_2 \), then \( f|A \simeq 1 \).

O. Define \( h_1 : V_1 \cap V_2 \to S^1 \) in the following way: Given \( A \in V_1 \cap V_2 \), take a map \( g_A : A \to \mathbb{R} \) such that \( e \circ g_A = f|A \). Define \( h_1(A) = e(\min g_A(A)) \). Then \( h_1 \) is well defined and continuous.

Indeed, it is easy to prove that \( h_1 \) is well defined. To prove that \( h_1 \) is continuous, take a sequence \( (D_n)_n \) in \( V_1 \cap V_2 \) such that \( D_n \to A \in V_1 \cap V_2 \). Let \( g_A : A \to \mathbb{R} \) be a map such that \( e \circ g_A = f|A \). Let \( U_1 \) be an open subset of \( X \) such that \( A \subset U_1 \subset U \) and \( f|U_1 \simeq 1 \). Let \( g : U_1 \to \mathbb{R} \) be a map such that \( e \circ g = f|U_1 \). Since \( D_n \to A \), there exists \( N \) such that \( D_n \subset U \) for all \( n \geq N \).

Then, for all \( n \geq N \), \( h_1(D_n) = e(\min g(D_n)) \to e(\min g(A)) = h_1(A) \).

P. Choose \( \delta > 0 \) such that \( A \subset B \) and \( |\omega(A) - \omega(B)| < \delta \) imply that \( \mathcal{H}(A, B) < \varepsilon \). Choose \( s* \in (0, t_0) \) such that \( t_0 - s* < \delta \) and \( \omega(A_0), \omega(B_0) < s* \). Choose \( p_0 \in H_0 \) and \( q_0 \in K_0 \). Finally, choose maps \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) from \( I \) to \( C(X) \) such that \( \alpha_1(0) = \{p_0\} = \beta_1(0), \alpha_2(0) = \{q_0\} = \beta_2(0), \alpha_1(1) = A_0 = \alpha_2(1), \beta_1(1) = B_0 = \beta_2(1) \), and, for \( i = 1, 2, s < t \) implies that \( \alpha_i(s) \) is properly contained in \( \alpha_i(t) \) (resp. \( \beta_i(s) \)) and \( \omega(\alpha_i(s)) \) is properly contained in \( \alpha_i(t) \) (resp. \( \beta_i(t) \)) (see [12, Thm. 1.8]).

Q. Choose \( r_1 \in I \) such that \( \omega(B_0 \cup \alpha_2(r_1)) = s* \). Define \( \gamma : [0, 4] \to C(X) \) by

\[
\gamma = \begin{cases} 
\alpha_2((1-t)r_1 + t) \cup \beta_2(w(t)) & \text{if } t \in [0, 1], \\
\beta_1((2-t)(1-w(1))) \cup A_0 \cup \beta_1(x(t)) & \text{if } t \in [1, 2], \\
\beta_1((3-t)(x(2)) + t - 2) \cup \alpha_1(y(t)) & \text{if } t \in [2, 3], \\
\alpha_1((4-t)y(3)) \cup B_0 \cup \alpha_2(z(t)) & \text{if } t \in [3, 4].
\end{cases}
\]

Here \( w(t), x(t), y(t), z(t) \in I \), for \( t \) in the respective intervals, are consecutively chosen in such a way that \( \omega(\gamma(t)) = s* \) for all \( t \in [0, 4] \). Then \( \gamma \) is well defined, continuous, \( \gamma(0) = \gamma(4) \) and \( \gamma(t) \in \omega^{-1}(s*) \cap C(E) \cap V_1 \cap V_2 \) for every \( t \in [0, 4] \).

R. Define \( \lambda : S^1 \to \omega^{-1}(s*) \cap V_1 \cap V_2 \) by \( \lambda(\cos t, \sin t) = \gamma(2t + \pi)/\pi \) if \( t \in [-\pi, \pi] \). Then \( \lambda \) is well defined, continuous and \( h_1 \circ \lambda \) is not homotopic to a constant.

To see that \( h_1 \circ \lambda \) cannot be lifted, we first show that, for each \( z \in S^2 \), there exists a map \( g_z : \lambda(z) \to [-\pi, 2\pi) \) such that \( e \circ g_z = f|\lambda(z) \) and \( 0 \in \text{Im} g_z \). Set \( z = (\cos t, \sin t) \) with \( t \in [-\pi, \pi] \). If \( t \in [-\pi, -\pi/2] \), then \( s = 2(t + \pi)/\pi \in [0, 1] \) and \( \lambda(z) = \gamma(s) = \alpha_2((1 - s)r_1 + s) \cup \beta_2(w(s)) \). If \( \beta_2(w(s)) \) is a proper subset of \( A_0 \) since \( s < t_0 \), the minimality of \( A_0 \) implies that \( \alpha_2((1 - s)r_1 + s) \cap H_0 = \emptyset \). Thus \( f(\alpha_2((1 - s)r_1 + s)) \) is a compact subset of \( S^*-\{(-1, 0)\} \) and, since
$f(\beta_2(w(s)))$ is contained in $S^-$, there exists a map $g_z : \lambda(z) \to [-\pi, \pi)$ such that $f(\lambda(z)) = e \circ g_z$. Since $(1, 0) = f(q_0) \in f(\lambda(z))$, we have $0 \in \text{Im } g_z$. If $\beta_2(w(s))$ is a proper subset of $B_0$, the minimality of $B_0$ implies that $\beta_2(w(s)) \cap H_0 = \emptyset$, so that $f(\beta_2(w(s)))$ is a compact subset of $S^- - \{(1, 0)\}$. Thus there exists a map $g_z : \lambda(z) \to (-\pi, \pi]$ such that $e \circ g_z = f(\lambda(z))$. In the case that $t \in [-\pi/2, 0]$, similar considerations lead to the existence of $g_z$.

Similarly, for each $z \in S^+$, there exists a map $g_z : \lambda(z) \to [0, 3\pi)$ such that $e \circ g_z = f(\lambda(z)$ and $0 \in \text{Im } g_z$.

If $z \in S^-$, then $h_1(\lambda(z)) = e(\min g_z(\lambda(z))) = e([-\pi, 0]) = S^-$, so $h_1(\lambda(z)) \in S^-$ for each $z \in S^-$. Since $\lambda((-1, 0)) = \gamma(0) = \alpha_2(r_1) \cup \beta_2(w(0)) = \alpha_2(r_1) \cup B_0$ and $f(p_0) = (-1, 0)$, it follows that $-\pi$ is in the image of the map $g((-1, 0)) : \lambda((-1, 0)) \to [-\pi, \pi]$. Then $h_1(\lambda((-1, 0))) = e(-\pi) = (1, 0)$. Similarly $h_1(\lambda((1, 0))) = (1, 0)$.

Thus $h_1 \circ \lambda$ is a map from $S^1$ to $S^2$ sending $S^+$ into $S^+$, $S^-$ into $S^-$, $(1, 0)$ into $(1, 0)$ and $(0, 1)$ into $(0, 1)$. This implies that $h_1 \circ \lambda$ cannot be lifted.

