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Relatively recursive expansions*

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Abstract. In this paper, we consider the following basic question. Let $\mathbf{A}$ be an L-structure and let ψ be an infinitary sentence in the language $L \cup \{R\}$, where R is a new relation symbol. When is it the case that for every $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and $R \leq_T D(\mathbf{B})$? We succeed in giving necessary and sufficient conditions in the case where ψ is a "recursive" infinitary Π_2 sentence. (A *recursive* infinitary formula is an infinitary formula with recursive disjunctions and conjunctions.) We consider also some variants of the basic question, in which R is r.e., Δ^0_α, or Σ_α instead of recursive relative to $D(\mathbf{B})$.

§ 0. Introduction. In [AN] it was shown that the syntactical characterization by Goncharov of the recursive structures that are "recursively stable" could be deduced quickly (without repeating the priority argument) from a characterization of the "intrinsically recursive" relations on recursive structures. The same method can be used to deduce the syntactical characterization of "relatively recursively stable" structures (obtained in [AKMS] and, independently, in [C]) from a characterization of "relatively intrinsically recursive" relations, without repeating the forcing argument. Our aim here is to obtain a general result on expansions which has as one consequence the syntactical characterization of "relatively recursively categorical" structures (also obtained in [AKMS] and [C]).

In this section, we give some examples to suggest the form of our result on the basic question. At the end of the present section, we give a more complete definition of the recursive infinitary formulas. Our result on the basic question stated in the abstract will be proved in Section 1, and the variants of the basic question are considered in Section 2. Section 3 has some applications.

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All structures here are assumed to have universe ω, and all languages are recursive. Formulas are identified with their Gödel numbers. Then for any structure $\mathbf{A}$, the open diagram $D(\mathbf{A})$ has a Turing degree. If $D(\mathbf{A})$ is recursive, then $\mathbf{A}$ is said to be *recursive*. A structure $\mathbf{A}$ is *relatively recursively categorical* if for each pair $\mathbf{B}$, $\mathbf{C}$ of isomorphic copies of $\mathbf{A}$, there exists $f \leq_T D(\mathbf{B}) \oplus D(\mathbf{C})$ such that $B \cong_f \mathbf{C}$. The theorem below gives the syntactical characterization in the special case where $\mathbf{A}$ is a recursive structure. (The more general result is obtained by relativizing.)

THEOREM 0.1 ([AKMS] and [C]). *Let* $\mathbf{A}$ *be a recursive structure. Then* $\mathbf{A}$ *is relatively recursively categorical iff for some finite sequence* $\mathbf{c}$ *from* $\mathbf{A}$, *there is a recursive sequence of existential formulas* $(\varphi_i(\mathbf{c}, \mathbf{x}_i))_{i\in\omega}$ (*all having parameters* $\mathbf{c}$, *but with different sequences* $\mathbf{x}_i$ *of free variables*) *such that*

(i) *for each finite sequence* $\mathbf{a}$ *from* $\mathbf{A}$, *there is at least one* i *such that* $A \models \varphi_i(\mathbf{c}, \mathbf{a})$,

(ii) *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are two sequences such that for some* i, $\mathbf{A} \models \varphi_i(\mathbf{c}, \mathbf{a})$ $\& \varphi_i(\mathbf{c}, \mathbf{b})$, *then* $(\mathbf{A}, \mathbf{a}) \cong (\mathbf{A}, \mathbf{b})$.

There is a restatement of Theorem 0.1 involving "cardinal sums", suggested by [AN]. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be structures for the same language L. A *cardinal sum* of $\mathbf{B}_1$ and $\mathbf{B}_2$ is a structure $\mathbf{B}$ such that

(1) the language of $\mathbf{B}$ is $L \cup \{U_1, U_2\}$ (where the U_i are unary predicate symbols not in L),

(2) there exist functions f_i mapping ω one-one onto $U_i^{\mathbf{B}}$, where the sets $U_i^{\mathbf{B}}$ partition the universe,

(3) for each relation symbol P from L, $P^{\mathbf{B}}$ consists of the tuples $f_i(\mathbf{b})$ such that $f_i(\mathbf{b}) \in P^{\mathbf{B}_i}$ (for $i = 1, 2$).

Let L be the language of cardinal sums of L_0-structures, and let ψ say of a new relation symbol R that it is an isomorphism between the first part of the sum and the second. We can take ψ to be a recursive infinitary Π_2 sentence—finitary if L_0 is finite.

The lemma below follows immediately from the definitions.

LEMMA 0.2. *An* L_0*-structure* $\mathbf{C}$ *is relatively recursively categorical iff for every cardinal sum* $\mathbf{B}$ *of* $\mathbf{C}$ *with itself*, *there is a function* R *such that* $R \leq_T D(\mathbf{B})$ *and* $(\mathbf{B}, R) \models \psi$.

Now, Theorem 0.1 can be restated as follows: If $\mathbf{C}$ is a recursive structure and $\mathbf{A}$ is a cardinal sum of $\mathbf{C}$ with itself, the following are equivalent:

(1) for each $\mathbf{B} \cong \mathbf{A}$, there exists $R \leq_T D(\mathbf{B})$ such that $(\mathbf{B}, R) \models \psi$,

(2) $\mathbf{C}$ satisfies the syntactical conditions from Theorem 0.1.

This suggests the following more focused version of the basic question.

QUESTION 1. Let L be a language and let R be a new relation symbol, not in L. Let $\mathbf{A}$ be a recursive L-structure, and let ψ be a sentence in the language $L \cup \{R\}$. What syntactical conditions (like those in Theorem 0.1) guarantee that for every $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and $R \leq_T D(\mathbf{B})$?

Below are two more examples (families of examples, really) involving recursive infinitary Π_2 sentences ψ.

EXAMPLE 1. Let $\mathbf{A}$ be a recursive L-structure, and let $\varphi(\mathbf{x}, y)$ be a recursive infinitary Σ_1 formula such that $A \models \forall \mathbf{x}\, \exists \mathbf{y}\, \varphi(\mathbf{x}, \mathbf{y})$. Let R be a new relation symbol, and let ψ say that R is a Skolem function for φ. It is easy to see that for any $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and $R \leq_T D(\mathbf{B})$. The value of the Skolem function R at $\mathbf{x} = \mathbf{b}$ can be determined by searching, using the diagram, for $\mathbf{c}$ and other witnesses showing that $\mathbf{B} \models \varphi(\mathbf{b}, \mathbf{c})$. If $\mathbf{A}$ is an ordering of type η and $\varphi(x, y)$ is $x < y$, then there will not be a definable Skolem function.

EXAMPLE 2. Let $\mathbf{A}$ be a recursive structure for a finite relational language. Let ψ say of a new binary relation symbol R that it is a non-trivial automorphism of $\mathbf{A}$. For some structures, every copy has a non-trivial automorphism that is recursive in the diagram. This is true, for example, of equivalence relations and also of orderings of type η. By contrast, an ordering of type $\omega^* + \omega$ has non-trivial automorphisms, but there is a recursive ordering of this type for which no non-trivial automorphism is recursive.

We now define the *recursive* infinitary formulas, together with indices for the formulas, in terms of ordinal notations. The definition is the same as in [A$_1$] except for the choice of indices. Let L be a recursive language. For each $a \in O$, we shall determine sets of indices S_a^{Σ} and S_a^{Π}, and for the ordinal α with notation a, we shall describe the recursive infinitary Σ_α and Π_α formulas with indices in these sets. The advantage of the present system of indexing over the system in [A$_1$] is that the sets S_a^{Σ} and S_a^{Π} are recursive.

The recursive infinitary Σ_0 and Π_0 formulas are the finitary open L-formulas. If φ is a finitary open formula with Gödel number g and variables among $\mathbf{x}$, then $(1, g, \mathbf{x})$ is an index of φ in both S_1^{Σ} and S_1^{Π}. (We use 1 in the first component because this is the index for 0 in O.) The set of these indices is $S_1^{\Sigma} = S_1^{\Pi}$. If a is a notation for $\alpha \geq 1$, let $S_a^{\Sigma} = \{(a, e, \Sigma, \mathbf{x}) : e \in \omega\}$ and $S_a^{\Pi} = \{(a, e, \Pi, x) : e \in \omega\}$. The recursive infinitary Σ_α formula with index $(a, e, \Sigma, \mathbf{x})$ is $\bigvee\!\!\bigvee_i \exists \mathbf{v}_i\ \beta_i(\mathbf{v}_i, \mathbf{x})$, where the disjunction is taken over those $i \in W_e$ such that i is a pair (i_0, i_1) with first component i_0 a finite sequence of variables $\mathbf{v}_i$ and second component i_1 an index in $\bigcup_{b <_O a} S_b^{\Pi}$ for a formula with free variables among $\mathbf{v}_i{}^\frown\mathbf{x}$. Similarly, the recursive infinitary Π_α formula with index $(a, e, \Pi, \mathbf{x})$ is $\bigwedge\!\!\bigwedge_i \forall \mathbf{v}_i\ \beta_i(\mathbf{v}_i, \mathbf{x})$, where the conjunction is taken over those $i \in W_e$ such that i is a pair (i_0, i_1) with first component i_0 a finite sequence of variables $\mathbf{v}_i$ and second component i_1 an index in $\bigcup_{b<_O a} S_b^\Sigma$ for a formula with free variables among $\mathbf{v}_i{}^\wedge\mathbf{x}$.

Having indicated how ordinal notation is involved in infinitary recursive formulas and their indices, we shall identify ordinals with their notations, and we shall not distinguish between formulas and their indices.

§ 1. Expansion families. In Theorem 1.1 of this section, we give our result on Question 1. The syntactical condition of the theorem involves the notion of an "expansion family". The relevant definitions are given before the statement of the theorem. The proof of Theorem 1.1 is a forcing argument similar to those in [K], [AKMS], and [C]. At the end of the section, we give two simple corollaries of Theorem 1.1.

