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Composant-like decompositions of spaces

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Abstract. The body of this paper falls into two independent sections. The first deals with the existence of cross-sections in F_σ-decompositions. The second deals with the extensions of the results on accessibility in the plane.

1. Introduction. The composants of an indecomposable metric continuum X are pairwise disjoint, continuum connected, first category, dense F_σ-subsets of X. Mazurkiewicz [8] proved that each indecomposable metric continuum has c composants by showing that there exists a Cantor set which is a partial cross-section for the composants of X.

There are number of useful results concerning the position of composant in an indecomposable continuum embedded in the plane. Mazurkiewicz [9] and Krasinkiewicz [3]–[5] have proved that most composants of a planar indecomposable continuum X are not accessible.

The purpose of this paper is to extend the above results to decompositions of separable metric spaces with only mild additional conditions.

Let X be a separable metric space and $R \subsetneq X \times X$ an equivalence relation on X. For $x \in X$ let $R(x)$ denote the R-equivalence class of x. For $A \subset X$ let $R(A) = \bigcup\{R(x) : x \in A\}$.

By a *continuum* we mean a compact, connected, metric space. A continuum is *decomposable* if it is the union of two proper subcontinua, otherwise it is *indecomposable*. A set A is *continuum connected* if every pair of points of A lies in a subcontinuum of A.

2. Cross-section for F_σ-decompositions. In this section X will be a separable metric space and $R = \bigcup_{i=1}^{\infty} R_i \subsetneq X \times X$ an equivalence relation such that each R_i is closed in $X \times X$.

2.1. Example. Let X be a non-degenerate, indecomposable metric continuum. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of proper open sets of X.

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Let $R_i = \{(x,y) \in X \times X : x$ and y lie in a component of $X - U_i\}$ and $R = \bigcup_{i=1}^{\infty} R_i$. If $x \in X$ then $R(x)$ is the composant of x, i.e. the union of all proper subcontinua of X which contain x (see [6]). The sets R_i are closed in $X \times X$.

2.2. Example. Let $f : X \times (-\infty, \infty) \to X$ be a flow on a space X and let R be the equivalence relation on X whose equivalence classes are the orbits of points of X under f. For each positive integer i let

$$R_i = \{(x, f(x,t)) : (x,t) \in X \times [-i,i]\}.$$

Then R_i is closed and $R = \bigcup_{i=1}^{\infty} R_i$.

The results of this section, which are valid for arbitrary separable metric spaces, were first obtained by Cook [1] and by Mazurkiewicz [8] for the case of indecomposable continua.

2.3. Proposition. *If K is a compact subset of a metric separable space X, then $R(K)$ is an F_σ-subset of X.*

Proof. Let $L_i = \{y \in X : (x,y) \in R_i$ for some $x \in K\}$. Let $y \in \mathrm{Cl}(L_i)$ and let $\{y_n\}_{n=1}^{\infty}$ be a sequence in L_i converging to y. For each n let $x_n \in K$ be such that $(x_n, y_n) \in R_i$. We may pass to a subsequence by compactness of K, and we may suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x in K. Then $(x,y) \in R_i$ since R_i is closed in $X \times X$. So, $y \in L_i$ and L_i is closed. Clearly, $R(K) = \bigcup_{i=1}^{\infty} L_i$.

2.4. Corollary. *If X is of second category and each R-equivalence class has empty interior in X then the set of R-equivalence classes is uncountable.*

2.5. Corollary. *Let each R-equivalence class be dense in X. If $K = \bigcup_{i=1}^{\infty} F_i$, where each F_i is a compact subset of X such that $R(F_i) \neq X$, then $R(K)$ is a first category F_σ-set in X.*

2.6. Theorem. *Let X be of second category and let each R-equivalence class be dense in X. If K is a compact subset of X such that $R(K) = X$ then there is a non-empty closed subset L of K such that* $\mathrm{Cl}(R(x) \cap L) = L$ *for each $x \in X$.*

Proof. Let $L = \bigcup\{\mathrm{Cl}(K \cap R(x)) : x \in X\}$.