S. $\lambda : S^1 \to V_1$ can be extended to a map $\bar{\lambda} : D^2 \to V_1$.

To see this, let $F : S^1 \times I \to C(S^1) (= D^2)$ be a map such that, for each $x \in S^1$, $F(x, 0) = \{x\}$, $F(x, 1) = S^1$ and $s \leq t$ implies that $F(x, s) \subset F(x, t)$. Define $\bar{\lambda} : S^1 \times I \to C(X)$ by $\bar{\lambda}(x, s) = \bigcup \{\lambda(z) \in C(X) : z \in F(x, s)\}$. Then $\bar{\lambda}$ is continuous, $\bar{\lambda}(x, 0) = \lambda(x)$ and $\bar{\lambda}(x, 1) = \bigcup \{\lambda(z) \in C(X) : z \in S^1\} = E$ for all $x \in S^1$. Identifying $D^2$ with $(S^1 \times I)/(S^1 \times \{1\})$, we deduce that $\bar{\lambda}$ is an extension of $\lambda$ to $D^2$. If $x \in S^1$ and $s \in I$, $\lambda(x) = \bar{\lambda}(x, s) \subset \bar{\lambda}(x, s) \subset E$, then $\text{ht}(\bar{\lambda}(x, s), E) \leq \text{ht}(\lambda(x), E) < \varepsilon$ and so $\bar{\lambda}(x, s) \in V_1$ for every $x \in S^1$ and $s \in I$.

T. $\lambda : S^1 \to V_2$ can be extended to a map $\lambda' : D^2 \to V_2$.

This follows from the fact that $\text{Im } \lambda \subset \omega^{-1}(s^*) \subset V_2$ and every map from $S^1$ into $\omega^{-1}(t_1)$ is homotopic to a constant.

This completes the construction of $\omega, t_0, V_1, V_2, \lambda$ and $h_1$. Now we consider the Mayer–Vietoris sequences for the triads $(V_1 \cup V_2, V_1, V_2)$ and $(S^2, S^2_t, S^2_\lambda)$ where $S^2_t = \{(x, y, z) \in S^2 : z \geq 0\}$ and $S^2_\lambda = \{(x, y, z) \in S^2 : z \leq 0\}$. Consider the diagram

\[
\begin{array}{cccc}
0 = H_2(S^2_t) \oplus H_2(S^2_\lambda) & \longrightarrow & H_2(S^2) & \xrightarrow{\partial} & H_1(S^1) & \longrightarrow & 0 \\
\downarrow \Lambda_s & & \Lambda & & \Lambda_s & & \\
H_2(V_1) \oplus H_2(V_2) & \longrightarrow & H_2(V_1 \cup V_2) & \xrightarrow{\partial} & H_1(V_1 \cap V_2)
\end{array}
\]

where $\Lambda : S^2 \to V_1 \cup V_2 = \omega^{-1}([0, t_0])$ is defined in such a way that $\Lambda|S^1 = \lambda, \Lambda|S^2_t = \bar{\lambda}$ and $\Lambda|S^2_\lambda = \lambda'$.
By Lemma 1.1, $\Lambda$ is homotopic to a map $A_0 : S^2 \to \omega^{-1}([0,t_0])$ such that $\text{Im}A_0 \subset \omega^{-1}(t_0)$. Since $\omega^{-1}(t_0)$ is a positive Whitney level, $A_0$ is homotopic to a constant. Therefore $A_\ast$ is the zero homomorphism. This implies that so is $\lambda_\ast$, and hence also the composition $h_1 \circ \lambda_\ast = (h_1 \circ \lambda)_\ast$. This is a contradiction since $h_1 \circ \lambda : S^1 \to S^1$ is not homotopic to a constant. Therefore $X$ is hereditarily unicoherent.

Remark. If $Y$ is a hereditarily indecomposable continuum then every Whitney level $\mathcal{A}$ in $C(Y)$ is hereditarily indecomposable (see [12, Thm. 14.1]); thus every map from $S^n$ into $\mathcal{A}$ is constant for each $n \in \mathbb{N}$. Therefore it is not enough to suppose that the maps from $n$-spheres ($n \geq 1$) into positive Whitney levels in $C(X)$ are null homotopic to conclude that $X$ is a dendroid. On the other hand [11, Example 3], it is not enough to suppose that every positive Whitney level $\mathcal{A}$ in $C(Z)$ is pathwise connected to conclude that $Z$ is pathwise connected. However, as shown below, it suffices to add the assumption that $Z$ is hereditarily unicoherent.

1.3. Lemma. Suppose that $Z$ is a hereditarily unicoherent continuum with the following property: If $p, q \in Z$ and $\varepsilon > 0$, then there exist $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in C(Z)$ such that $p \in A_1$, $q \in A_n$, $A_1 \cap A_2 \neq \emptyset$, $\ldots$, $A_{n-1} \cap A_n \neq \emptyset$ and $\text{diam}(A_i) < \varepsilon$ for each $i$. Then $Z$ is pathwise connected.

Proof. Let $p$ and $q$ be two different points in $Z$ and let $A = \bigcap\{B \in C(Z) : p, q \in B\}$. Since $Z$ is hereditarily unicoherent, we have $A \in C(Z)$. We will prove that $A$ is connected in kleinen at each point. Let $a \in A$ and let $\varepsilon > 0$. Take $A_1, \ldots, A_n \in C(Z)$ such that $p \in A_1$, $q \in A_n$, $A_1 \cap A_2 \neq \emptyset$, $\ldots$, $A_{n-1} \cap A_n \neq \emptyset$ and $\text{diam}(A_i) < \varepsilon$ for each $i$. Let $D = \bigcup\{A_1 : a \in A_i\}$ and let $W = A - \bigcup\{A_1 : a \notin A_i\}$. Then $D \in C(Z)$, $A \subset A_1 \cup \ldots \cup A_n$, $W$ is an open subset of $A$ and $a \in W \subset D \subset B(\{a\}, \varepsilon)$. Hence $A$ is connected in kleinen at $a$. Therefore $A$ is a locally connected continuum. Thus $A$ is pathwise connected (in fact, this implies that $A$ is an arc). Hence $Z$ is pathwise connected.

1.4. Theorem. If $Z$ is hereditarily unicoherent and all its positive Whitney levels are pathwise connected, then $Z$ is pathwise connected.

Proof. Let $p, q \in Z$ and let $\varepsilon > 0$. Fix a Whitney map $\mu$ for $C(Z)$. Let $0 < \delta < 1$ be such that if $A, B \in C(Z)$, $|\mu(A) - \mu(B)| < \delta$ and $A \subset B$, then $\mathcal{H}(A, B) < \varepsilon$. Let $0 < t < \delta/2$. Choose $A, B \in \mu^{-1}(t)$ such that $p \in A$ and $q \in B$. Let $\alpha : I \to \mu^{-1}(t)$ be a map such that $\alpha(0) = A$ and $\alpha(1) = B$. Let $\lambda > 0$ be such that $|t - s| < \lambda$ implies that $\mathcal{H}(\alpha(t), \alpha(s)) < \varepsilon/3$. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $I$ such that $t_1 - t_0 < \delta$ for all $i \geq 1$. For $i \geq 1$, define $A_i = \bigcup\{\alpha(t) : t_{i-1} \leq t \leq t_i\}$. Then $A_1, \ldots, A_n \in C(Z)$, $\text{diam}(A_i) < \varepsilon$ for all $i$, $p \in A_1$, $q \in A_n$ and $A_1 \cap A_2 \neq \emptyset$, $\ldots$, $A_{n-1} \cap A_n \neq \emptyset$. Therefore $Z$ is pathwise connected.
1.5. THEOREM. If each positive Whitney level in \( C(X) \) is 2-connected, then \( X \) is a dendroid.

1.6. COROLLARY. If every positive Whitney level in \( C(X) \) is contractible, then \( X \) is a dendroid.

2. If \( X \) is a dendroid then every positive Whitney level in \( C(X) \) is \( \infty \)-connected. In [12, Thm. 14.8], it was shown that if \( X \) is pathwise connected then every Whitney level for \( C(X) \) is also pathwise connected. So we concentrate our attention on the null homotopy of maps from \( n \)-spheres \( (n \geq 1) \) into positive Whitney levels.