Let L be a fixed language, and let R be an r-placed relation symbol not in L. Let ψ be a recursive infinitary Π_2 sentence in the language $L \cup \{R\}$. An *R-formula* is a conjunction of finitely many formulas of the form $R(\mathbf{u})$ or $\neg R(\mathbf{u})$, where the variables appearing in a conjunct need not be distinct. When we write $\varrho(x)$ for an R-formula, we assume that $\mathbf{x}$ is a sequence of distinct variables, including all variables that appear in the formula.

Let $\mathbf{A}$ be an L structure, and let $L(\mathbf{A})$ be the extension of the language L obtained by adding the elements of $\mathbf{A}$ as constants. Let $\mathbf{A}^{\mathbf{x}}$ denote the set of *assignments* of values in $\mathbf{A}$ to the variables in $\mathbf{x}$. Formally, such an assignment is a function from the set of variables in $\mathbf{x}$ into the universe of $\mathbf{A}$, but we identify this function with the obvious sequence $\mathbf{a}$ of constants from $\mathbf{A}$. An *R-sentence for* $\mathbf{A}$ is an $L(\mathbf{A})$-sentence $\varrho(\mathbf{a})$ obtained from an R-formula $\varrho(\mathbf{x})$ and an assignment $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$ by replacing all occurrences of variables in $\varrho(\mathbf{x})$ by the corresponding constants from $\mathbf{a}$. Note that if σ is an R-sentence for $\mathbf{A}$, then $\sigma \vdash R(\mathbf{a})$ iff $R(\mathbf{a})$ is a conjunct of σ, and similarly for $\neg R(\mathbf{a})$. Hence, σ is consistent iff σ has no pair of explicitly contradictory conjuncts $R(\mathbf{a})$ and $\neg R(\mathbf{a})$.

If $\psi = \bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j}\ \beta_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j})$ is a recursive infinitary Π_2 sentence in the language $L \cup \{R\}$, then ψ is logically equivalent to a recursive infinitary Π_2 sentence $\bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j}\ (\varphi_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}) \& \varrho_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}))$, where $\varphi_{i,j}$ is a quantifier-free L-formula, and $\varrho_{i,j}$ is an R-formula. By suitable "padding", we may assume that the conjunction and disjunctions are taken over ω. We shall assume that our recursive infinitary Π_2 sentences all come in this *standard* form.

Let $\psi = \bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j}\ (\varphi_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}) \& \varrho_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}))$ be a recursive infinitary Π_2 sentence, as above. An *expansion family* for ψ on $\mathbf{A}$ is a family $\mathcal{S}$ of consistent R-sentences for $\mathbf{A}$ satisfying the following conditions:

(1) $\mathcal{S} \neq \emptyset$,

(2) if $\sigma \in \mathcal{S}$ and $\mathbf{a}$ is an r-tuple from $\mathbf{A}$, then there exists $\tau \in \mathcal{S}$ such that $\tau \vdash \sigma$ and either $\tau \vdash R(\mathbf{a})$ or $\tau \vdash \neg R(\mathbf{a})$,

(3) for each $\sigma \in \mathcal{S}$, $i \in \omega$, and $\mathbf{c} \in \mathbf{A}^{\mathbf{u}_i}$, there exist $\tau \in \mathcal{S}$, $j \in \omega$, and $\mathbf{d} \in \mathbf{A}^{\mathbf{v}_{i,j}}$ such that $\tau \vdash \sigma$, $\tau \vdash \varrho_{i,j}(\mathbf{c}, \mathbf{d})$, and $\mathbf{A} \models \varphi_{i,j}(\mathbf{c}, \mathbf{d})$.

An expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$ is said to be *formally* Σ_1^0 if for some finite sequence $\mathbf{c}$ of elements of $\mathbf{A}$, there is a recursive function assigning to each R-formula $\sigma(\mathbf{x})$ a recursive infinitary Σ_1 formula $\alpha_\sigma(\mathbf{c}, \mathbf{x})$, in the language L augmented by parameters $\mathbf{c}$, such that for each $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, $\sigma(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\sigma(\mathbf{c}, \mathbf{a})$.

We are now in a position to state the result on Question 1.

THEOREM 1.1. *Let* $\mathbf{A}$ *be a recursive* L*-structure*, *let* R *be a new relation symbol*, *and let* ψ *be a recursive infinitary* Π_2 *sentence in the language* $L \cup \{R\}$. *Then the following are equivalent*:

(1) *for each* $\mathbf{B} \cong \mathbf{A}$, *there is a relation* R *such that* $(\mathbf{B}, R) \models \psi$ *and* $R \leq_T D(\mathbf{B})$,

(2) *there is a formally* Σ_1^0 *expansion family for* ψ *on* $\mathbf{A}$.

Proof. Throughout, we have in mind a fixed recursive infinitary Π_2 sentence $\psi = \bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j}\ (\varphi_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}) \,\&\, \varrho_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}))$. The proof will be broken into several lemmas.

LEMMA 1.2. *Let* $\mathcal{S}$ *be an expansion family for* ψ *on* $\mathbf{A}$, *and suppose that* $\mathcal{S}$ *is r.e. in* $D(\mathbf{A})$. *Then there exists* $R \leq_T D(\mathbf{A})$ *such that* $(\mathbf{A}, R) \models \psi$.

Proof. It follows from Conditions (1), (2), and (3) in the definition of "expansion family" that there is a procedure recursive in $D(\mathbf{A})$ for choosing a sequence $(\sigma_k)_{k\in\omega}$ from $\mathcal{S}$ such that

(a) $\sigma_{k+1} \vdash \sigma_k$,

(b) for each r-tuple $\mathbf{a}$ from $\mathbf{A}$, there exists k such that either $\sigma_k \vdash R(\mathbf{a})$ or $\sigma_k \vdash \neg R(\mathbf{a})$,

(c) for each $i \in \omega$ and each $\mathbf{c} \in \mathbf{A}^{\mathbf{u}_i}$, there exist $k, j \in \omega$ and $\mathbf{d} \in \mathbf{A}^{\mathbf{v}_{i,j}}$ such that $\sigma_k \vdash \varrho_{i,j}(\mathbf{c}, \mathbf{d})$ and $\mathbf{A} \models \varphi_{i,j}(\mathbf{c}, \mathbf{d})$.

Let $R = \{\mathbf{a} : \sigma_k \vdash R(\mathbf{a}) \text{ for some } k\}$. By (b) and the fact that each σ_k is consistent, $R \leq_T D(\mathbf{A})$. Then by (c), we have $(\mathbf{A}, R_{\mathbf{A}}) \models \psi$. This completes the proof of Lemma 1.2.

Let F be an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$. If $\sigma = \varrho(\mathbf{a})$ is an R-sentence for $\mathbf{A}$, let $F(\sigma)$ denote the corresponding R-sentence for $\mathbf{B}$ (i.e., $F(\sigma) = \varrho(\mathbf{b})$, where $\mathbf{b} = F(\mathbf{a})$). If $\mathcal{S}$ is an expansion family for ψ on $\mathbf{A}$, let $F(\mathcal{S})$ denote the corresponding family $\{F(\sigma) : \sigma \in \mathcal{S}\}$ of R-sentences for $\mathbf{B}$. Clearly, $F(\mathcal{S})$ is an expansion family for ψ on $\mathbf{B}$. In general, $F(\mathcal{S})$ will not be r.e. in $D(\mathbf{B})$. The next lemma says that $F(\mathcal{S})$ is r.e. in $D(\mathbf{B})$ if $\mathcal{S}$ is formally Σ_1^0.

LEMMA 1.3. *If $\mathcal{S}$ is a formally Σ^0_1 expansion family for ψ on* **A** *and* $\mathbf{A} \cong_F \mathbf{B}$, *then the expansion family $F(\mathcal{S})$ for ψ on* **B** *is r.e. in $D(\mathbf{B})$.*

Proof. There is a recursive function assigning recursive infinitary Σ_1 formulas to R-formulas such that if $\alpha_\sigma(\mathbf{c},\mathbf{x})$ is the recursive infinitary Σ_1 formula assigned to the R-formula $\sigma(\mathbf{x})$, then for all $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, $\sigma(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\sigma(\mathbf{c},\mathbf{a})$. Then for all $\mathbf{b} \in \mathbf{B}^{\mathbf{x}}$, $\sigma(\mathbf{b}) \in F(\mathcal{S})$ iff $\mathbf{B} \models \alpha_\sigma(F(\mathbf{c}),\mathbf{b})$. It is clear that we can enumerate, recursively in $D(\mathbf{B})$, the R-sentences $\sigma(\mathbf{b})$ for **B** such that $\mathbf{B} \models \alpha_\sigma(F(\mathbf{c}),\mathbf{b})$. Therefore, $F(\mathcal{S})$ is r.e. in $D(\mathbf{B})$. This completes the proof of Lemma 1.3.

Lemmas 1.2 and 1.3 combine to give one direction of Theorem 1.1. Suppose there is a formally Σ^0_1 expansion family $\mathcal{S}$ for ψ on **A**. If $\mathbf{B} \cong \mathbf{A}$, then by Lemma 1.3, there is an expansion family $\mathcal{S}_{\mathbf{B}}$ for ψ on **B** such that $\mathcal{S}_{\mathbf{B}}$ is r.e. in $D(\mathbf{B})$. By Lemma 1.2, there exists $R \leq_T D(\mathbf{B})$ such that $(\mathbf{B}, R) \models \psi$. This shows that (2) implies (1). To show that (1) implies (2), we consider a generic copy **B** of **A**, determined as in [K], [AKMS], or [C] by a permutation g of ω such that $\mathbf{B} \cong_g \mathbf{A}$. Here we shall give only a brief description of the kind of forcing used, summarizing some facts proved in [AKMS].