Choose a countable subcover $\mathcal{V}$ of the open cover $\{K - \mathrm{Cl}(K \cap R(x)) : x \in X\}$ of $K - L$. The set $K - \mathrm{Cl}(K \cap R(x))$, $x \in X$, is σ-compact (being an open subset of the compact set K) and misses $R(x)$. By Corollary 2.5, $R(K - \mathrm{Cl}(K \cap R(x)))$ is a first category F_σ-set in X. Hence, $R(K - L)$, being equal to $\bigcup\{R(V) : V \in \mathcal{V}\}$, is a first category F_σ-set in X. Observe that, since $R(K) = X$, the set L is non-empty.

It remains to prove the equality $L = \mathrm{Cl}(R(x) \cap L)$ for each $x \in X$. To do this, let U be an open subset of X having non-empty intersection with L. By the definition of L, the set U meets each of the sets $K \cap R(x)$. Hence, $R(U \cap K) = X$ and, in consequence, $R(U \cap L)$ is of second category, since $R(K - L)$ is of first category. But $U \cap L$ is σ-compact. By Corollary 2.5, $R(U \cap L) = X$. Hence, $U \cap L \cap R(x) \neq 0$ for each $x \in X$. So $\mathrm{Cl}(L \cap R(x)) = L$ for each $x \in X$.

2.7. Corollary. *Let X be of second category, let each R-equivalence class be dense in X and let $R \neq X \times X$. Then there do not exist σ-compact subsets of X which are full cross-sections for the family of R-equivalence classes.*

Proof. Suppose K is σ-compact, i.e. $K = \bigcup_{i=1}^{\infty} F_i$, where each F_i is compact, and K is a full cross-section for the family of R-equivalence classes. Then, by Corollary 2.5, there exists i such that $R(F_i) = X$. But K is a full cross-section, hence, $K = F_i$. So K is compact. This contradicts Theorem 2.6, since $L \subset K$ and $R(x) \cap L \neq 0$ for all x implies $L = K$, but $\mathrm{Cl}(R(x) \cap K) \neq K$, because $R(x) \cap K$ is a single point.

2.8. Lemma. *If the set of R-equivalence classes is uncountable then X contains a non-empty G_δ-set $X' = R(X')$ such that each open and non-empty set in X' meets uncountably many R-equivalence classes.*

Proof. Let $\mathcal{U} = \{U : U$ is an open set meeting only countably many R-equivalence classes$\}$ and let $X' = X - R(\bigcup \mathcal{U})$. Since $\mathcal{U}$ has a countable subcollection covering $\bigcup \mathcal{U}$, $\bigcup \mathcal{U}$ meets only countably many R-equivalence classes. So if U is an open set meeting X' then U meets uncountably many R-equivalence classes and, in consequence, $U \cap X'$ meets uncountably many R-equivalence classes. Clearly, since, by Proposition 2.3, each R-equivalence class is an F_σ-set, X' is an G_δ-set.

2.9. Theorem (cf. Kuratowski [7]). *Suppose X is a topologically complete, separable, metric space and the set of R-equivalence classes is uncountable. Then X contains a Cantor set L such that $L \cap R(x)$ contains at most one point for each $x \in X$.*

Proof. By Lemma 2.8, we may suppose that each non-empty open subset of X meets uncountably many R-equivalence classes.

Let ϱ be a complete metric for X. We construct for each positive integer n a family $\mathcal{A}_n = \{A(d_1, \ldots, d_n) : d_i \in \{0, 1\}, i = 1, \ldots, n\}$ of disjoint, regularly closed, non-empty subsets of X such that

(1) $A(d_1, \ldots, d_n) \times A(d'_1, \ldots, d'_n) \cap R_m = \emptyset$ if $(d_1, \ldots, d_n) \neq (d'_1, \ldots, d'_n)$ and $m \le n$,

(2) diameter $A(d_1, \ldots, d_n) < 2^{-n}$,

(3) $A(d_1, \ldots, d_n) \subset \mathrm{Int}(A(d_1, \ldots, d_{n-1}))$ for $n > 1$.