Throughout this section we will suppose that \( X \) is a dendroid. Fix a Whitney map \( \mu \), a number \( t_0 \in (0,1] \) and an integer \( N \in \mathbb{N} \). We will show that every map \( G : S^N \to \mu^{-1}(t_0) \) is null homotopic. To do this, we will need to define a strong form of convergence in \( C(X) \).

2.1. DEFINITION. Given \( x \not= y \in X \), the unique arc joining \( x \) and \( y \) in \( X \) will be denoted by \( \overrightarrow{xy} \). The set \( \{x\} \) will be denoted by \( \overrightarrow{x} \). Define \( \overrightarrow{L : C(X) \times X \to C(X)} \) by \( L(A,x) = \overrightarrow{ax} \) where \( a \) is the unique element in \( A \) such that \( \overrightarrow{a} \cap A = \{a\} \). Given a sequence \( (A_n)_n \) in \( C(X) \) and an element \( A \in C(X) \), we say that \( (A_n)_n \) strongly converges to \( A \) \( (A_n \xrightarrow{s} A) \) if \( A_n \to A \) and \( L(A_n,a) \to \{a\} \) for each \( a \in A \).

The following lemma is easy to prove.

2.2. LEMMA. (a) If \( A_n \xrightarrow{s} A \), \( B_n \xrightarrow{s} B \) and \( A_n \cap B_n \neq \emptyset \) for each \( n \), then \( A_n \cup B_n \xrightarrow{s} A \cup B \).

(b) Let \( (A_n)_n \subset C(X) \) and \( A \in C(X) \) be such that, for each infinite subset \( S \) of \( \mathbb{N} \), there exists a subsequence \( (A_{n_k})_k \) such that \( n_k \in S \) for every \( k \) and \( A_{n_k} \xrightarrow{s} A \). Then \( A_n \xrightarrow{s} A \).

Define \( J : C(X) \times C(X) \to C(X) \) by
\[
J(A,B) = \begin{cases} 
A \cap B & \text{if } A \cap B \neq \emptyset, \\
\{b\} & \text{if } A \cap B = \emptyset,
\end{cases}
\]
where \( b \) is the unique point in \( B \) such that \( \overrightarrow{ab} \cap B = \{b\} \) for each \( a \in A \).

2.3. LEMMA. If \( A_n \xrightarrow{s} A \) and \( B_n \xrightarrow{s} B \), then \( J(A_n,B_n) \xrightarrow{s} J(A,B) \).

Proof. Case 1: \( A \cap B = \emptyset \). Then there exists \( M \) such that \( A_n \cap B_n = \emptyset \) for all \( n \geq M \). Let \( \{a\} = J(B,A) \) and \( \{b\} = J(A,B) \). For each \( n \geq M \), let \( \{a_n\} = J(B_n,A_n) \), \( \{b_n\} = J(A_n,B_n) \) and let \( c_n \in A_n \) and \( d_n \in B_n \) be such that \( \overrightarrow{ac_n} = L(A_n,a) \) and \( \overrightarrow{bd_n} = L(B_n,b) \). Since the set \( \overrightarrow{ac_n} \cup \overrightarrow{ab} \cup \overrightarrow{bd_n} \) is connected and intersects \( A_n \) and \( B_n \), it contains \( \overrightarrow{a_n b_n} \). In particular, \( b_n \in \overrightarrow{ac_n} \cup \overrightarrow{ab} \cup \overrightarrow{bd_n} \to \overrightarrow{ab} \). Thus the limit points of the sequence \( (b_n)_n \) are in \( \overrightarrow{ab} \cap B = \{b\} \). Therefore \( b_n \to b \). Hence \( J(A_n,B_n) \to J(A,B) \).
Since $c_m \to \{a\}$, there exists $M_1 \geq M$ such that $b_n \not\in c_m$ for every $n \geq M_1$. Thus $b_n \in \overline{ab} \cup \overline{bd}$ for all $n \geq M_1$. It follows that $\overline{b_n b} \to \{b\}$. So $L(J(A_n, B_n), b) \to \{b\}$. Thus $J(A_n, B_n) \to J(A, B)$.

Case 2: $A \cap B \neq \emptyset$. First we will prove that $\limsup J(A_n, B_n) \subset J(A, B)$. Let $x \in \limsup J(A_n, B_n)$. Then there exists a subsequence $(n_k)_k$ of $(n)_n$ and, for each $k$, there exists $x_k \in J(A_{n_k}, B_{n_k})$ such that $x_k \to x$. If $A_{n_k} \cap B_{n_k} \neq \emptyset$ for an infinite number of $k$'s, then $x \in A \cap B = J(A, B)$ (in this case). Thus we may suppose that $A_{n_k} \cap B_{n_k} = \emptyset$ for every $k$.

If there exist $z, y \in A \cap B$ such that $z \neq y$, choose $p \in \overline{zy} - \{z, y\}$. For each $k \in \mathbb{N}$, let $a_k, b_k \in A_{n_k}$ be such that $L(A_{n_k}, z) = \overline{ak}$ and $L(A_{n_k}, y) = \overline{bk}$. Since $\overline{ak} \to \{z\}$ and $\overline{bk} \to \{y\}$, there exists $K \in \mathbb{N}$ such that, for all $k \geq K$, $\overline{ak} \cap \overline{bk} = \emptyset$, $\overline{ak} \cap \overline{py} = \emptyset$ and $\overline{bk} \cap \overline{py} = \emptyset$. Given $k \geq K$, $\overline{ak} \subset A_{n_k}$ and $(\overline{ak} \cup \overline{tp} \cup \overline{py} \cup \overline{yr})$ and $(\overline{bk} \cup \overline{tp}) \cap (\overline{py} \cap \overline{yr} = \{p\}$. Therefore $p \in \overline{ak} \cup \overline{bk}$. Hence $p \in A_{n_k}$ for all $k \geq K$. Similarly, there exists $K_1$ such that $p \in B_{n_k}$ for all $k \geq K_1$. This contradicts our assumption. Therefore $A \cap B$ consists of a single point $a_0$.

For each $k \in \mathbb{N}$, let $a_k \in A_{n_k}$ and $b_k \in B_{n_k}$ be such that $\overline{ak} \cup \overline{bk} \cap A_{n_k} = \{a_k\}$ and $a_k b_k \cap B_{n_k} = \{b_k\}$. Then $\{b_k\} = J(A_{n_k}, B_{n_k})$. So $x_k \to b_k$. Suppose that $L(A_{n_k}, a_0) = \overline{ak} a_0$ and $L(B_{n_k}, a_0) = \overline{bk} a_0$ with $\gamma \in A_{n_k}$ and $d_k \in B_{n_k}$. Then $x_k \in a_k b_k \subset \overline{ak} a_0 \cup \overline{bk} d_k \to \{a_0\}$. Therefore $x = a_0 \in A \cap B = J(A, B)$. Hence $\limsup J(A_n, B_n) \subset J(A, B)$.

Now take a point $x \in J(A, B) = A \cap B$. For each $n$, let $a_n \in A_n$ and $b_n \in B_n$ be such that $L(A_n, x) = \overline{an}$ and $L(B_n, x) = \overline{bn}$. If $A_n \cap B_n \neq \emptyset$, then $a_n b_n \subset A_n \cup B_n$. Thus $a_n b_n \cap A_n \cap B_n \neq \emptyset$. Hence $(\overline{an} \cup \overline{bn}) \cap A_n \cap B_n \neq \emptyset$. This implies that $L(A_n \cap B_n, x) \subset \overline{an} \cup \overline{bn}$. If $A_n \cap B_n = \emptyset$, let $\{d_n\} = J(A_n, B_n)$. Then $d_n \in \overline{an} \cup \overline{bn}$ and $L(J(A_n, B_n), x) \subset \overline{an} \cup \overline{bn}$. Therefore $L(J(A_n, B_n), x) \subset \overline{an} \cup \overline{bn}$ for all $n$. Since $\overline{an} \cap \overline{bn} \to \{x\}$, we have $L(J(A_n, B_n), x) \to \{x\}$. Thus $x \in \liminf J(A_n, B_n)$ and we conclude that $J(A_n, B_n) \to J(A, B)$.