The forcing conditions are finite partial permutations $p, q, \ldots$ of ω. The forcing language in [AKMS] includes a function symbol f, standing for the permutation g, in addition to the symbols of $L(\mathbf{A})$. The formulas are to be interpreted in the structure $(\mathbf{A}, g)$. However, the structure we really want to talk about is **B**. We must have sentences with the following meanings:

(1) $\mathbf{B} \models \psi(\mathbf{b})$, where $\psi(\mathbf{x})$ is a quantifier-free L-formula and $\mathbf{b} \in \mathbf{B}^{\mathbf{x}}$,
(2) $\varphi_e^{D(\mathbf{B})}(k) = j$,
(3) $\varphi_e^{D(\mathbf{B})}(k) \downarrow$,
(4) $\varphi_e^{D(\mathbf{B})}(k) \uparrow$,
(5) $\varphi_e^{D(\mathbf{B})}(k)$ is total,
(6) $\varphi_e^{D(\mathbf{B})}(k)$ is $\{0,1\}$-valued.

The sentence of type (1) with the intended meaning $\mathbf{B} \models \psi(\mathbf{b})$, where $\psi(\mathbf{x})$ is a quantifier-free L-formula and $\mathbf{b} \in \mathbf{B}^{\mathbf{x}}$, is $\psi(f(\mathbf{b}))$. The sentence of type (2) with intended meaning $\varphi_e^{D(\mathbf{B})}(k) = j$ is a recursive disjunction of sentences of type (1). There is one disjunct for each computation which follows procedure e, starts with input k, and halts with output j, having used answers to finitely many questions about membership in $D(\mathbf{B})$. For simplicity, we shall write $\psi(\mathbf{b}) \in D$, $\varphi_e^D(k) = j$, $\varphi_e^D(k) \downarrow$, $\varphi_e^D(k) \uparrow$, φ_e^D is total, and φ_e^D is $\{0,1\}$-valued, for the sentences of types (1), (2), (3), (4), (5), and (6), respectively.

Let σ be an R-sentence, with constants which we think of as naming elements of **B**. There is a recursive infinitary sentence σ^e such that whenever

$\mathbf{B} \cong_g \mathbf{A}$, if $\varphi_e^{D(\mathbf{B})}$ is the characteristic function of a relation R, then we have $(\mathbf{B}, R) \models \sigma$ iff $(\mathbf{A}, g) \models \sigma^e$. (If $\varphi_e^{D(\mathbf{B})}$ is the characteristic function of a relation, then $(\mathbf{A}, g) \models \varphi_e^D$ is total $\& \varphi_e^D$ is $\{0,1\}$-valued.) We can take σ^e to be a Σ_1 sentence logically equivalent to the sentence that results from replacing the conjuncts in σ of form $R(\mathbf{b})$ by $\varphi_e^D(\mathbf{b}) = 1$ and those of form $\neg R(\mathbf{b})$ by $\varphi_e^D(\mathbf{b}) = 0$. The sentence σ^e is determined effectively from σ and e; the same σ^e works for all g and $\mathbf{B}$ such that $\mathbf{B} \cong_g \mathbf{A}$.

Forcing for recursive infinitary sentences is defined in a standard way, except for sentences of type (1) above. If $\psi(\mathbf{x})$ is a quantifier-free L-formula and $b \in \mathbf{B}^{\mathbf{x}}$, then $p \Vdash \psi(\mathbf{b}) \in D$ iff the elements of $\mathbf{b}$ are all in $\mathrm{dom}(p)$ and p maps $\mathbf{b}$ to some $\mathbf{a}$ such that $\mathbf{A} \models \psi(\mathbf{a})$. Thus, the formula $\psi(\mathbf{x})$ defines in $\mathbf{A}$ the set of tuples $\mathbf{a}$ such that if p maps $\mathbf{b}$ to $\mathbf{a}$, then $p \Vdash \psi(\mathbf{b}) \in D$. For a type (2) sentence $\varphi_e^D(k) = j$ and a fixed $\mathbf{b}$, we can find a recursive infinitary Σ_1 formula $\theta(\mathbf{x})$ of L which defines in $\mathbf{A}$ the set of tuples $\mathbf{a}$ such that if p has domain $\mathbf{b}$ and p maps $\mathbf{b}$ to $\mathbf{a}$, then $p \Vdash \varphi_e^D(k) = j$. Recall that the sentence saying $\varphi_e^D(k) = j$ has the form $\bigvee\!\!\!\bigvee_i \psi_i(\mathbf{b}_i) \in D$, where for each i, $\psi_i(\mathbf{x}_i)$ is a quantifier-free L-formula and $\mathbf{b}_i$ is a finite sequence of constants. Let $\theta(\mathbf{x})$ be the disjunction of those $\psi_i(\mathbf{x}_i)$ for which $\mathbf{b}_i \subseteq \mathbf{b}$. While the sentence $\varphi_e^D(k) = j$ involves infinitely many constants, the formula $\theta(\mathbf{x})$ has only finitely many free variables.

If σ is an R-sentence on $\mathbf{B}$, then the sentence σ^e (described above) involves infinitely many constants. However, for any R-formula $\varrho(\mathbf{x})$, with variables among $\mathbf{x}$, and any $\mathbf{b} \in \mathbf{B}^{\mathbf{x}}$, we can find an infinitary Σ_1 formula $\varrho^e_{\mathbf{b}}(\mathbf{x})$ in the language L, having free variables among $\mathbf{x}$ and no extra constants, such that $\mathbf{A} \models \varrho^e_{\mathbf{b}}(\mathbf{a})$ iff for the forcing condition p with domain $\mathbf{b}$ such that p maps $\mathbf{b}$ to $\mathbf{a}$, we have $p \Vdash \varrho(\mathbf{b})^e$. Each conjunct of $\varrho(\mathbf{b})$ gives rise to a conjunct of $\varrho(\mathbf{b})^e$ of the form $\varphi_e^D(\mathbf{d}) = k$ (for $\mathbf{d} \subseteq \mathbf{b}$), and each conjunct of $\varrho(\mathbf{x})$ gives rise to a recursive infinitary Σ_1 formula $\theta(\mathbf{x})$ in the language L, as above. We let $\varrho^e_{\mathbf{b}}(\mathbf{x})$ be a Σ_1 formula logically equivalent to the conjunction of these finitely many $\theta(\mathbf{x})$.

We choose an increasing sequence $(p_k)_{k \in \omega}$ of forcing conditions deciding all of the recursive infinitary formulas (or all that could possibly matter). There is a countable fragment containing these formulas. The function g will be $\bigcup_k p_k$. For the structure $(\mathbf{A}, g)$, we have the usual "truth and forcing" lemma.

Suppose that Condition (1) (from the statement of Theorem 1.1) holds. Then, in particular, for the generic copy $\mathbf{B}$ such that $\mathbf{B} \cong_g \mathbf{A}$, there exists R such that $R \le_T D(\mathbf{B})$ and $(\mathbf{B}, R) \models \psi$. For some e, R has characteristic function $\varphi_e^{D(\mathbf{B})}$.

The following sentences are true in $(\mathbf{A}, g)$:

(a) $\bigwedge\!\!\!\bigwedge_\sigma (\sigma^e \vee (\neg\sigma)^e)$, where the disjunction here is taken over all R-

sentences σ on $\mathbf{B}$,

(b) $\bigwedge\!\!\!\bigwedge_{i\in\omega,\mathbf{c}\in\mathbf{B}^{\mathbf{u}_i}} \bigvee\!\!\!\bigvee_{j\in\omega,\mathbf{d}\in\mathbf{B}^{\mathbf{v}_{i,j}}} (\varphi_{i,j}(f(\mathbf{c}),f(\mathbf{d}))\&\varrho_{i,j}(\mathbf{c},\mathbf{d})^e)$.

Therefore, some forcing condition p from the sequence $(p_k)_{k\in\omega}$ forces both of these sentences. Let $\mathcal{S}_p$ be the family of R-sentences $\varrho(\mathbf{a})$ for $\mathbf{A}$ such that for some $q\supseteq p$ and some $\mathbf{b}$, $q(\mathbf{b})=\mathbf{a}$ and $q\Vdash\varrho(\mathbf{b})^e$.

Lemma 1.4. *The set $\mathcal{S}_p$ is a formally Σ^0_1 expansion family for ψ on $\mathbf{A}$.*

Proof. We begin by checking that $\mathcal{S}_p$ has the properties of an expansion family for ψ on $\mathbf{A}$. The first thing to check is that if $\gamma\in\mathcal{S}_p$, then γ is consistent. If not, then for some r-tuple $\mathbf{a}$, both $R(\mathbf{a})$ and $\neg R(\mathbf{a})$ are conjuncts of γ. Then there exist q and $\mathbf{b}$ such that $q\supseteq p$, $\mathbf{b}\subseteq\mathrm{dom}(q)$, $q(\mathbf{b})=\mathbf{a}$, and q forces both $R(\mathbf{b})^e$ and $(\neg R(\mathbf{b}))^e$. By putting q into our forcing sequence, we obtain g, $\mathbf{B}$, and R such that both $R(\mathbf{b})$ and $\neg R(\mathbf{b})$ are true in $(\mathbf{B},R)$, a contradiction. The next thing to check is that $\mathcal{S}_p\neq\emptyset$. For any r-tuple $\mathbf{a}$ in $\mathbf{A}$, either $R(\mathbf{a})$ or $\neg R(\mathbf{a})$ must be in $\mathcal{S}_p$, since there exists $q\supseteq p$ such that for some $\mathbf{b}$, $q(\mathbf{b})=\mathbf{a}$, and either $q\Vdash R(\mathbf{b})^e$ or $q\Vdash(\neg R(\mathbf{b}))^e$. A similar argument shows that if $\sigma\in\mathcal{S}_p$, then for any r-tuple $\mathbf{a}$ in $\mathbf{A}$, there exists $\tau\in\mathcal{S}_p$ such that $\tau\vdash\sigma$ and either $\tau\vdash R(\mathbf{a})$ or $\tau\vdash\neg R(\mathbf{a})$. The final thing to check is that for each $\sigma\in\mathcal{S}_p$ and each $i\in\omega$ and $\mathbf{c}\in\mathbf{A}^{\mathbf{u}_i}$, there exist $\tau\in\mathcal{S}_p$, $j\in\omega$, and $\mathbf{d}\in\mathbf{A}^{\mathbf{v}_{i,j}}$ such that $\tau\vdash\sigma$, $\tau\vdash\varrho_{i,j}(\mathbf{c},\mathbf{d})$, and $\mathbf{A}\models\varphi_{i,j}(\mathbf{c},\mathbf{d})$. Let $\sigma=\varrho(\mathbf{a})$, and suppose $q\supseteq p$, where $q(\mathbf{b})=\mathbf{a}$ and $q\Vdash\varrho(\mathbf{b})^e$. Since p forces sentence (b), there exist n, $\mathbf{d}$, and $q'\supseteq q$ such that $\mathbf{c}$ and $\mathbf{d}$ are both in $\mathrm{ran}(q')$, $\mathbf{A}\models\varphi_{i,j}(\mathbf{c},\mathbf{d})$, and if $q'(\mathbf{c}')=\mathbf{c}$ and $q'(\mathbf{d}')=\mathbf{d}$, then $q'\Vdash\varrho_{i,j}(\mathbf{c}',\mathbf{d}')^e$. Let τ be $\sigma\&\varrho_{i,j}(\mathbf{c},\mathbf{d})$. Then $\tau\in\mathcal{S}_p$, as required.