Choose $x(0)$ and $x(1)$ in different R-equivalence classes. Clearly $(x(0), x(1))$ as well as $(x(1), x(0))$ do not belong to R_1. Since R_1 is closed, there exist disjoint regularly closed neighbourhoods $A(0)$ of $x(0)$ and $A(1)$ of $x(1)$ satisfying (1) and (2). Let $\mathcal{A}_1 = \{A(0), A(1)\}$.

Suppose $\mathcal{A}, \ldots, \mathcal{A}_n$ have been defined. Choose points $x(d_1, \ldots, d_{n+1})$ in different R-equivalence classes such that $x(d_1, \ldots, d_{n+1}) \in \mathrm{Int}(A(d_1, \ldots \ldots, d_n))$. This is possible since each set $\mathrm{Int}(A(d_1, \ldots, d_n))$ meets infinitely many R-equivalence classes. Clearly, $(x(d_1, \ldots, d_{n+1}), x(d'_1, \ldots, d'_{n+1}))$ does not belong to R_m for any m if $(d_1, \ldots, d_{n+1}) \neq (d'_1, \ldots, d'_{n+1})$. Since R_m is closed there exist disjoint regularly closed neighbourhoods $A(d_1, \ldots, d_{n+1})$ of $x(d_1, \ldots, d_{n+1})$ satisfying (1)–(3). Let $\mathcal{A}_{n+1} = \{A(d_1, \ldots, d_{n+1}) : d_i \in \{0, 1\}, i = 1, \ldots, n+1\}$.

By induction, the family $\{\mathcal{A}_n\}_{n=1}^{\infty}$ is defined. Let $L = \bigcap_{n=1}^{\infty}(\bigcup \mathcal{A}_n)$. Since X is complete, L is a Cantor set. Since $R = \bigcup_{i=1}^{\infty} R_i$, the assumption that x and y are points of L lying in one R-equivalence class implies that x and y are in the same element of $\mathcal{A}_n$ for large n. Hence, $x = y$.

2.10. Corollary. *Suppose X is a topologically complete, separable metric space such that the set of R-equivalence classes is uncountable. Then the set of R-equivalence classes has cardinality c.*

2.11. Corollary. *Suppose X is a topologically complete metric space and each R-equivalence class is a proper, dense set in X. Then the set of R-equivalence classes has cardinality c.*

3. External R-equivalence classes in the plane. Throughout this section X will be a subset of the 2-sphere S^2 and R will be an equivalence relation on X such that each R-equivalence class is continuum connected. The results of this section were proved by Krasinkiewicz for the case of indecomposable continua in the plane.

We say $R(x)$ is an *external* R-equivalence class if there exists a continuum $L \subset S^2$ with $L \cap R(x) \neq \emptyset$, $L \not\subset \mathrm{Cl}(X)$ and $L \cap R(y) = \emptyset$ for some $y \in X$. If $R(x)$ is not external then it is said to be *internal*.

The following lemma for separable spaces is well known and easy to prove (Whyburn [12], p. 43, Th. (1.5)).

3.1. Lemma. *Let $\mathcal{A}$ be an uncountable family of disjoint closed connected sets in a connected space Y such that each of them disconnects Y. Then there exists $A, B, C \in \mathcal{A}$ such that A separates B from C in Y.*

The following lemma is based on one in [9].

3.2. Lemma. *Let $K \subset S^2$ be a continuum and let U and V be disjoint open discs meeting K such that $U \cap K$ is contained in no component of*

$K - V$. Then there exists a continuum in $K - U$ which disconnects $S^2 - U$ and meets V.

Proof. Let $K - V = P \cup Q$, where P and Q are disjoint closed sets which both meet U. Let $F = P \cap \mathrm{Bd}(U)$ and $G = Q \cap \mathrm{Bd}(U)$. Since K is a continuum, both sets F and G are non-empty and there exists a component C of $K - (F \cup G)$ such that $F \cap \mathrm{Cl}(C) \neq \emptyset \neq G \cap \mathrm{Cl}(C)$. Since $C \not\subset U$, we have $C \subset K - \mathrm{Cl}(U)$. Let $p \in F \cap \mathrm{Cl}(C)$ and $q \in G \cap \mathrm{Cl}(C)$ and let L be a continuum in $\mathrm{Cl}(C)$ irreducible from p to q. Since p and q lie in different components of $K - V$, the continuum L meets V. By a theorem of Janiszewski [6, §61, Th. 2] the continuum L disconnects $S^2 - U$.