In order to give a “uniform” parametrization of the arcs in $X$, we define, for $a, b \in X$, the function $\gamma(a, b) : I \to \overline{ab}$ by $\gamma(a, b)(t) = x$ if $\mu(\overline{ab}) = t \mu(\overline{ab})$ and $x \in \overline{ab}$. Then we have:

2.4. Lemma. For each $a, b \in X$, $\gamma(a, b)$ is a map, $\gamma(a, b)(0) = a$, $\gamma(a, b)(1) = b$ and, if $a \neq b$, then $\gamma(a, b)$ is injective.

2.5. Lemma. If $\{a_n\} \to \{a\}$, $\{b_n\} \to \{b\}$, then $\gamma(a_n, b_n)(r_n) \to \gamma(a, b)(r)$ and $\{\gamma(a_n, b_n)(r_n)\} \to \{\gamma(a, b)(r)\}$.

Proof. Let $\gamma_r = \gamma(a_n, b_n)$ and $\gamma = \gamma(a, b)$. Since $\overline{an} \in \overline{an} \cup \overline{ab} \cup \overline{bn}$ and $\overline{ab} \subset \overline{an} \cup \overline{bn} \cup \overline{bn}$, we have $\overline{an} \to \overline{ab}$ and $\overline{bn} \to \overline{ab}$. First, we will show that $\{\gamma_r(r)\} \to \{\gamma(r)\}$.\[\]
If \( r = 0 \) or \( a = b \), then \( \overrightarrow{\eta_n(r_n)} \to a \), since \( \mu(a_n \gamma_n(r_n)) = r_n \mu(\overrightarrow{a_n b_n}) \to 0 \) and \( a_n \to a \). Since \( L(\gamma_n(r_n)), \gamma(r) = a \gamma_n(r_n) \subset \overrightarrow{a_n a} \cup a_n \gamma_n(r_n) \to \{a\} \), we have \( \{\gamma_n(r_n)\} \xrightarrow{\alpha} \{\gamma(r)\} \).

If \( r = 1 \) and \( a \neq b \), then for \( p \in \overrightarrow{ab} - \{a, b\} \), \( a_n \gamma_n(r_n) \subset \overrightarrow{a_n a} \cup \overrightarrow{ap} \cup \overrightarrow{pb} \cup \overrightarrow{bb} \). Since \( \mu(\overrightarrow{ap} \cup \overrightarrow{mb}) \to \mu(\overrightarrow{mp}) < \mu(\overrightarrow{ab}) \) and \( \mu(a_n \gamma_n(r_n)) = r_n \mu(\overrightarrow{a_n b_n}) \to \mu(\overrightarrow{ab}) \), there exists \( M \) such that \( \gamma_n(r_n) \notin \overrightarrow{a_n a} \cup \overrightarrow{ap} \) for all \( n \geq M \). Thus \( \gamma_n(r_n) \in \overrightarrow{pb} \cup \overrightarrow{pb} \) for all \( n \geq M \). This implies that \( \{\gamma_n(r_n)\} \xrightarrow{\alpha} \{\gamma(r)\} \).

If \( 0 < r < 1 \) and \( a \neq b \), then for \( p \in A \gamma - \{\gamma(r)\} \) and \( q \in \gamma(r) b - \{\gamma(r)\} \), \( a_n \gamma_n(r_n) \subset \overrightarrow{a_n a} \cup \overrightarrow{ap} \cup \overrightarrow{aq} \cup q \overrightarrow{b} \cup \overrightarrow{bb} \). Proceeding as above, there exists \( M \) such that \( \gamma_n(r_n) \notin \overrightarrow{a_n a} \cup \overrightarrow{ap} \) for all \( n \geq M \). If there exists a subsequence \( (\gamma_{n_k}(r_{n_k}))_k \) of \( (\gamma_n(r_n))_n \) such that \( \gamma_{n_k}(r_{n_k}) \to x \) for some \( x \in \overrightarrow{ab} \) and \( a_{n_k} \gamma_{n_k}(r_{n_k}) \to A \) for some \( A \in C(X) \). Then \( a, x \in A, \mu(a_{n_k} \gamma_{n_k}(r_{n_k})) \to r \mu(\overrightarrow{ab}) = \mu(a \gamma(r)) < \mu(\overrightarrow{ap}) \leq \mu(\overrightarrow{a}) \leq \mu(A) = \lim \mu(a_{n_k} \gamma_{n_k}(r_{n_k})). \) This contradiction proves that there exists \( M \in \mathbb{N} \) such that \( \gamma_n(r_n) \notin \overrightarrow{a} \) for all \( n \geq M \). It follows that \( \{\gamma_n(r_n)\} \xrightarrow{\alpha} \{\gamma(r)\} \).

Now we will prove that \( \overrightarrow{\gamma_n(r_n) \gamma_n(t_n)} \xrightarrow{\alpha} \overrightarrow{\gamma(r) \gamma(t)} \). Notice that \( \overrightarrow{\gamma_n(r_n) \gamma_n(t_n)} \to \overrightarrow{\gamma(r) \gamma(t)} \). Given \( p = \gamma(s) \in \overrightarrow{\gamma(r) \gamma(t)} \), there exists a sequence \( (s_n)_n \subset I \) such that \( s_n \to s \) and \( s_n \) is between \( r_n \) and \( t_n \). Then \( \gamma(s_n) \xrightarrow{\alpha} \gamma(s) \). Since \( L(\gamma_n(r_n)) \gamma_n(t_n), \gamma(s) \subset \gamma_n(r_n), \gamma(s) \to \gamma(s) \), we obtain \( \overrightarrow{\gamma_n(r_n) \gamma_n(t_n)} \xrightarrow{\alpha} \overrightarrow{\gamma(r) \gamma(t)} \).

Define \( \mathfrak{A} = \{(A, B) \in C(X) \times C(X) : A \subset B\} \) and \( F : \mathfrak{A} \times I \to C(X) \) by \( F(A, B, t) = \bigcup \{\overrightarrow{\mu(p)} \in C(X) : a \in A, x \in B \text{ and } \mu(\overrightarrow{p}) \leq t\} \).

2.6. Lemma. (a) \( F \) is well defined.
(b) \( F((A, B)) \times I \) is continuous for every \( (A, B) \in \mathfrak{A} \).
(c) \( F(A, B, 0) = A \) and \( F(A, B, 1) = B \).
(d) If \( s \leq t \), then \( F(A, B, s) \subset F(A, B, t) \).

Proof. We only prove (b). Let \( (A, B) \in \mathfrak{A} \) and let \( \varepsilon > 0 \). Let \( \delta > 0 \) be such that if \( A_1 \subset B_1 \) and \( |\mu(A_1) - \mu(B_1)| < \delta \), then \( H(A_1, B_1) < \varepsilon \). It is easy to check that if \( |s - t| < \delta \), then \( H(F(A, B, t), F(A, B, s)) < \varepsilon \). Thus \( F((A, B)) \times I \) is continuous.