We have shown that $\mathcal{S}_p$ is an expansion family for ψ on $\mathbf{A}$. It remains to show that $\mathcal{S}_p$ is formally Σ^0_1 on $\mathbf{A}$. Suppose that p maps $\mathbf{b}_0$ to $\mathbf{a}_0$. For any R-formula $\varrho(\mathbf{x})$ and any $\mathbf{a}\in\mathbf{A}^{\mathbf{x}}$, we have $\varrho(\mathbf{a})\in\mathcal{S}_p$ iff there exist $q\supseteq p$, $\mathbf{b}$, and possibly further elements $\mathbf{b}_1$ and $\mathbf{a}_1$, such that q takes $\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1$ to $\mathbf{a}_0{}^\wedge\mathbf{a}^\wedge\mathbf{a}_1$, and $q\Vdash\varrho(\mathbf{b})^e$. For a particular $\mathbf{b}$ and $\mathbf{b}_1$, we have the recursive infinitary Σ_1 formula $\varrho^e_{\mathbf{b}_0,\mathbf{b},\mathbf{b}_1}(\mathbf{y},\mathbf{x},\mathbf{z})$, in the language L, such that if q maps $\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1$ to $\mathbf{a}_0{}^\wedge\mathbf{a}^\wedge\mathbf{a}_1$, then $q\Vdash\varrho(\mathbf{b})^e$ iff $\mathbf{A}\models\varrho^e_{\mathbf{b}_0,\mathbf{b},\mathbf{b}_1}(\mathbf{a}_0,\mathbf{a},\mathbf{a}')$.

There is an effective procedure for determining, given ϱ and e, an index for a recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{y},\mathbf{x})$ which is logically equivalent to the disjunction of the formulas $\exists\mathbf{z}\ \varrho^e_{\mathbf{b}_0,\mathbf{b},\mathbf{b}'}(\mathbf{y},\mathbf{x},\mathbf{z})$. Then the formula $\alpha_\varrho(\mathbf{a}_0,\mathbf{x})$ is the one we assign to the R-formula $\varrho(\mathbf{x})$. Note that the fixed sequence of parameters $\mathbf{a}_0$ (the range of the forcing condition p) serves for all ϱ. This completes the proof of Lemma 1.4, which was all that remained in the proof of Theorem 1.1.

In the introduction, we indicated how Theorem 1.1 was inspired by Theorem 0.1. Now, we obtain the non-trivial direction of Theorem 0.1 as a corollary of Theorem 1.1.

Corollary 1.5. *Let* $\mathbf{C}$ *be a recursive structure which is relatively recursively categorical. Then for some finite sequence* $\mathbf{c}$ *from* $\mathbf{C}$, *there is a recursive sequence of existential formulas* $(\varphi_i(\mathbf{c},\mathbf{x}_i))_{i\in\omega}$ *such that*

(i) *for each finite sequence* $\mathbf{a}$ *from* $\mathbf{C}$, *there is at least one* i *such that* $\varphi_i(\mathbf{c},\mathbf{a})$ *holds*,

(ii) *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are two sequences such that for some* i, $\varphi_i(\mathbf{c},\mathbf{a})$, $\varphi_i(\mathbf{c},\mathbf{b})$ *both hold*, *then* $(\mathbf{C},\mathbf{a})\cong(\mathbf{C},\mathbf{b})$.

Proof. Let $\mathbf{A}$ be a recursive cardinal sum of $\mathbf{C}$ with itself. Let ψ be the sentence saying of a new relation symbol R that it is an isomorphism between the two parts of $\mathbf{A}$. Applying Theorem 1.1, we get a formally Σ^0_1 expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$. Let $\mathbf{c}$ be the sequence of constants from the expansion family. For each n, let $\mathbf{x}=(x_0,\ldots,x_{n-1})$ and $\mathbf{y}=(y_0,\ldots,y_{n-1})$ be sequences of distinct variables, and let $\sigma_n(\mathbf{x},\mathbf{y})$ be the R-formula saying that $R(x_k)=y_k$ for $k<n$. For each R-formula $\varrho(\mathbf{x},\mathbf{y},\mathbf{z})$ such that $\vdash\varrho(\mathbf{x},\mathbf{y},\mathbf{z})\to\sigma_n(\mathbf{x},\mathbf{y})$, we have an effectively assigned recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{c},\mathbf{x},\mathbf{y})$ such that $\varrho(\mathbf{a},\mathbf{b},\mathbf{d})\in\mathcal{S}$ iff $\mathbf{A}\models\ \alpha_\varrho(\mathbf{c},\mathbf{a},\mathbf{b},\mathbf{d})$. For each n, we can effectively determine a recursive infinitary Σ_1 formula $\alpha_n(\mathbf{c},\mathbf{x},\mathbf{y})$ that is logically equivalent to the disjunction of the formulas $\exists\mathbf{z}$ $\alpha_\varrho(\mathbf{c},\mathbf{x},\mathbf{y},\mathbf{z})$. Note that if $\mathbf{A}\models\alpha_n(\mathbf{c},\mathbf{a},\mathbf{b})$ then $\mathbf{a}$ and $\mathbf{b}$ sit isomorphically in the two parts of $\mathbf{A}$, and for any $\mathbf{a}$ in the first part of $\mathbf{A}$ (or $\mathbf{b}$ in the second part), there exist n and $\mathbf{b}$ (or $\mathbf{a}$) such that $\mathbf{A}\models\alpha_n(\mathbf{c},\mathbf{a},\mathbf{b})$.

Let $\mathbf{c}=\mathbf{c}_1{}^\wedge\mathbf{c}_2$, where $\mathbf{c}_1$ lies in one part of $\mathbf{A}$ and $\mathbf{c}_2$ lies in the other. For each n, we may assume that $\alpha_n(\mathbf{c},\mathbf{x},\mathbf{y})$ is a disjunction of formulas $\gamma_{n,m}(\mathbf{c}_1,\mathbf{x})\,\&\delta_{n,m}(\mathbf{c}_2,\mathbf{y})$, where $\gamma_{n,m}$ and $\delta_{n,m}$ are finitary existential formulas, relativized to the first and second parts of the cardinal sum, respectively. (That is, we may apply the Feferman–Vaught "splitting" procedure to effectively determine a formula of this form that is equivalent to $\alpha_n(\mathbf{c},\mathbf{x},\mathbf{y})$ in $\mathbf{A}$.) If $\mathbf{A}\models\gamma_{n,m}(\mathbf{c}_1,\mathbf{a}_1)\,\&\delta_{n,m}(\mathbf{c}_2,\mathbf{b})$, then $\mathbf{a}_1$ and $\mathbf{b}$ sit isomorphically in the two parts of $\mathbf{A}$. If, in addition, $\mathbf{A}\models\gamma_{n,m}(\mathbf{c}_1,\mathbf{a}_2)$, then $\mathbf{A}\models\gamma_{n,m}(\mathbf{c}_1,\mathbf{a}_2)\,\&\delta_{n,m}(\mathbf{c}_2,\mathbf{b})$, so $\mathbf{a}_1$ and $\mathbf{a}_2$ must sit isomorphically in the first part of $\mathbf{A}$.

We can recursively list the formulas $\gamma_{n,m}(\mathbf{c}_1,\mathbf{x})$ such that $\gamma_{n,m}(\mathbf{c}_1,\mathbf{x})$ $\&\delta_{n,m}(\mathbf{c}_2,\mathbf{y})$ is satisfiable in $\mathbf{A}$. Let $\varphi_i(\mathbf{c}_1,\mathbf{x})$ be the unrelativized formula corresponding to the ith formula $\gamma_{n,m}(\mathbf{c}_1,\mathbf{x})$ on the list, and think of $\mathbf{C}$ as the part of $\mathbf{A}$ that includes $\mathbf{c}_1$. Clearly, each finite sequence $\mathbf{a}$ from $\mathbf{C}$ satisfies one of the formulas $\varphi_i(\mathbf{c}_1,\mathbf{x})$, and the argument above shows that if $\mathbf{a}_1$ and $\mathbf{a}_2$ are two sequences such that for some i, $\mathbf{C}\models\varphi_i(\mathbf{c},\mathbf{a}_1)\,\&\varphi_i(\mathbf{c},\mathbf{a}_2)$ both hold, then $(\mathbf{C},\mathbf{a}_1)\cong(\mathbf{C},\mathbf{a}_2)$. This completes the proof.

The next result is related to Beth's Theorem.