3.3. Lemma. *Let $X \subset S^2$ and let $R \subsetneq X \times X$ be an equivalence relation on X such that each R-equivalence class is continuum connected. For all but countably many R-equivalence classes $R(x)$ and for all open disjoint discs U and V which meet X, if $U \cap R(x)$ is contained in no continuum component of $R(x) - V$ then there exists a continuum $K \subset R(x) - U$ such that K separates two points of $V \cap X$ in $S^2 - U$.*

Proof. Consider the family $\{U_1, U_2, \ldots\}$ of all open discs the centres of which lie in a certain countable dense subset of S^2 and whose diameters are rational.

Let U_n and U_m be disjoint discs meeting X. We shall show first that for all but countably many R-equivalence classes $R(x)$, if $K \subset R(x) - U_n$ is a continuum disconnecting $S^2 - U_n$ and meeting U_m then K separates two points of $U_m \cap X$ in $S^2 - U_n$.

Suppose to the contrary that there exists an uncountable family $\mathcal{K}$ of subcontinua of $S^2 - U_n$, disconnecting $S^2 - U_n$ and meeting the set $U_m \cap X$ but separating no two points of this set in $S^2 - U_n$ and such that any two elements of $\mathcal{K}$ are contained in different R-equivalence classes. By Lemma 3.1, there exist $L, M, N \in \mathcal{K}$ such that L separates M from N in $S^2 - U_n$. Since M and N meet $U_m \cap X$, L separates two points of $U_m \cap X$ in $S^2 - U_n$, which is a contradiction.

So, for all but countably many R-equivalence classes $R(x)$ and for any two disjoint discs U_n and U_m meeting X, if $K \subset R(x) - U_n$ is a continuum disconnecting $S^2 - U_n$ and meeting $U_m \cap X$ then K separates two points of $U_m \cap X$ in $S^2 - U_n$.

Now, let $R(x)$ be an R-equivalence class as above. Let U and V be disjoint open discs meeting X such that $U \cap X$ is contained in no continuum component of $R(x) - V$. Let $a, b \in U \cap X$ be points lying in different continuum components of $R(x) - V$. Let n be a positive integer such that $a, b \in U_n \subset U$. Such an n exists since the diameters of U_n run over all positive rationals and their centres run over a dense subset of S^2. Let $K \subset R(x)$ be a continuum joining a and b. Then a and b lie in different components of $K - V$. By Lemma 3.2, there exists a continuum $L \subset K - U_n$ which disconnects $S^2 - U_n$ and meets V. Let m be a positive integer such that $U_m \subset V$ and $U_m \cap L \neq \emptyset$. Hence, by the choice of $R(x)$, L separates two points of $U_m \cap X$ in $S^2 - U_n$. So, the continuum L separates two points of $V \cap X$ in $S^2 - U_n$. Hence the compact set $L - U$ separates two points of $V \cap X$ in $S^2 - U$. Then a component of $L - U$ separates two points of $V \cap X$ in $S^2 - U$, since S^2 is locally connected and unicoherent (see [6]).

3.4. Theorem. *Let A, B, C and D be continua in S^2 such that $A \cap B = \emptyset$ and $C \cap D = \emptyset$. If $A \cap C$ is contained in a component of $S^2 - (B \cap D)$ then $B \cap D$ is contained in a component of $S^2 - (A \cup C)$.*

Proof. Since the sphere is locally arcwise connected we may assume that $A \cap C$ is connected. Since neither A nor C separates two points of $B \cap D$ neither does $A \cup C$ by the first theorem of Janiszewski [6], p. 507, Th. 7.