2.7. Lemma. If \( A_n \xrightarrow{\alpha} A, B_n \xrightarrow{\alpha} B \) and \( t_n \to t \) with \( (A_n, B_n) \in \mathfrak{A} \) for each \( n \), then \( F(A_n, B_n, t_n) \xrightarrow{\alpha} F(A, B, t) \).

Proof. Take \( x \in \lim \sup F(A_n, B_n, t_n) \). Then \( x = \lim x_k \) where \( x_k \in F(A_n_k, B_n_k, t_{n_k}) \) and \( (n_k)_k \) is a subsequence of \( (n)_n \). For each \( k \), there exists \( a_k \in A_{n_k} \) and \( b_k \in B_{n_k} \) such that \( x_k \in \overrightarrow{a_k b_k} \) and \( \mu(\overrightarrow{a_k b_k}) \leq t_{n_k} \). We may suppose that \( a_k \to a \) for some \( a \in A \) and \( \overrightarrow{a_k b_k} \to C \) for some \( C \in C(X) \). Then \( \overrightarrow{a} \subseteq C \subseteq B \) and \( \mu(\overrightarrow{a}) \leq \mu(C) \leq t \). Hence \( x \in F(A, B, t) \). Therefore \( \lim \sup F(A_n, B_n, t_n) \subset F(A, B, t) \).
Now take \( x \in F(A, B, t) \). Then \( x \in B \) and there exists \( a \in A \) such that \( \mu(\overline{tx}) \leq t \). Let \( s = \mu(\overline{tx}) \). Then there exists a sequence \((s_n)\) with \( 0 \leq s_n \leq t_n \) for all \( s \) and \( s_n \to s \). For each \( n \in \mathbb{N} \), let \( a_n \in A_n \) and \( x_n \in B_n \) be such that \( L(A_n, a) = \overline{a_n}x \) and \( L(B_n, x) = \overline{x_n}x \). Let \( y_n \in F(A_n, B_n, t_n) \) be such that \( L(F(A_n, B_n, t_n), x) = \overline{y_n}x \). If \( \mu(\overline{a_n}x) \leq s_n \), define \( z_n = x_n \). If \( \mu(\overline{a_n}x) \geq s_n \), let \( z_n \) be the unique element in \( \overline{a_n}x_n \) such that \( \mu(\overline{a_n}z_n) = s_n \). Then \( z_n \in F(A_n, B_n, t_n) \).

If \( x = a \), then \( L(F(A_n, B_n, t_n), x) = \overline{y_n}a \subset \overline{a_n}a \to \{a\} \). Therefore \( L(F(A_n, B_n, t_n), x) \to \{x\} \). Now suppose that \( x \neq a \). Given \( p \in \overline{px} = \{a, x\} \), \( z_n \in \overline{a_n}x \subset \overline{a_n}a \cap \overline{ap} \cup \overline{ax} \cup \overline{ax_n} \). Since \( \mu(\overline{a_n}a \cap \overline{ap}) \to \mu(\overline{ap}) < s \), there exists \( M \) such that \( z_n \in \overline{ap} \cup \overline{ax_n} \) for all \( n \geq M \). This implies that \( \overline{ap} \to \{x\} \). Since \( \overline{ap} \subset \overline{ax} \), we have \( L(F(A_n, B_n, t_n), x) \to \{x\} \). It follows that \( F(A_n, B_n, t_n) \to F(A, B, t) \).

Now we “uniformize” the map \( F \). Define \( G : \mathfrak{A} \times I \to C(\mathcal{X}) \) by \( G(A, B, t) = F(A, B, s) \) where \( s \) is chosen in such a way that \( \mu(G(A, B, t)) = \mu(A) + t(\mu(B) - \mu(A)) \).

2.8. Lemma. (a) \( G(A, B, 0) = A \) and \( G(A, B, 1) = B \).

(b) If \( s \leq t \), then \( G(A, B, s) \subset G(A, B, t) \).

(c) If \( A_n \to A, B_n \to B \) and \( t_n \to t \) with \( (A_n, B_n) \in \mathfrak{A} \) for each \( n \), then \( G(A_n, B_n, t_n) \to G(A, B, t) \).

(d) \( G(\{(A, B)\}) \times I \) is continuous for every \((A, B) \in \mathfrak{A} \).

Proof. We only prove (c). We will use Lemma 2.2(b). Let \( S \) be an infinite subset of \( \mathbb{N} \). For each \( n \in S \), let \( G(A_n, B_n, t_n) = F(A_n, B_n, s_n) \) with \( s_n \in I \). Let \( G(A, B, t) = F(A, B, s) \). Take a subsequence \((n_k)_k \) of \((n)_n \) such that \( n_k \in S \) for all \( k \) and \( s_{n_k} \to s^* \) for some \( s^* \in I \). Then \( G(A_{n_k}, B_{n_k}, t_{n_k}) \to F(A, B, s^*) \). This yields \( \mu(F(A, B, s^*)) = \lim(\mu(A_{n_k}) + t_{n_k}(\mu(B_{n_k}) - \mu(A_{n_k}))) = \mu(G(A, B, t)) = \mu(F(A, B, s)) \). It follows that \( F(A, B, s^*) = F(A, B, s) \). Hence \( G(A_{n_k}, B_{n_k}, t_{n_k}) \to G(A, B, t) \). Therefore \( G(A_n, B_n, t_n) \to G(A, B, t) \).

Now we define “standard” arcs joining elements in \( \mu^{-1}(t_0) \). Define \( \alpha : \mu^{-1}(t_0) \times \mu^{-1}(t_0) \times I \to \mu^{-1}(t_0) \) in the following way:

A. If \( A \cap B = \emptyset \), let \( \{a\} = J(B, A) \), \( \{b\} = J(A, B) \) and \( \gamma = \gamma(a, b) \).

A.1. If \( \mu(ab) \leq t_0 \), let \( s_0 \) be the unique number in \( I \) such that \( \mu(\overline{ab} \cup G(\{a\}, A, s_0)) = t_0 \), then define:

\[
\alpha(A, B, t) = \begin{cases} 
\overline{a\gamma(3t)} \cup G(\{a\}, A, s) & \text{if } 0 \leq t \leq 1/3, \\
G(\{a\}, A, (2 - 3t)s_0) \cup \overline{ab} \cup G(\{b\}, B, s) & \text{if } 1/3 \leq t \leq 2/3, \\
\overline{\gamma(3t - 2)b} \cup G(\{b\}, B, s) & \text{if } 2/3 \leq t \leq 1.
\end{cases}
\]
In the three cases the element \( s \in I \) is chosen in such a way that \( \mu(\alpha(A, B, t)) = t_0. \)

2. If \( \mu(ab) \geq t_0, \) let \( s_0 \) and \( r_0 \) be the unique elements in \( I \) such that
\[
\mu(\alpha_0(s_0)) = t_0 = \mu(\gamma(r_0)b).
\]
Then define
\[
\alpha(A, B, t) = \begin{cases} 
\alpha_0(3t_0s_0) \cup G(\{a\}, A, s) & \text{if } 0 \leq t \leq 1/3, \\
\gamma(s)\gamma((2 - 3t)\epsilon + 3t - 1) & \text{if } 1/3 \leq t \leq 2/3, \\
\gamma((3t - 2 + (3 - 3t)\epsilon)b) \cup G(\{b\}, B, s) & \text{if } 2/3 \leq t \leq 1,
\end{cases}
\]

with \( s \) chosen as above.

B. If \( A \cap B \neq \emptyset, \) define
\[
\alpha(A, B, t) = \begin{cases} 
A & \text{if } 0 \leq t \leq 1/3, \\
G(A \cap B, A, 2 - 3t) \cup G(A \cap B, B, s) & \text{if } 1/3 \leq t \leq 2/3, \\
B & \text{if } 2/3 \leq t \leq 1,
\end{cases}
\]

with \( s \) chosen in the same way.