Corollary 1.6. *Let* $\mathbf{A}$ *be a recursive* L*-structure*, *let* R *be a new relation* *symbol, and let ψ be a recursive infinitary Π_2 sentence in the language $L \cup \{R\}$. Suppose that for each $\mathbf{B} \cong \mathbf{A}$, there is a unique relation R such that $(\mathbf{B}, R) \models \psi$, and this relation R is always recursive in $D(\mathbf{B})$. Then for the relation R on $\mathbf{A}$ such that $(\mathbf{A}, R) \models \psi$, both R and its complement are definable in $\mathbf{A}$ by recursive infinitary Σ_1 formulas in the language L (with finitely many parameters from $\mathbf{A}$).*

Proof. By Theorem 1.1, there is a formally Σ_1^0 expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$, with parameters $\mathbf{c}$, say. We consider the R-formulas $\varrho(\mathbf{x}, \mathbf{y})$ such that $\varrho(\mathbf{x}, \mathbf{y}) \vdash R(\mathbf{x})$ or $\varrho(\mathbf{x}, \mathbf{y}) \vdash \neg R(\mathbf{x})$, with the corresponding recursive Σ_1 L-formulas $\alpha_\varrho(\mathbf{c}, \mathbf{x}, \mathbf{y})$ such that $\varrho(\mathbf{a}, \mathbf{b}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\varrho(\mathbf{c}, \mathbf{a}, \mathbf{b})$. Let $\alpha(\mathbf{c}, \mathbf{x})$ be the disjunction of $\exists \mathbf{y}\, \alpha_\varrho(\mathbf{c}, \mathbf{x}, \mathbf{y})$ for those R-formulas $\varrho(\mathbf{x}, \mathbf{y})$ such that $\varrho(\mathbf{x}, \mathbf{y}) \vdash R(\mathbf{x})$, and let $\beta(\mathbf{c}, \mathbf{x})$ be the disjunction of $\exists \mathbf{y}\, \alpha_\varrho(\mathbf{c}, \mathbf{x}, \mathbf{y})$ for those R-formulas $\varrho(\mathbf{x}, \mathbf{y})$ such that $\varrho(\mathbf{x}, \mathbf{y}) \vdash \neg R(\mathbf{x})$. For each $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, there is some $\sigma \in \mathcal{S}$ such that $\sigma \vdash R(\mathbf{a})$ or $\sigma \vdash \neg R(\mathbf{a})$. For any $\sigma \in \mathcal{S}$, we could use the expansion family to produce a relation R on $\mathbf{A}$ satisfying ψ and agreeing with σ. Since the relation R on $\mathbf{A}$ is unique, $\alpha(\mathbf{c}, \mathbf{x})$ and $\beta(\mathbf{c}, \mathbf{x})$ are the desired defining formulas.

The results in [AKMS] provide a different proof of Corollary 1.6, which works for arbitrary infinitary sentences ψ, not just for recursive Π_2 sentences.

In the next section, we consider some variants of Question 1.

§ 2. Related results. The first variant of the basic question concerns expansions which are relatively Δ_α^0 instead of relatively recursive.

QUESTION 2. For α a recursive ordinal, when is it the case that for every $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and R is Δ_α^0 relative to $D(\mathbf{B})$?

Recall Example 2 from Section 1. In this example, L is a finite relational language, and ψ is a finitary Π_2 sentence in the language $L \cup \{R\}$ saying that R is a non-trivial automorphism of the L-reduct. Let $\mathbf{A}$ be an ordering of type $\omega^* + \omega$. There is a non-trivial automorphism defined by the formula $\varphi(x, y) = x < y \,\&\, \forall z\, \neg(x < z \,\&\, z < y)$. It follows that for any $\mathbf{B} \cong \mathbf{A}$, there is a non-trivial automorphism recursive in $D(\mathbf{B})'$. Or, let $\mathbf{A}$ be an ordering of type $\omega \cdot \eta$. Then no non-trivial automorphism is definable, but for any $\mathbf{B} \cong \mathbf{A}$, there is a non-trivial automorphism recursive in $D(\mathbf{B})''$.

The next variant of the basic question concerns expansions which are relatively r.e.

QUESTION 3. When is it the case that for every $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and R is r.e. in $D(\mathbf{B})$?

If there is a relation R such that $(\mathbf{A}, R) \models \psi$ and R is definable by a recursive infinitary Σ_1 formula, then the answer to Question 3 is positive. However, definability is not necessary. The two examples below both have the feature that for each $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and R is r.e. in $D(\mathbf{B})$, but for some $\mathbf{B}$, R cannot be taken to be definable or recursive in $D(\mathbf{B})$.

EXAMPLE 3. Let $\mathbf{A}$ be a structure of the form (ω, T), where T is a ternary relation on ω obtained as follows: Let U, V be disjoint infinite subsets of ω (such that the complement of $U \cup V$ is also infinite), let X be a partition of V into two-element sets, and let F be a function mapping U one-one onto X. Now, let T consist of the triples (u, x, y) for which $F(u) = \{x, y\}$. Note that $(u, x, y) \in T$ iff $(u, y, x) \in T$. Let ψ be a sentence saying of a new unary relation symbol R that for each triple in T, R contains exactly one of the last two components. Using T as a name for itself, we have $\psi = \forall u \forall x \forall y\ [T(u, x, y) \rightarrow (R(x) \leftrightarrow \neg R(y))]$. For any $\mathbf{B} \cong \mathbf{A}$, there is some R, r.e. in $D(\mathbf{B})$, such that $(\mathbf{B}, R) \models \psi$. (To enumerate R, we enumerate $T^{\mathbf{B}}$, and as each triple (a, b, c) appears, we put b into R unless c is already in.) Clearly, there is no definable relation satisfying ψ. We can produce $\mathbf{B} \cong \mathbf{A}$ such that $D(\mathbf{B})$ is recursive and there is no recursive R such that $(\mathbf{B}, R) \models \psi$.

The example below is a modification of Example 1.

EXAMPLE 4. Let $\mathbf{A}$ be a recursive L-structure, let $\varphi(\mathbf{x}, \mathbf{y})$ be a recursive infinitary Σ_1 L-formula, and let ψ be a recursive infinitary Π_2 sentence saying that R is a function, not necessarily total, such that $\forall \mathbf{x} \forall \mathbf{y}\ (R(\mathbf{x}, \mathbf{y}) \rightarrow \varphi(\mathbf{x}, \mathbf{y}))$ and $\forall \mathbf{x} \forall \mathbf{y}\ (\varphi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z}\ R(\mathbf{x}, \mathbf{z}))$. Then for any $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and R is r.e. in $D(\mathbf{B})$. There is a recursive structure $\mathbf{A} = (\omega, \mathbf{C})$, where $\mathbf{C}$ is a ternary relation—resembling the relation $\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : \mathbf{z}$ is a halting computation for $\varphi_{\mathbf{x}}(\mathbf{y})\}$—such that if $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the formula saying $\exists \mathbf{z}\ \mathbf{C}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, and we form ψ as above, then there is no recursive or definable R such that $(\mathbf{A}, R) \models \psi$.

Initially, we obtained a result on Question 3 for recursive infinitary Π_1 sentences, such as the sentence ψ from Example 3. We then extended the result to a special class of recursive infinitary Π_2 sentences, including the sentence ψ from Example 4. Before stating the results on Questions 2 and 3, we define some hierarchies of formulas, similar to the one in [A_2].

As always, R is an r-placed relation symbol not in the language L. For each fixed ordinal $\alpha > 0$, we define sets of infinitary $L \cup \{R\}$-formulas $\Sigma_\beta|(R \in \Sigma_\alpha)$, $\Pi_\beta|(R \in \Sigma_\alpha)$, $\Sigma_\beta|(R \in \Delta_\alpha)$, and $\Pi_\beta|(R \in \Delta_\alpha)$, for all β. If $\beta < \alpha$, then $\Sigma_\beta|(R \in \Sigma_\alpha)$ and $\Sigma_\beta|(R \in \Delta_\alpha)$ are both equal to the set of Σ_β L-formulas, and $\Pi_\beta|(R \in \Sigma_\alpha)$ and $\Pi_\beta|(R \in \Delta_\alpha)$ are both equal to the set of Π_β L-formulas. Let $\beta \geq \alpha$. Then $\Sigma_\beta|(R \in \Sigma_\alpha)$ is the set of infinitary formulas of the form $\bigvee\!\!\!\bigvee_n \exists \mathbf{y}_n\ (\varphi_n(\mathbf{x},\mathbf{y}_n)\,\&\,\psi_n(\mathbf{x},\mathbf{y}_n))$, where for each n, φ_n is a finite conjunction of formulas of the form $R(\mathbf{u})$ and ψ_n is $\Pi_\gamma|(R \in \Sigma_\alpha)$ for some $\gamma < \beta$; $\Pi_\beta|(R \in \Sigma_\alpha)$ is the set of infinitary formulas of the form $\bigwedge\!\!\!\bigwedge_n \forall \mathbf{y}_n\ (\varphi_n(\mathbf{x},\mathbf{y}_n) \vee \psi_n(\mathbf{x},\mathbf{y}_n))$, where for each n, φ_n is a finite disjunction of formulas of the form $\neg R(\mathbf{u})$ and ψ_n is $\Sigma_\gamma|(R \in \Sigma_\alpha)$ for some $\gamma < \beta$; $\Sigma_\beta|(R \in \Delta_\alpha)$ is the set of infinitary formulas of the form $\bigvee\!\!\!\bigvee_n \exists \mathbf{y}_n\ (\varphi_n(\mathbf{x},\mathbf{y}_n)\,\&\,\psi_n(\mathbf{x},\mathbf{y}_n))$, where for each n, φ_n is an R-formula and ψ_n is $\Pi_\gamma|(R \in \Delta_\alpha)$ for some $\gamma < \beta$; and $\Pi_\beta|(R \in \Delta_\alpha)$ is the set of infinitary formulas of the form $\bigwedge\!\!\!\bigwedge_n \forall \mathbf{y}_n\ (\varphi_n(\mathbf{x},\mathbf{y}_n) \vee \psi_n(\mathbf{x},\mathbf{y}_n))$, where for each n, φ_n is a finite disjunction of formulas $R(\mathbf{u})$ or $\neg R(\mathbf{u})$ and ψ_n is $\Sigma_\gamma|(R \in \Delta_\alpha)$ for some $\gamma < \beta$.