3.5. Lemma. *Let $X \subset S^2$ and $R \subseteq X \times X$ be an equivalence relation on X such that each R-equivalence class is continuum connected. Let P be a non-empty open set in X such that each R-equivalence class is of first category and dense in P, and such that for uncountably many R-equivalence classes $R(x)$ and for each open non-empty subset U of P each continuum component of $R(x) - U$ has empty interior with respect to $R(x) \cap P$. Then the union E of external R-equivalence classes of X is of first category in P. Moreover, E is an F_σ-set if P is of second category.*

Proof. Let $\{U_1, U_2, \ldots\}$ be a basis of open discs for the topology on S^2.

If i and k are positive integers such that $U_i \cap P \neq \emptyset$, $U_i \cap X \subset P$, $\mathrm{Cl}(U_k) \cap \mathrm{Cl}(X) = \emptyset$ and $U_i \cap U_k = \emptyset$, then define $L_{i,k}$ to be the union of sets $L \cap X$, where L runs over all those subcontinua of $S^2 - U_i$ such that $L \cap \mathrm{Cl}(U_k) \neq \emptyset$ and $R(L \cap X) \neq X$. If i and k do not satisfy the above-mentioned conditions, define $L_{i,k}$ to be the empty set.

Clearly, $L_{i,k} \subset E$ for all i and k. Now, let $R(x)$ be an external R-equivalence class of X and let L be a continuum in S^2 such that $L \cap R(x) \neq \emptyset$, $L \not\subset \mathrm{Cl}(X)$ and $R(L \cap X) \neq X$. Let K be a continuum in $R(x)$ such that $x \in K$ and $K \cap L \neq \emptyset$. Since $L \not\subset \mathrm{Cl}(X)$, there exists $y \in L - \mathrm{Cl}(X)$. Since $P \not\subset R(L \cap X)$, there exists $z \in P - R(L \cap X)$. Clearly, $z \notin K$. Since L and K are closed, there exist positive integers i and k such that $z \in U_i$, $y \in U_k$, $U_i \cap X \subset P$, $U_i \cap (K \cup L) = \emptyset$, $\mathrm{Cl}(U_k) \cap \mathrm{Cl}(X) = \emptyset$ and $U_i \cap U_k = \emptyset$. Then $x \in (K \cup L) \cap X \subset L_{i,k}$. So, $E = \bigcup_{i,k=1}^{\infty} L_{i,k}$.

To prove that E is of first category in P, it suffices to prove that $L_{i,k}$ is nowhere dense in P. Assume $L_{i,k} \neq \emptyset$.

Let U be an open disc such that $\mathrm{Cl}(U) \subset U_i$ and $U \cap P \neq \emptyset$. Let V be an open disc such that $V \cap P \neq \emptyset$, $V \cap X \subset P$, $V \cap U = \emptyset$ and $V \cap U_k = \emptyset$.

We shall show that $V \cap X$ is not contained in $\mathrm{Cl}(L_{i,k})$.

Let $R(x)$ be an R-equivalence class guaranteed by Lemma 3.3 and let $K \subset R(x)-U$ be a continuum which separates two points of $V \cap P$ in S^2-U. Let x_1 be a point of $V \cap P$ which K separates in S^2-U from U_k and let W be an open disc such that $x_1 \in W \subset V-K$. By Lemma 3.3, there exist $y \in X-R(x)$ and a continuum $L \subset R(y)-U$ which separates two points of $W \cap P$ in S^2-U. Let x_2 be a point of $W \cap P$ which L separates in S^2-U from U_k and let G be an open disc such that $x_2 \in G \subset W-L$. Then there exists a continuum $M \subset R(x)-U$ which separates two points of $G \cap P$ in S^2-U; clearly, L separates M from K in S^2-U. Let x_3 be a point of $G \cap P$ which M separates in S^2-U from U_k and let C be a component of $(S^2-U)-M$ to which x_3 belongs. It suffices to prove that $C \cap L_{i,k}=\emptyset$.