It is easy to check that \( \alpha \) is well defined, \( \alpha(A, B, 0) = A \) and \( \alpha(A, B, 1) = B \) for all \( (A, B) \in \mu^{-1}(t_0) \times \mu^{-1}(t_0) \) and if \( A, B \subset A_0 \subset C(X), \) then \( \alpha(A, B, t) \subset A_0 \) for each \( t \in I. \)

2.9. Lemma. If \( A_n \xrightarrow{\alpha} A, B_n \xrightarrow{\alpha} B \) and \( t_n \rightarrow t, \) then \( \alpha(A_n, B_n, t_n) \xrightarrow{\alpha} \alpha(A, B, t) \) \( (A_n, B_n, A \text{ and } B \text{ in } \mu^{-1}(t_0)). \)

Proof. We will use Lemma 2.2(b). Let \( S \) be an infinite subset of \( \mathbb{N}. \) We need to analyze several cases.

1. \( A \cap B \neq \emptyset. \)

1.1. \( A_{n_k} \cap B_{n_k} = \emptyset \) for infinitely many elements \( n_1 < n_2 < \ldots \) in \( S. \) For each \( k, \) let \( \{a_{n_k}\} = J(B_{n_k}, A_{n_k}) \) and \( \{b_{n_k}\} = J(A_{n_k}, B_{n_k}). \) Since \( \{a_{n_k}\} = J(A_{n_k}, B_{n_k}) \xrightarrow{\alpha} J(A, B) = A \cap B, \) \( A \cap B \) consists of a single point \( a_0. \) Then \( \{a_{n_k}\} = J(B_{n_k}, A_{n_k}) \xrightarrow{\alpha} \{a_0\}. \) For each \( k, \) let \( \gamma_k = \gamma(a_{n_k}, b_{n_k}). \) It follows that, for all sequences \( (r_k)_k \) and \( (m_k)_k \) in \( \mathcal{I}, \) \( \gamma_k(r_k)\gamma_k(m_k) \xrightarrow{\alpha} \{a_0\}. \)

1.1.1. \( t_0 = 0. \) Then \( \mu(a_{n_k}b_{n_k}) \geq t_0, \) so \( \alpha(A_{n_k}, B_{n_k}, t_{n_k}) \) is equal to either \( \{a_{n_k}\}, \) a point in \( \gamma_k(0)\gamma_k(1) = a_{n_k}b_{n_k} \) or \( \{b_{n_k}\}. \) Thus \( \alpha(A_{n_k}, B_{n_k}, t_{n_k}) \xrightarrow{\alpha} \{a_0\} = A = B = \alpha(A, B, t). \)

1.1.2. \( t_0 > 0. \) We may suppose that \( \mu(a_{n_k}b_{n_k}) < t_0 \) for every \( k. \) For each \( k, \) let \( s^k \in I \) be such that \( \mu(a_{n_k}b_{n_k} \cup G(\{a_{n_k}\}, A_{n_k}, s^k)) = t_0 \) and let \( s_k \) be the number chosen so that \( \mu(\alpha(A_{n_k}, B_{n_k}, t_{n_k})) = t_0. \) We may suppose that \( s_k \rightarrow s^* \) for some \( s^* \in I \) and \( s^k \rightarrow s^* \) for some \( s^* \in I. \) Then \( \mu(a_{n_k}b_{n_k}) = \mu(s^* \mu(\alpha_{n_k}b_{n_k})) = \mu(\{a_{n_k}\} + s^* \mu(A) - \mu(\{a_{n_k}\})), \) and so \( s^* = 1. \) We may suppose that one of the following three cases holds:
1.1.2.1. \( t_{nk} \in [0, 1/3] \) for every \( k \). Then \( t \in [0, 1/3] \) and \( \alpha(A_{nk}, B_{nk}, t_{nk}) \xrightarrow{s} G(\{a_0\}, A, s') = A = \alpha(A, B, t) \).

1.1.2.2. \( t_{nk} \in [1/3, 2/3] \) for every \( k \). Then \( t \in [1/3, 2/3] \) and we have \( \alpha(A_{nk}, B_{nk}, t_{nk}) \xrightarrow{s} G(\{a_0\}, A, (2 - 3t)s') \cup G(\{a_0\}, B, s') = \alpha(A, B, t) \).

1.1.2.3. \( t_{nk} \in [2/3, 1] \) for every \( k \). Then \( t \in [2/3, 1] \) and \( \alpha(A_{nk}, B_{nk}, t_{nk}) \xrightarrow{s} G(\{a_0\}, B, s') = B = \alpha(A, B, t) \).

This completes Subcase 1.1.

1.2. \( A_{nk} \cap B_{nk} \neq \emptyset \) for infinitely many elements \( n_1 < n_2 < \ldots \) in \( S \). Then we may suppose that one of the following three cases holds:

1.2.1. \( t_{nk} \in [0, 1/3] \) for all \( k \). Then \( \alpha(A_{nk}, B_{nk}, t_{nk}) = A_{nk} \xrightarrow{s} A = \alpha(A, B, t) \).

1.2.2. \( t_{nk} \in [1/3, 2/3] \) for all \( k \). So \( \alpha(A_{nk}, B_{nk}, t_{nk}) = B_{nk} \xrightarrow{s} B = \alpha(A, B, t) \).

1.2.3. \( t_{nk} \in [2/3, 1] \) for every \( k \). Then \( \alpha(A_{nk}, B_{nk}, t_{nk}) = G(A_{nk} \cap B_{nk}, A_{nk}, 2 - 3t_{nk}) \cup G(A_{nk} \cap B_{nk}, B_{nk}, s_k) \), where \( s_k \in I \), and we may suppose that \( s_k \rightarrow s' \) for some \( s' \in I \). Then \( \alpha(A_{nk}, B_{nk}, t_{nk}) \xrightarrow{s} G(J(A, B), A, 2 - 3t) \cup G(J(A, B), B, s') = \alpha(A, B, t) \).

This completes the proof of Case 1.

2. \( A \cap B = \emptyset \). Then we may suppose that \( A_n \cap B_n = \emptyset \) for every \( n \in S \).

Here it is necessary to consider the following cases:

2.1. \( \mu(\overline{a_n b_n}) \geq t_0 \) for infinitely many elements \( n_1 < n_2 < \ldots \) in \( S \).

2.1.1. \( t_{nk} \in [0, 1/3] \) for every \( k \).

2.1.2. \( t_{nk} \in [1/3, 2/3] \) for every \( k \).

2.1.3. \( t_{nk} \in [2/3, 1] \) for every \( k \).

2.2. \( \mu(\overline{a_n b_n}) \leq t_0 \) for infinitely many elements \( n_1 < n_2 < \ldots \) in \( S \).

2.2.1. \( t_{nk} \in [0, 1/3] \) for every \( k \).

2.2.2. \( t_{nk} \in [1/3, 2/3] \) for every \( k \).

2.2.3. \( t_{nk} \in [2/3, 1] \) for every \( k \).

All of them can be treated similarly to Case 1.

Hence, in each one of the cases, infinitely many elements \( n_1 < n_2 < \ldots \) of \( S \) can be obtained such that \( \alpha(A_{nk}, B_{nk}, t_{nk}) \xrightarrow{s} \alpha(A, B, t) \).

Therefore \( \alpha(A_n, B_n, t_n) \xrightarrow{s} \alpha(A, B, t) \).