Our result on Question 2 is for sentences ψ in $\Pi_{\alpha+1}|(R \in \Delta_\alpha)$. If ψ is such a sentence, then we may assume that ψ is in the "standard form" $\bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j}\ (\varphi_{i,j}(\mathbf{u}_i,\mathbf{v}_{i,j})\,\&\,\varrho_{i,j}(\mathbf{u}_i,\mathbf{v}_{i,j}))$, where for each i and j, $\varphi_{i,j}$ is a Π_β formula of L, for some $\beta < \alpha$, and $\varrho_{i,j}$ is an R-formula. When we defined "expansion family" earlier, we were thinking of recursive infinitary sentences ψ in $\Pi_2|(R \in \Delta_1)$. The same definition serves for ψ in $\Pi_{\alpha+1}|(R \in \Delta_\alpha)$. An expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$ is said to be *formally* Σ^0_α if for some finite sequence $\mathbf{c}$ of elements of $\mathbf{A}$, there is a recursive function assigning to each R-formula $\varrho(\mathbf{x})$ a recursive infinitary Σ_α-formula $\alpha_\varrho(\mathbf{c},\mathbf{x})$, in the language L with parameters $\mathbf{c}$, such that for each $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, $\varrho(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\varrho(\mathbf{c},\mathbf{a})$.

THEOREM 2.1. *Suppose* $\mathbf{A}$ *is a recursive* L*-structure. Let* R *be a new relation symbol, and suppose* ψ *is a recursive infinitary sentence in* $\Pi_{\alpha+1}|(R \in \Delta_\alpha)$*. Then for any recursive ordinal* $\alpha \geq 1$*, the following are equivalent*:

(1) *for all* $\mathbf{B} \cong \mathbf{A}$ *there exists* R *such that* $(\mathbf{B}, R) \models \psi$ *and* R *is* Δ^0_α *relative to* $D(\mathbf{B})$,

(2) *there exists a formally* Σ^0_a *expansion family for* ψ *on* $\mathbf{A}$.

The proof of Theorem 2.1 is a straightforward generalization of the proof of Theorem 1.1, so we shall omit it.

There is a natural extension of Corollary 1.6 that follows from Theorem 2.1.

COROLLARY 2.2. *Let* $\mathbf{A}$ *be a recursive* L*-structure, let* R *be a new relation symbol, and let* ψ *be a recursive infinitary sentence in* $\Pi_{\alpha+1}|(R \in \Delta_\alpha)$*. Suppose that for each* $\mathbf{B} \cong \mathbf{A}$*, there is a unique relation* R *such that* $(\mathbf{B}, R) \models \psi$*, and suppose that this* R *is always* Δ^0_α *relative to* $D(\mathbf{B})$*. Then for the relation* R *on* $\mathbf{A}$*,* R *and its complement are both definable in* $\mathbf{A}$ *by recursive infinitary* Σ_α *formulas in the language* L *(with finitely many parameters from* $\mathbf{A}$*).*

We turn to Question 3. Our result is for the case where ψ is a recursive infinitary sentence in $\Pi_2|(R \in \Sigma_1)$. The sentence ψ from Example 4 can be taken to be of this form. If ϱ is an R-formula, or an R-sentence, let ϱ^+ denote the positive part. If ϱ has no positive conjuncts, then $\varrho^+ = \top$. Note that every positive R-formula is consistent, and every R-sentence on a structure $\mathbf{A}$ is consistent with the sentences true in $\mathbf{A}$. Let ψ be a recursive infinitary sentence in $\Pi_2|(R \in \Sigma_1)$. We may assume that ψ has the form $\bigwedge\!\!\!\bigwedge_i \forall \mathbf{u}_i \bigvee\!\!\!\bigvee_j \exists \mathbf{v}_{i,j} \, (\varphi_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}) \& \varrho_{i,j}(\mathbf{u}_i, \mathbf{v}_{i,j}))$, where for each i and j, $\varphi_{i,j}$ is an open L-formula and $\varrho_{i,j}$ is an R-formula such that for any conjunct of the form $\neg R(\mathbf{x})$, $\mathbf{x} \subseteq \mathbf{u}_i$.

A *positive expansion family* for ψ on $\mathbf{A}$ is a family $\mathcal{S}$ of positive R-sentences for $\mathbf{A}$ such that

(1) $\mathcal{S} \neq \emptyset$,

(2) for each $\sigma \in \mathcal{S}$, $i \in \omega$, and $\mathbf{c} \in \mathbf{A}^{\mathbf{u}_i}$, there exist $j \in \omega$, $d \in \mathbf{A}^{\mathbf{v}_{i,j}}$, and τ a consistent R-sentence for $\mathbf{A}$ such that $\tau^+ \in \mathcal{S}$, $\tau^+ \vdash \sigma$, $\tau \vdash \varrho_{i,j}(\mathbf{c}, \mathbf{d})$, and $\mathbf{A} \models \varphi_{i,j}(\mathbf{c}, \mathbf{d})$.

A positive expansion family for ψ on $\mathbf{A}$ is said to be *formally* Σ_1^0 if for some finite sequence $\mathbf{c}$ of elements of $\mathbf{A}$, there is a recursive function assigning to each positive R-formula $\sigma(\mathbf{x})$ a recursive infinitary Σ_1 formula $\alpha_\sigma(\mathbf{c}, \mathbf{x})$, in the language L augmented by parameters $\mathbf{c}$, such that for each $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, $\sigma(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\sigma(\mathbf{c}, \mathbf{a})$.

Here is the result on Question 3.

THEOREM 2.3. *For* $\mathbf{A}$ *a recursive* L*-structure*, R *a new relation symbol*, *and* ψ *a recursive infinitary sentence in* $\Pi_2|(R \in \Sigma_1)$, *the following are equivalent*:

(1) *for all* $\mathbf{B} \cong \mathbf{A}$ *there exists* R *such that* $(\mathbf{B}, R) \models \psi$ *and* R *is r.e. in* $D(\mathbf{B})$,

(2) *there exists a formally* Σ_1^0 *positive expansion family for* ψ *on* $\mathbf{A}$.

Proof. The proof that (2) implies (1) has the same pattern as in Theorem 1.1.

LEMMA 2.4. *Let* $\mathcal{S}$ *be a positive expansion family for* ψ *on* $\mathbf{A}$, *and suppose that* $\mathcal{S}$ *is r.e. in* $D(\mathbf{A})$. *Then there exists* R *r.e. in* $D(\mathbf{A})$ *such that* $(A, R) \models \psi$.

Proof. Fix a recursive enumeration $(i_k, \mathbf{c}_k)_{k \in \omega}$ of the pairs $(i, \mathbf{c})$ such that $\mathbf{c} \in \mathbf{A}^{\mathbf{u}_i}$.

CLAIM. *There is a procedure recursive in* $D(\mathbf{A})$ *for choosing a sequence* $(\sigma_n)_{n \in \omega}$ *of elements of* $\mathcal{S}$ *such that*

(1) $\sigma_{n+1} \vdash \sigma_n$,

(2) *for each* n, *there is a consistent* R*-sentence* σ *for* $\mathbf{A}$ *such that* $\sigma^+ = \sigma_{n+1}$ *and for each* $k < n+1$, *there exist* j *and* $\mathbf{d}$ *such that* $\sigma \vdash \varrho_{i_k,j}(\mathbf{c}_k, \mathbf{d})$ *and* $\mathbf{A} \models \varphi_{i_k,j}(\mathbf{c}_k, \mathbf{d})$.

Proof of Claim. Let σ_0 be the first element of $\mathcal{S}$ that appears in an enumeration. Given σ_n, we search for σ_{n+1} with properties (1) and (2). The search will succeed provided there is some $\sigma \in \mathcal{S}$ with these properties. We first take $\sigma' \in \mathcal{S}$ such that $\sigma' \vdash \sigma_n$ and the set $\{\mathbf{a} : \sigma' \vdash R(a) \,\&\, \exists k < n+1 \ \mathbf{a} \subseteq \mathbf{c}_k\}$ is maximal. For each $k < n+1$, there is a consistent R-sentence τ_k on $\mathbf{A}$ such that $\tau_k \vdash \sigma'$, $\tau_k^+ \in \mathcal{S}$, and for some j and $\mathbf{d}$, we have $\mathbf{A} \models \varphi_{i_k,j}(\mathbf{c}_k, \mathbf{d})$ and $\tau_k \vdash \varrho_{i_k,j}(\mathbf{c}_k, \mathbf{d})$. We may assume that the negative conjuncts in any τ_k all appear in σ', and we may choose τ_{k+1}^+ to imply τ_k^+, which means that $\tau_{k+1} \vdash \tau_k$. Then $\sigma = \tau_n^+$ has the desired properties, so we have the claim.

To complete the proof of Lemma 2.4, let $R = \{\mathbf{a} : \mathbf{a}$ is an r-tuple from $\mathbf{A}$ and $\sigma_n \vdash R(\mathbf{a})$ for some $n\}$. Clearly, R is r.e. in $D(\mathbf{A})$, and $(\mathbf{A}, R) \models \psi$.