Suppose I is a continuum in S^2-U joining $\mathrm{Cl}(U_k)$ and $C \cap X$ such that $R(I \cap X) \neq X$. Clearly I meets K, L and M. Let $z \in X-R(I \cap X)$. Then $R(z) \neq R(x)$. Let J be a continuum contained in $R(x)$ and containing both continua K and M. Hence, $J \cap I$ has points in two components of $S^2-(L \cup \mathrm{Cl}(U))$. By Theorem 3.4, $L \cap \mathrm{Cl}(U)$ has points in two components of $S^2-(I \cup J)$. Since $R(z)$ is dense in P it has points in two components of $S^2-(I \cup J)$ also. But $R(z)$ is connected and disjoint from $I \cup J$, which is a contradiction. Hence, $L_{i,k}$ is nowhere dense in P and E is of first category in P.

Now, assume that P is of second category. To prove E is an F_σ-set it suffices to prove $L_{i,k}$ is closed. Let $x \in \mathrm{Cl}(L_{i,k})$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $L_{i,k}$ which converges to $x \in X$. For each n let $L_n \subset S^2-U_i$ be a continuum such that $x_n \in L_n$, $L_n \cap \mathrm{Cl}(U_k) \neq \emptyset$ and $R(L_n \cap X) \neq X$. The sequence L_n has a convergent subsequence with respect to the Hausdorff metric. We may suppose L_n is such a subsequence. It converges to a continuum $L \subset S^2-U_i$. Then $x \in L$ and $L \cap \mathrm{Cl}(U_k) \neq \emptyset$.

It remains to prove that L misses some R-equivalence class. Let $y \in U_i$ such that $R(y)$ is an internal equivalence class and $R(y)$ satisfies the conclusion of Lemma 3.3. Let $\{U_{n_j}\}_{j=1}^\infty$ be the basic neighbourhoods of y such that each $U_{n_j} \subset U_i$. Just suppose L meets $R(y)$.

For each $R(z) \neq R(y)$ and for each positive integer j let $A(z,j)$ be the union of the continua in $R(z)-U_{n_j}$ which meet L. Let $Q_j = \bigcup_{z \in X-R(y)} A(z,j)$. Then $X-R(y)=\bigcup_{j=1}^\infty Q_j$.

Since $X-R(y)$ is of second category in X, there exists an integer m and a basic neighbourhood $U_r \subset \mathrm{Cl}(U_r) \subset S^2-\mathrm{Cl}(U_{n_m})$ such that Q_m is dense in $U_r \cap X \neq \emptyset$ and $U_r \cap X \subset P$.

By Lemma 3.3 there is a continuum $B \subset R(y)-U_{n_m}$ such that B separates two points of $U_r \cap X$ in $S^2-U_{n_m}$. Since Q_m is dense $U_r \cap X$, B separates two points a and b of Q_m in $S^2-U_{n_m}$. By the definition of $A(a,m)$ there is a continuum K_a in $R(a)-U_{n_m}$ from a to L and there is a continuum K_b in $R(b)-U_{n_m}$ from b to L. Since K_a and K_b miss B, it follows that B separates two points of L in $S^2-U_{n_m}$. Since the sequence $\{L_n\}_{n=1}^{\infty}$ converges to L, there is a positive integer p such that B separates two points of L_p in $S^2-U_{n_m}$. This is a contradiction since L_p misses every internal R-equivalence class of X. Thus, L misses each internal R-equivalence class, $x\in L_{i,k}$ and $L_{i,k}$ is closed.

3.6. Theorem. *Let $X\subset S^2$ and let $R\subsetneq X\times X$ be an equivalence relation on X such that each R-equivalence class $R(x)$ is a continuum connected first category set in X, and for each open non-empty set U in X each continuum component of $R(x)-U$ has empty interior in $R(x)$. Then the union E of external R-equivalence classes of X is a first category set in X. Moreover, E is an F_σ-set in X if X is of second category.*

Proof. We can restrict our considerations to the case in which the set of R-equivalence classes is uncountable since otherwise the theorem is obvious. Let P be X in Lemma 3.5. It suffices to prove that each $R(x)$ is dense in X. But if $U\subset X$ is open and non-empty then each continuum component of $R(x)-U$ has empty interior in $R(x)$. Hence, $R(x)\cap U\neq\emptyset$.

A point y of a set Y is said to be a *terminal point* if for each pair of continua K and L in Y with $y\in K\cap L$ we have either $K\subset L$ or $L\subset K$.