2.10. Construction. For each \( r \in \mathbb{N} \), let \( S_r = \{\{0, 1\}\}^r \). For each set \( E = \{A_{\sigma} \in \mu^{-1}(t_0) : \sigma \in S_N\} \) define \( f_E : I^N \rightarrow \mu^{-1}(t_0) \) through the following steps:

- \( f_E(a_1, \sigma_1) = \alpha(A_{(0, \sigma_1)}, A_{(1, \sigma_1)}, a_1) \) if \( a_1 \in I \) and \( \sigma_1 \in S_{N-1} \).
- \( f_E(a_1, a_2, \sigma_2) = \alpha(f_E(a_1, 0, \sigma_2), f_E(a_1, 1, \sigma_2), a_2) \) if \( a_1, a_2 \in I \) and \( \sigma_2 \in S_{N-2} \).
If \(2 \leq r < N\), then \(f_E(a_1, \ldots, a_r, \sigma_r) = \alpha(f_E(a_1, \ldots, a_{r-1}, 0, \sigma_r), \sigma_E(a_1, \ldots, a_r)\) for \(a_1, \ldots, a_r \in I\) and \(\sigma_r \in S_{N-r}\).

If \(r = N\), then we set \(f_E(a_1, \ldots, a_N) = \alpha(f_E(a_1, \ldots, a_{N-1}, 0), f_E(a_1, \ldots, a_{N-1}, 1), a_N)\) for \(a_1, \ldots, a_N \in I\).

The following lemma is easy to prove.

2.11. **Lemma.** (a) \(f_E\) is well defined.

(b) If \((a_n)_n \subset I^N\) and \(a \in I^N\) are such that \(a_n \to a\) then \(f_E(a_n) \to f_E(a)\).

(c) If \(A \subset A \subset C(X)\) for each \(\sigma \in S_N\), then \(f_E(a) \subset A\) for every \(a \in I^N\).

2.12. **Lemma.** Let \(p,q \in \{0,1\}\). Let \(E = A_\sigma : \sigma \in S_N\) and \(D = \{B_\sigma : \sigma \in S_N\}\) and let \(r \in \{1,\ldots,N\}\) be such that \(A_{(\sigma_1,p,\sigma_2)} = B_{(\sigma_1,q,\sigma_2)}\) for each \(\sigma_1 \in S_{r-1}\) and \(\sigma_2 \in S_{N-r}\). Then \(f_E(a_1,p,a_2) = f_D(a_1,q,a_2)\) for every \(a_1 \in I^{r-1}\) and \(a_2 \in I^{N-r}\).

**Proof.** Let \(x = (x_1,\ldots,x_N), y = (y_1,\ldots,y_N) \in I^N\) be such that \(x_r = y_r, y_i = q\) and \(x_i = y_i\) for all \(i \neq r\). We will show, by induction on \(k\), that if \(x_{k+1},\ldots,x_N, y_{k+1},\ldots,y_N \in \{0,1\}\) then \(f_E(x) = f_D(y)\).

Suppose that \(k = 1\). Let \(\sigma = (x_2,\ldots,x_N)\) and \(q = (y_2,\ldots,y_N) \in S_{N-1}\). If \(r > 1\), then \(A_{(0,\sigma)} = B_{(0,\sigma)}\), \(A_{(1,\sigma)} = B_{(1,\sigma)}\) and \(x_1 = y_1\). Then \(f_E(x) = \alpha(A_{(0,\sigma)}, A_{(1,\sigma)}, x_1) = \alpha(B_{(0,\sigma)}, B_{(1,\sigma)}, y_1) = f_D(y)\). If \(r = 1\), then \(\sigma = q\). Notice that \(f_E(x) = A_{(p,\sigma)}\) and \(f_D(y) = B_{(q,\sigma)}\). Thus \(f_E(x) = f_D(y)\).

Suppose that the assertion holds for \(k < n\). Suppose that \(x_{k+2},\ldots,x_N, y_{k+2},\ldots,y_N \in \{0,1\}\). Then \(f_E(x) = \alpha(f_E(x_1,\ldots,x_k,0,x_{k+2},\ldots,x_N), f_E(x_1,\ldots,x_k,1,x_{k+2},\ldots,x_N)\), \(x_{k+1} = \ast\). If \(k + 1 \neq r\), the induction hypothesis implies that \(\ast = f_D(y)\), and if \(k + 1 = r\), then \(f_E(x) = f_D(y)\), which, by the induction hypothesis, is equal to \(f_D(y_1,\ldots,y_k,q,y_{k+2},\ldots,y_N) = f_D(y)\).

This completes the induction. Then the theorem follows by taking \(k = N\).

2.13. **Construction.** Let \(g : I^N \to \mu^{-1}(t_0)\) be a map. Given \(m \in \mathbb{N} \cup \{0\}\) and \(x = (x_1,\ldots,x_N) \in \{0,1,\ldots,10^m - 1\}\). Define \(Q(x) = [x_1/10^m, (x_1 + 1)/10^m] \times \cdots \times [x_N/10^m, (x_N + 1)/10^m]\) and \(E(x) = \{A_\sigma : \sigma \in S_N\}\) where \(A_\sigma = g((x + \sigma)/10^m)\) for every \(\sigma \in S_N\). Next, define \(h_x : Q(x) \to \mu^{-1}(t_0)\) by \(h_x(a) = f_E(x)(10^m(a - x/10^m))\). Then \(h_x\) is well defined. Now define \(h_m : I^N \to \mu^{-1}(t_0)\) by \(h_m(a) = h_x(a)\) if \(a \in Q(x)\). Finally, define \(h : I^{N+1} \to \mu^{-1}(t_0)\) by

\[h(a,t) = \begin{cases} g(a) & \text{if } t = 0, \\ \alpha(h_{m+1}(a), h_m(a), 2^{m+1}(t - 1/2^{m+1})) & \text{if } t \in [1/2^{m+1}, 1/2^m]. \end{cases} \]
2.14. **Lemma.** For each $m$, $h_m$ is well defined and, if $a_n \to a$, then $h_m(a_n) \xrightarrow{s} h_m(a)$.

**Proof.** To see that $h_m$ is well defined take a point $a \in Q(x) \cap Q(y)$. First suppose that $x$ and $y$ differ just in one coordinate $r$. Suppose that $x_r < y_r$. Then $a_r 10^m = y_r = x_r + 1$. Then $h_m(a)$ can be defined as $f_{E(x)}(10^m(a - x/10^m))$ and $f_{E(y)}(10^m(a - y/10^m))$ where $E(x) = \{g((x + \sigma)/10^m) : \sigma \in S_N\}$ and $E(y) = \{g((y + \sigma)/10^m) : \sigma \in S_N\}$.

We will apply Lemma 2.12. Let $c = 10^m(a - x/10^m)$ and $d = 10^m(a - y/10^m)$. Then $c_r = 1$ and $d_r = 0$. Let $p = 1$ and $q = 0$. For $\sigma_1 \in S_{r_1}$ and $\sigma_2 \in S_{r_2}$, we have $g((x + (\sigma_1, p, \sigma_2))/10^m) = g((y + (\sigma_1, q, \sigma_2))/10^m)$. Hence, by Lemma 2.12, $f_{E(x)}(c) = f_{E(y)}(d)$. Thus $f_{E(x)}(10^m(a - x/10^m)) = f_{E(y)}(10^m(a - y/10^m))$.

If $x$ and $y$ differ in more than one coordinate, considering the vectors $(x_1, y_2, \ldots, y_N) \cup (x_1, x_2, y_3, \ldots, y_N) \cup \ldots (x_1, \ldots, x_{N-1}, y_N)$, we conclude that $h_m$ is well defined.

The second part of the lemma follows from Lemma 2.11(b).