The analogue of Lemma 1.3 for positive expansion families is clearly true, and this combines with Lemma 2.4 to give half of Theorem 2.3—the fact that (2) implies (1). To show that (1) implies (2), we again use forcing. Recall the sentences $\varphi_e^D(k)\downarrow$ from our forcing language. In terms of these, we get, for each $e \in \omega$ and each R-sentence σ on $\mathbf{B}$, a sentence with the meaning $(\mathbf{B}, W_e^{D(\mathbf{B})}) \models \sigma$. We now let σ^e denote the sentence with this meaning, forgetting entirely the old use of the notation. We have $(\mathbf{B}, W_e^{D(\mathbf{B})}) \models \sigma$ iff $(\mathbf{A}, g) \models \sigma^e$. In general, σ^e will be a recursive infinitary Π_2 sentence with infinitely many constants. If σ is positive, then σ^e will be a recursive infinitary Σ_1 sentence, still with infinitely many constants. Given a positive R-formula $\varrho(\mathbf{x})$ and a sequence $\mathbf{b}$ of distinct elements from $\mathbf{B}$, we can find a recursive infinitary Σ_1 formula $\varrho_{\mathbf{b}}^e(\mathbf{x})$, in the language L, with no extra constants, such that for all forcing conditions p with domain $\mathbf{b}$, if p takes $\mathbf{b}$ to $\mathbf{a}$, then $p \Vdash \varrho(\mathbf{b})^e$ iff $\mathbf{A} \models \varrho_{\mathbf{b}}^e(\mathbf{a})$.

We construct a generic copy $\mathbf{B}$ of $\mathbf{A}$. If (1) holds, then there is some e such that $(\mathbf{B}, W_e^{D(\mathbf{B})}) \models \psi$, and some forcing condition p forces the following sentences:

(a) $\bigwedge\!\!\!\bigwedge_\sigma (\sigma^e \vee (\neg\sigma)^e)$, where the disjunction here is taken over all R-sentences σ for $\mathbf{B}$,

(b) $\bigwedge\!\!\!\bigwedge_{i\in\omega, \mathbf{c}\in \mathbf{B}^{\mathbf{u}_i}} \bigvee\!\!\!\bigvee_{j\in\omega, \mathbf{d}\in\mathbf{B}^{\mathbf{v}_{i,j}}} (\varphi_{i,j}(f(\mathbf{c}), f(\mathbf{d})) \,\&\, \varrho_{i,j}(\mathbf{c}, \mathbf{d})^e)$.

Let $\mathcal{S}_p$ be the family of positive R-sentences $\varrho(\mathbf{a})$ for $\mathbf{A}$ with the following feature: for some $q \supseteq p$ and some $\mathbf{b}$, we have $q(\mathbf{b}) = \mathbf{a}$ and $q \Vdash \varrho(\mathbf{b})^e$. The following lemma will complete the proof of Theorem 2.3.

Lemma 2.5. *The set* $\mathcal{S}_p$ *is a formally* Σ_1^0 *positive expansion family for* ψ *on* $\mathbf{A}$.

P r o o f. Clearly, $\top \in \mathcal{S}_p$, so $\mathcal{S}_p \neq \emptyset$. It must be shown that for each $\sigma \in \mathcal{S}_p$, $i \in \omega$, and $\mathbf{c} \in \mathbf{A}^{\mathbf{u}_i}$, there exist $j \in \omega$, $\mathbf{d} \in \mathbf{A}^{\mathbf{v}_{i,j}}$, and τ a consistent R-sentence on $\mathbf{A}$ such that $\tau^+ \in \mathcal{S}_p$, $\tau^+ \vdash \sigma$, $\tau \vdash \varrho_{i,j}(\mathbf{c}, \mathbf{d})$, and $\mathbf{A} \models \varphi_{i,j}(\mathbf{c}, \mathbf{d})$. Suppose σ arises from $p' \supseteq p$. There exist $q \supseteq p'$, $\mathbf{c}' \in \mathbf{B}^{\mathbf{u}_i}$, $j \in \omega$, $\mathbf{d} \in \mathbf{A}^{\mathbf{v}_{i,j}}$, and $\mathbf{d}' \in \mathbf{B}^{\mathbf{v}_{i,j}}$ such that $q(\mathbf{c}') = \mathbf{c}$, $q(\mathbf{d}') = \mathbf{d}$, and $q \Vdash \varphi_{i,j}(f(\mathbf{c}'), f(\mathbf{d}')) \,\&(\varrho_{i,j}(\mathbf{c}', \mathbf{d}'))^e$. Let τ be the result of adding to σ all of the conjuncts in $\varrho_{i,j}(\mathbf{c}, \mathbf{d})$. This τ has the feature we want. Therefore, $\mathcal{S}_p$ is a positive expansion family for ψ on $\mathbf{A}$.

It must be shown that $\mathcal{S}_p$ is formally Σ^0_1 on $\mathbf{A}$. Suppose that p maps $\mathbf{b}_0$ to $\mathbf{a}_0$. For any positive R-formula $\varrho(\mathbf{x})$ and any $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, we have $\varrho(\mathbf{a}) \in \mathcal{S}_p$ iff there exist $\mathbf{b}$, and possibly further elements $\mathbf{b}_1$ and $\mathbf{a}_1$, and $q \supseteq p$, such that q takes $\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1$ to $\mathbf{a}_0{}^\wedge\mathbf{a}^\wedge\mathbf{a}_1$, and $q \Vdash \varrho(\mathbf{b})^e$. For a particular $\mathbf{b}$ and $\mathbf{b}_1$, we have the recursive infinitary Σ_1 formula $\varrho^e_{\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1}(\mathbf{y}, \mathbf{x}, \mathbf{z})$, in the language L, such that if q maps $\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1$ to $\mathbf{a}_0{}^\wedge\mathbf{a}^\wedge\mathbf{a}_1$, then $q \Vdash \varrho(\mathbf{b})^e$ iff $\mathbf{A} \models \sigma^e_{\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}_1}(\mathbf{a}_0, \mathbf{a}, \mathbf{a}')$. There is an effective procedure for determining, given ϱ and e, an index for a recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{y}, \mathbf{x})$ which is logically equivalent to the disjunction of the formulas $\exists \mathbf{z}\ \varrho^e_{\mathbf{b}_0{}^\wedge\mathbf{b}^\wedge\mathbf{b}'}(\mathbf{y}, \mathbf{x}, \mathbf{z})$. The formula $\alpha_\varrho(\mathbf{a}_0, \mathbf{x})$ is the one that we assign to $\varrho(\mathbf{x})$. This completes the proof.

QUESTION 4. For α a recursive ordinal, when is it the case that for every $\mathbf{B} \cong \mathbf{A}$, there is a relation R such that $(\mathbf{B}, R) \models \psi$ and R is Σ_α relative to $D(\mathbf{B})$?

If ψ is Π_1, this question is not interesting for $\alpha > 1$. It is easy to see that if $D(\mathbf{A})$ is consistent with ψ, where ψ is a recursive infinitary Π_1 sentence, then there exists $R \leq_T D(\mathbf{A})'$ such that $(\mathbf{A}, R) \models \psi$. Our result on Question 4 is for ψ in $\Pi_{\alpha+1}|(R \in \Sigma_\alpha)$. The definition of "positive expansion family" is the same as for ψ in $\Pi_2|(R \in \Sigma_1)$. A positive expansion family for ψ is *formally* Σ^0_α if for some finite sequence $\mathbf{c}$ of elements of $\mathbf{A}$, there is a recursive function assigning to each positive R-formula $\sigma(\mathbf{x})$ a recursive infinitary Σ_α formula $\alpha_\sigma(\mathbf{c}, \mathbf{x})$, in the language L augmented by parameters $\mathbf{c}$, such that for each $\mathbf{a} \in \mathbf{A}^{\mathbf{x}}$, $\sigma(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\sigma(\mathbf{c}, \mathbf{a})$. The result below generalizes Theorem 2.3. We omit the proof, as it involves no new ideas.

THEOREM 2.6. *For $\mathbf{A}$ a recursive L-structure, R a new relation symbol, and ψ a recursive infinitary sentence in $\Pi_{\alpha+1}|(R \in \Sigma_\alpha)$, the following are equivalent*:

(1) *for all $\mathbf{B} \cong \mathbf{A}$, there exists R such that $(\mathbf{B}, R) \models \psi$ and R is Σ_α relative to $D(\mathbf{B})$*,

(2) *there exists a formally Σ^0_α positive expansion family for ψ on $\mathbf{A}$.*

§ 3. Some applications. Theorem 1.1 has an application involving paths through trees. Let $T \subseteq \omega^{<\omega}$ be a tree, with level n consisting of functions having domain n. (If $\sigma \in T$ and $m < \mathrm{dom}(\sigma)$, then $\sigma|m \in T$.) Let S be the successor relation on T (for σ, $\tau \in T$, $(\sigma,\tau) \in S$ iff $\tau = \sigma^\wedge n$ for some n). Let ψ say of a new unary relation symbol R that it is a path (i.e., an infinite branch) through T. We can take ψ to be a finitary Π_2 sentence logically equivalent to the conjunction of the following:

(1) $\forall x \forall y\, [(S(x,y) \,\&\, R(y)) \to R(x)]$,

(2) $\forall x \forall y \forall z\, [(S(x,y) \,\&\, S(x,z) \,\&\, R(y) \,\&\, R(z)) \to y = z]$,

(3) $\forall x\, [R(x) \to \exists z\, (R(z) \,\&\, S(x,z))]$.

Corollary 3.1. *If* $\mathbf{A} \cong (T,S)$ *is a recursive tree*, *as above*, *then the following are equivalent*:

(1) *for each* $\mathbf{B} \cong \mathbf{A}$, *there is a path* R *such that* $R \le_T D(\mathbf{B})$,

(2) *there is a non-empty subtree* T_1 *of* T (*with level* n *consisting of functions having domain* n, *as above*) *such that* T_1 *is definable in* $\mathbf{A}$ *by a recursive infinitary* Σ_1 *formula* $\varphi(\mathbf{c},\mathbf{x})$ *and every node in* T_1 *extends to a path in* T_1.

Proof. It is obvious that (2) implies (1). We must show that (1) implies (2). If (1) holds, then by Theorem 1.1, there is a formally Σ_1^0 expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$. We have fixed parameters $\mathbf{c}$ and a recursive function assigning to each R-formula $\varrho(\mathbf{x})$ a recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{c},\mathbf{x})$ such that $\varrho(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\varrho(\mathbf{c},\mathbf{a})$. Let $\mathbf{C}$ be the set of R-formulas $\varrho(\mathbf{y},x)$ such that $\varrho(\mathbf{y},x) \vdash R(x)$. Let $\varphi(\mathbf{c},x)$ be a recursive infinitary Σ_1 formula logically equivalent to $\bigvee\!\!\!\bigvee_{\varrho\in C} \exists \mathbf{y}\ \alpha_\varrho(\mathbf{c},\mathbf{y},x)$. This formula defines the desired subtree.