3.7. Lemma. *Let U and V be disjoint open non-empty sets in the continuum connected space Y such that for each continuum K in Y, $U-K\neq\emptyset$ and $V-K\neq\emptyset$. Suppose $y\in Y-(U\cup V)$ is a terminal point of Y. Then U is contained in no continuum component of $Y-V$.*

Proof. Let $a\in U$ and let A be a continuum joining a and y. Let $b\in V-A$ and let B be a continuum joining b and y. Then $A\subset B$ since $b\in B-A$ and y is a terminal point in Y. Let $c\in U-B$. Now, let C be any continuum joining a and c. Then $A\cup C$ is a continuum joining c and y. So $B\subset A\cup C$. Since $b\in B-A\subset C$, C meets the set V. Hence, a and c are two points of U which lie in different continuum components of $Y-V$.

A continuum K is said to be a *triod* if $K-L$ has at least three components for some subcontinuum L of K. It is a well-known theorem of Moore (cf. [10]) that the 2-sphere does not contain an uncountable collection of pairwise disjoint triods.

3.8. Theorem. *Let $X\subset S^2$ be such that each non-empty open subset of X is of second category. Let $R\subsetneq X\times X$ be an equivalence relation on X such that each R-equivalence class is continuum connected, of first category and dense in X. Then the union of external R-equivalence classes of X is a first category F_σ-set in X.*

P r o o f. By Theorem 3.6 it suffices to prove that for each open, nonempty set U in X and each R-equivalence class $R(x)$ each continuum component of $R(x) - U$ has empty interior in $R(x)$.

Just suppose there exist disjoint discs U and V each of which meets X and there exists an R-equivalence class $R(x)$ such that $R(x) \cap V$ is contained in a continuum component of $R(x) - U$. If $y \in X - R(x)$ then no continuum $K \subset R(y) - U$ separates some two points of $X \cap V$ in $S^2 - U$; otherwise, since $R(x)$ is dense in X, the continuum K would separate some two points of $R(x) \cap V$ in $S^2 - U$ and such two points would lie in different continuum components of $R(x) - U$. Hence, by Lemma 3.3, for all but countably many R-equivalence classes $R(y)$, $V \cap R(y)$ is contained in a continuum component of $R(y) - U$.

Let $X' = \{y \in X - U : R(y) \cap V$ is contained in the continuum component of y in $R(y) - U\}$. Let $R' = R|X' \times X'$. Then each R'-equivalence class is continuum connected, dense in $V \cap X'$ and of first category in $V \cap X'$. Also $V \cap X'$ is of second category, since $V - X'$ is contained in the sum of countably many R-equivalence classes. Each R'-equivalence class is external since it meets the boundary of the disc U which is disjoint from X'.

We shall show that in uncountably many R'-equivalence classes there exist terminal points outside of V. If $a \in R'(y) \cap \mathrm{Bd}(U)$ is not a terminal point then there exist continua K and L contained in $R'(y)$ such that $K \not\subset L$, $L \not\subset K$ and $a \in K \cap L$. Let M be the interval joining the point a and the middle point in the radius of U beginning at the point a. The union $K \cup L \cup M$ is a triod. Such triods for different R'-equivalence classes are disjoint. Since on the plane each family of disjoint triods is countable, for all but countably many R'-equivalence classes, and in consequence for uncountably many R'-equivalence classes $R'(x)$, there exists a terminal point of $R'(x)$ in $\mathrm{Bd}(U)$.

Observe that if K is a continuum in an R'-equivalence class $R'(y)$ then the interior of K in $R'(y)$ is disjoint from V; otherwise, since $R'(y)$ is dense in $V \cap X'$, K and hence $R'(y)$ would have interior in $V \cap X'$, which would contradict the fact that each R'-equivalence class is dense in $V \cap X'$.

Taking X' in place of X, $V \cap X'$ in place of P and R' in place of R we see, using Lemma 3.7, that the assumptions of Lemma 3.5 are satisfied. Hence, the sum of R'-equivalence classes, i.e. the set X', is of first category, which is a contradiction.

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