2.15. **Lemma.** $h$ is well defined and continuous.

**Proof.** It is easy to check that $h$ is well defined. From Lemma 2.13 it follows that if $(a_n, t_n) \to (a, t)$ and $t > 0$ then $h(a_n, t_n) \xrightarrow{s} h(a, t)$. Thus $h$ is continuous at $(a, t)$ if $t > 0$.

Now take a point $(a, 0) \in I^{N+1}$; we will check that $h$ is continuous at this point. Let $\varepsilon > 0$. Consider the metric $d_0$ in $I^N$ defined by $d_0(a, c) = \max\{|a_i - c_i| : 1 \leq i \leq N\}$. Let $\delta > 0$ be such that $d_0(a, b) < \delta$ implies that $H(g(a), g(b)) < \varepsilon$. Let $A_0 = [a_1 - \delta, a_1 + \delta] \times \ldots \times [a_N - \delta, a_N + \delta]$ and let $A = \bigcup (g(b) : b \in A_0 \cap I^N)$. Then $A$ is a subcontinuum of $X$ and $A \subset N(\varepsilon, g(a))$. Fix $M \in \mathbb{N}$ such that $3/10^M < \delta$.

We will prove that $h(b, t) \subset N(\varepsilon, h(a, 0))$ for $(b, t) \in I^{N+1}$ such that $d_0(a, b) \leq 1/10^M$ and $t < 1/2^M$.

Given $m \geq M$, let $x \in \{(0, 1, \ldots, 10^m - 1)\}^N$ be such that $b \in Q(x)$. If $\sigma \in S_N$, then $d_0(a, (x + \sigma)/10^m) = \max\{|a_i - (x_i + \sigma_i)/10^m| : 1 \leq i \leq N\} < \delta$. Thus $g((x + \sigma)/10^m) \subset A$ for each $\sigma \in S_N$. By Lemma 2.11(c), $f_{E(x)}(10^m(b - x/10^m)) \subset A$. Therefore $h_m(b) \subset A$ for each $m \geq M$. It follows that $h(b, t) \subset A \subset N(\varepsilon, h(a, 0))$.

Now suppose that $h$ is not continuous at $(a, 0)$. Then there exists $B \in \mu^{-1}(t_0 - \{h(a, 0)\})$ and a sequence $(a_n, t_n)$ such that $(a_n, t_n) \to (a, 0)$ and $h(a_n, t_n) \to B$. By the paragraph above, for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $h(a_n, t_n) \subset N(\varepsilon, h(a, 0))$ for every $n \geq K$. This implies that $B \subset h(a, 0)$, so $B = h(a, 0)$. This contradiction completes the proof of the continuity of $h$. 
2.16. Lemma. Let \( g, g^* : I^N \to \mu^{-1}(t_0) \) be maps such that \( g|\text{Fr}(I^N) = g^*|\text{Fr}(I^N) \). Let \( h, h^* : I^{N+1} \to \mu^{-1}(t_0) \) be the maps constructed as in 2.13 for the maps \( g \) and \( g^* \) respectively. Then \( h|\text{Fr}(I^N) \times I = h^*|\text{Fr}(I^N) \times I \) and \( h|I^N \times \{1\} = h^*|I^N \times \{1\} \).

Proof. Consider \( h_m^*, E^*(x) \) and \( A_r^* \) constructed as in 2.13 for the map \( g^* \). Let \( (a, t) \in \text{Fr}(I^N) \times I \). If \( t = 0 \), then \( h(a, t) = g(a) = g^*(a) = h^*(a, t) \). Now suppose that \( t > 0 \). To prove that \( h(a, t) = h^*(a, t) \), it is enough to prove that \( h_m(a) = h_m^*(a) \) for every \( m \geq 0 \). Let \( x = (x_1, \ldots, x_N) \in (\{0, 1, \ldots, 10^m - 1\})^N \) be such that \( a \in Q(x) \). We have to prove that \( f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m)) \). Since \( a \in \text{Fr}(I^N) \), there exists \( r \in \{1, \ldots, N\} \) such that \( a_r = 0 \) or \( 1 \).

If \( a_r = 0 \), then \( x_r = 0 \). We will apply Lemma 2.13 to \( p = q = 0 \). Given \( \sigma_1 \in S_{r-1} \) and \( \sigma_2 \in S_{N-r}, A_{(\sigma_1, 0, \sigma_2)} = g((x + (\sigma_1, 0, \sigma_2))/10^m) = g^*((x + (\sigma_1, 0, \sigma_2))/10^m) = A_r(\sigma_1, 0, \sigma_2) \). Thus Lemma 2.13 implies that \( f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m)) \).

If \( a_r = 1 \), then \( x_r + 1 = 10^m \) and \( a_r - x_r/10^m = 1/10^m \). Set \( p = q = 1 \). Given \( \sigma_1 \in S_{r-1} \) and \( \sigma_2 \in S_{N-r}, A_{(\sigma_1, 1, \sigma_2)} = g((x + (\sigma_1, 1, \sigma_2))/10^m) = g^*((x + (\sigma_1, 1, \sigma_2))/10^m) = A_r(\sigma_1, 1, \sigma_2) \). Thus Lemma 2.13 implies that \( f_{E(x)}(10^m(a - x/10^m)) = f_{E^*(x)}(10^m(a - x/10^m)) \). Hence \( h(a, t) = h^*(a, t) \).

Now take \( a \in I^N \). We will prove that \( h(a, 1) = h^*(a, 1) \). Notice that \( h(a, 1) = h_0(a) = f_{E^*(0)}(a)\) and \( h^*(a, 1) = f_{E^*(0)}(a) \). Given \( \sigma \in S_N \subset \text{Fr}(I^N) \), we have \( A_r = g(\sigma) = g^*(\sigma) = A_r^* \). Thus \( f_{E^*(0)} = f_{E^*(0)} \). Therefore \( h(a, 1) = h^*(a, 1) \).

2.17. Theorem. Every map \( G : S^N \to \mu^{-1}(t_0) \) is null homotopic.

Proof. Let \( G : S^N \to \mu^{-1}(t_0) \) be a map. Let \( (S^N)^+ \) and \( (S^N)^- \) be the north and south hemispheres of \( S^N \) respectively. Let \( g = G|(S^N)^+ \) and \( g^* = G|(S^N)^- \). Then \( g|\text{Fr}((S^N)^+) = g^*|\text{Fr}((S^N)^-) \). Identifying \( (S^N)^+ \) and \( (S^N)^- \) with \( I^N \), we consider \( h \) and \( h^* \) as in Lemma 2.16. Then \( h|\text{Fr}((S^N)^+ \times I) \cup ((S^N)^- \times \{1\}) = h^*|\text{Fr}((S^N)^- \times I) \cup ((S^N)^+ \times \{1\}) \). We consider the \( (N + 1) \)-ball \( B^{N+1} \) as the space obtained by identifying, in the disjoint union \( ((S^N)^+ \times I) \cup ((S^N)^- \times I) \), the points of the set \( \text{Fr}((S^N)^+ \times I) \cup ((S^N)^+ \times \{1\}) \) with the points of the set \( h^*|\text{Fr}((S^N)^- \times I) \cup ((S^N)^- \times \{1\}) \) in the natural way. Then there exists a map \( \tilde{h} : B^{N+1} \to \mu^{-1}(t_0) \) which extends both \( h \) and \( h^* \). Thus \( \tilde{h} \) is an extension of \( G \). Hence \( G \) is null homotopic.

Remark. Related with this topic, the following question by A. Petrus ([13]) remains open: If \( X \) is a contractible dendroid, is then every Whitney level for \( C(X) \) contractible?
References


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