The form of the sentence ψ above is such that we could apply Theorem 2.3 as well as Theorem 1.1. In general, if we apply Theorems 1.1 and 2.3 to the same sentence, we expect different results. However, for these paths through trees, being relatively r.e. is equivalent to being relatively recursive.

Corollary 3.1 can be extended as follows, using Theorem 2.1. The proof is the obvious modification of the proof of Corollary 3.1.

Corollary 3.2. *If* $\mathbf{A} \cong (T,S)$ *is a recursive tree as above*, *and* $\alpha \ge 1$ *is a recursive ordinal*, *then the following are equivalent*:

(1) *for each* $\mathbf{B} \cong \mathbf{A}$, *there is a path* R *that is* Δ_α^0 *relative to* $D(\mathbf{B})$,

(2) *there is a subtree* T_1 *of* T, *definable in* $\mathbf{A}$ *by a recursive infinitary* Σ_α *formula* $\varphi(\mathbf{c},x)$, *such that every node in* T_1 *extends to a path in* T_1.

Theorem 2.3 has an application involving homogeneous subsets of a graph. Let $\mathbf{A} = (\omega, G)$ be a non-directed graph. There is a recursive infinitary sentence ψ saying of a new unary relation symbol R that it is an infinite homogeneous set. We can take ψ to be a sentence in $\Pi_2|(R \in \Sigma_1)$, logically equivalent to the conjunction of the following family of sentences ψ_n:

$$\psi_0 = \forall x \forall y \forall u \forall v \ [(x \neq y \& u \neq v \& R(x) \& R(y) \& R(u) \& R(v)) \\ \rightarrow (G(x,y) \leftrightarrow G(u,v))] ,$$

and for $n \geq 1$,

$$\psi_n = \forall x_1 \ldots \forall x_n \exists y \Big[\bigvee\!\!\!\bigvee_{1 \leq i < j \leq n} x_i = x_j \vee \Big(\bigwedge\!\!\!\bigwedge_{1 \leq i \leq n} x_i \neq y \& R(y) \Big) \Big] .$$

COROLLARY 3.3. *If* $\mathbf{A} = (\omega, G)$ *is a recursive non-directed graph, then the following are equivalent*:

(1) *for all* $\mathbf{B} \cong \mathbf{A}$, *there is an infinite homogeneous set* R *such that* R *is r.e. in* $D(\mathbf{B})$,

(2) *there is a recursive sequence of recursive infinitary* Σ_1 *formulas* $\delta_n(\mathbf{a}, x_1, \ldots, x_n)$, *in the language of* $\mathbf{A}$, *such that*

- (a) *if* $(\mathbf{a}_n)_{n \in \omega}$ *is a sequence such that* $\mathbf{A} \vDash \delta_n(\mathbf{a}, a_1, \ldots, a_n)$ *for all* n, *then* $\{a_n : n \in \omega\}$ *is an infinite homogeneous set*,
- (b) *there is* a_1 *such that* $\mathbf{A} \vDash \varphi_1(\mathbf{a}, x_1)$, *and if* $\mathbf{A} \vDash \delta_n(\mathbf{a}, a_1, \ldots, a_n)$, *then there exists* a_{n+1} *such that* $\mathbf{A} \vDash \delta_{n+1}(\mathbf{a}, a_1, \ldots, a_n)$.

Proof. Clearly, (2) implies (1) and (1) implies (3). To show that (1) implies (2), note that if (1) holds, then by Theorem 2.3, there is a formally Σ^0_1 positive expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$. We have a recursive function assigning to each positive R-formula $\varrho(\mathbf{x})$ a recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{a}, \mathbf{x})$ such that $\varrho(\mathbf{b}) \in \mathcal{S}$ iff $\mathbf{A} \vDash \alpha_\varrho(\mathbf{a}, \mathbf{b})$. For each $n \geq 1$, let C_n be the set of positive R-formulas $\varrho(\mathbf{u}, x_1, \ldots, x_n)$ such that $\varrho(\mathbf{u}, x_1, \ldots, x_n) \vdash R(x_1) \& \ldots \& R(x_n)$, and let $d_n(x_1, \ldots, x_n) = \bigwedge\!\!\!\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$. Let $\delta_n(\mathbf{c}, x_1, \ldots, x_n)$ be a recursive infinitary Σ_1 formula equivalent to the formula $d_n(x_1, \ldots, x_n) \& \bigvee\!\!\!\bigvee_{\varrho \in C_n} \exists \mathbf{u} \, \alpha_\varrho(\mathbf{a}, \mathbf{u}, x_1, \ldots, x_n)$. From the definition of "positive expansion family", and an examination of the meaning of the sentences ψ_n, it is clear that (b) holds. The proof of (a) reduces to the following claim:

CLAIM. *If* $\mathbf{A} \vDash \delta_n(\mathbf{a}, a_1, \ldots, a_n)$, *then* $\{a_1, \ldots, a_n\}$ *is homogeneous.*

Proof of Claim. We can put the sentence ψ_0 in the form $\exists \mathbf{u} \bigvee\!\!\!\bigvee_k (\varphi_k \& \sigma_k)$, where $\mathbf{u} = (x, y, u, v)$, the disjunction is finite, and for each k, φ_k is an open formula in the language of graphs, σ_k is an R-formula, and one of the following holds:

(a) $\varphi_k \vdash x = y \vee u = v$,

(b) $\sigma_k \vdash \neg R(x) \vee \neg R(y) \vee \neg R(u) \vee \neg R(v)$,

(c) $\varphi_k \vdash G(\mathbf{x}, y) \leftrightarrow G(u, v)$.

The elements $a_1, \ldots, a_n$ are distinct, and $\mathbf{A} \models \alpha_\varrho(\mathbf{a}, \mathbf{d}, a_1, \ldots, a_n)$ for some $\mathbf{d}$ in $\mathbf{A}$ and some $\varrho \in C_n$. Let $\mathbf{c} = (a_1, a_2, a_i, a_j)$, where $1 \leq i < j \leq n$. There exist k and a consistent R-sentence τ for $\mathbf{A}$ such that $\tau \vdash \varrho(\mathbf{d}, a_1, \ldots, a_n)$, $\tau^+ \in \mathcal{S}$, and for some k, we have $\tau \vdash \sigma_k(\mathbf{c})$ and $\mathbf{A} \models \varphi_k(\mathbf{c})$. For this k, (a) and (b) are impossible. Therefore, (c) must hold, and so $\mathbf{A} \models G(a_1, a_2) \leftrightarrow G(a_i, a_j)$. This proves the claim.

Remark. Suppose $\mathbf{A}$ is a recursive non-directed graph such that for each $\mathbf{B} \cong \mathbf{A}$, there is an infinite homogeneous set R such that R is r.e. in $D(\mathbf{B})$. Then the homogeneous set can be taken to be always of the same color; that is, either

(1) for all $\mathbf{B} \cong \mathbf{A}$, there is an infinite set R r.e. in $D(\mathbf{B})$ with all pairs (of distinct elements) in G, or else

(2) for all $\mathbf{B} \cong \mathbf{A}$, there is an infinite set R r.e. in $D(\mathbf{B})$ with no pairs in G.

The first author prefers to use Theorem 2.3 to prove this, while the second author prefers to use Corollary 3.3. Both proofs are easy, so we give neither.

There is another application of Theorem 2.3, involving linear orderings. Let L be the language of linear orderings, and let ψ say of a new unary relation symbol R that it is non-empty and has no first element. A linear ordering $\mathbf{A}$ has an infinite descending sequence of elements iff $\mathbf{A}$ can be expanded to a model of ψ.

COROLLARY 3.4. *Let* $\mathbf{A}$ *be a recursive linear ordering. Then the following are equivalent*:

(1) *for each* $\mathbf{B} \cong \mathbf{A}$, *there exists a descending sequence of elements* $(b_n)_{n\in\omega}$ *recursive in* $D(\mathbf{B})$,

(2) *for each* $\mathbf{B} \cong \mathbf{A}$, *there is a set* R *r.e. in* $D(\mathbf{B})$ *such that* $(\mathbf{B}, R) \models \psi$,

(3) *there is a recursive infinitary* Σ_1 *formula* $\varphi(\mathbf{c}, x)$ *defining a non-empty subset of* $\mathbf{A}$ *with no first element.*

Proof. It is easy to see that (3) implies (1) and (1) implies (2). If (2) holds, then by Theorem 2.6, there is a formally Σ^0_1 positive expansion family $\mathcal{S}$ for ψ on $\mathbf{A}$. We have a recursive function assigning to each positive R-formula $\varrho(\mathbf{x})$ a recursive infinitary Σ_1 formula $\alpha_\varrho(\mathbf{c}, x)$ such that $\varrho(\mathbf{a}) \in \mathcal{S}$ iff $\mathbf{A} \models \alpha_\varrho(\mathbf{c}, \mathbf{a})$. Let $\varphi(\mathbf{c}, x)$ be a recursive infinitary Σ_1 formula logically equivalent to the disjunction of the formulas $\exists \mathbf{u}\ \alpha_\varrho(\mathbf{c}, \mathbf{x}, \mathbf{u})$, where $\vdash \varrho(\mathbf{x}, \mathbf{u}) \rightarrow R(\mathbf{x})$. This formula clearly defines a non-empty subset of $\mathbf{A}$ with no first element.

§ 4. Conclusion. In the cases where ψ is more complicated than Π_2, it does not seem possible to extract from our forcing construction necessary and sufficient conditions (similar to those in Theorem 1.1) which answer Question 1. We therefore pose the problem of determining whether there exist such conditions when ψ is, for example, Π_3.

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