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On D-dimension of metrizable spaces*

by

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Abstract. For every cardinal τ and every ordinal α, we construct a metrizable space $M_\alpha(\tau)$ and a strongly countable-dimensional compact space $Z_\alpha(\tau)$ of weight τ such that $D(M_\alpha(\tau)) \le \alpha$, $D(Z_\alpha(\tau)) \le \alpha$ and each metrizable space X of weight τ such that $D(X) \le \alpha$ is homeomorphic to a subspace of $M_\alpha(\tau)$ and to a subspace of $Z_{\alpha+1}(\tau)$.

1. Introduction. Our notation and terminology follow [1] and [2]. By dimension we mean the covering dimension, by space a normal space, and by mapping a continuous mapping. We use the habitual convention that an ordinal α is the set of all ordinals less than α.

The symbol $|A|$ denotes the cardinality of the set A, the symbols $\mathcal{N}$ and I the set of non-negative integers and the closed unit interval, respectively; i, j, k, l, m, n denote natural numbers, α, β, γ, δ, ξ, η ordinals, λ a limit ordinal, τ an infinite cardinal, $\aleph_0$ the smallest infinite cardinal, and ϱ, σ metrics.

Let X be a space. We put $D(X) = -1$ whenever $X = \emptyset$. If $X \ne \emptyset$ and $\alpha = \lambda + n$, then $D(X) \le \alpha$ whenever there exists a closed covering $\{A_\beta : \beta \le \lambda\}$ of X consisting of finite-dimensional subsets such that:

(1.1) for every $\delta \le \lambda$, the set $\bigcup\{A_\beta : \delta \le \beta \le \lambda\}$ is closed,

(1.2) for every $x \in X$, there exists a greatest ordinal $\beta \le \lambda$ such that $x \in A_\beta$,

(1.3) $\dim A_\lambda \le n$.

If, apart from that, $A_\lambda = \emptyset$, then we write $D(X) <\le \lambda$. If there exists an α such that $D(X) \le \alpha$, then $D(X)$ is the smallest such α; in the opposite case, we set $D(X) = \Delta$, where $\Delta > \alpha$ and $\Delta + \alpha = \Delta$ for every ordinal α.

The existence of a closed covering $\{A_\beta : \beta \le \lambda\}$ of X by finite-dimensional subsets satisfying (1.1)–(1.3) implies the existence of such a covering

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$\{A'_\beta : \beta < \lambda\}$ satisfying (1.1)–(1.3) and

(1.4) if $\beta = \gamma + m < \lambda$, where γ is 0 or a limit ordinal, then $\dim A'_\beta \le m$.

Indeed, it suffices to put

$$A'_{\gamma+m} = \begin{cases} A_{\gamma+l} & \text{if } m = l + \dim A_{\gamma+1} + \ldots + \dim A_{\gamma+l}, \\ \emptyset & \text{for the remaining } m \in \mathcal{N} \end{cases}$$

for every γ which is either 0 or a limit ordinal less than λ.

The ordinal number $D(X)$ is called the *D-dimension* of X and was introduced by D. W. Henderson (see [4]). If X is a space of weight τ, then $|D(X)| \le \tau$ (see [4], Theorem 10).

A space is called *strongly countable-dimensional* if it is the union of a countable family of finite-dimensional closed subsets. One readily checks that a space X is strongly countable-dimensional iff $X = \bigcup\{X_n : n \in \mathcal{N}\}$, where X_n is a closed subspace of X and $\dim X_n \le n$ for every $n \in \mathcal{N}$.

In [10] a class of small spaces was defined and it was observed that in the realm of strongly hereditarily normal (in particular, metrizable) spaces this class coincides with the class of spaces X such that $D(X) < \Delta$; thus it follows from Theorems 3.2 and 3.8, and Corollary 3.4 in [10] that if X is a metrizable space, then $D(X) < \Delta$ iff X has a strongly countable-dimensional completion (see also [6]).

L. Luxemburg showed (see [8], Theorem 1.3) that

(1.5) for every ordinal α such that $|\alpha| \le \aleph_0$, there exists a universal space for metrizable spaces X of weight $\aleph_0$ such that $D(X) \le \alpha$.

It is also known (see Conjecture in [5], Theorem 2 in [6], Theorem 8.1 in [7]) that

(1.6) every metrizable space X of weight $\aleph_0$ has a strongly countable-dimensional compactification Z of weight $\aleph_0$ such that $D(Z) \le D(X) + 1$.

An analogous compactification theorem for metrizable spaces of arbitrarily large weight, i.e., (1.6) with $\aleph_0$ replaced by an arbitrary τ, follows from the results announced in I. M. Kozlovskiĭ's paper [6]; their proofs, however, have never been published by the author. This general compactification theorem seems to be particularly valuable in view of the following corollary (see [6], Corollary):

(1.7) every strongly countable-dimensional completely metrizable space of weight τ has a strongly countable-dimensional compactification of weight τ.

The aim of this paper is to generalize (1.5) to metrizable spaces of arbitrarily large weight. Simultaneously, we give a proof of I. M. Kozlovskiĭ's generalization of (1.6); in fact, we prove a little more—namely, the existence of a common such compactification for all spaces with fixed weight and D-dimension.

We denote by $J(\tau)$ the hedgehog of spininess τ, by d its standard metric (see [1], Example 4.1.5), and by $[J(\tau)]^{\aleph_0}$ its countable Cartesian power. The symbol $\mathcal{O}$ will often appear in our considerations; it will denote distinguished points of various spaces. In particular, we denote by $\mathcal{O}$ the "origin" of $J(\tau)$ and the point of $[J(\tau)]^{\aleph_0}$ with all coordinates equal to $\mathcal{O}$.

Every ordinal $\alpha > 0$ which is not the sum of two ordinals smaller than α is called a *prime component*. For every ordinal $\alpha > 0$, there is a unique representation $\alpha = \alpha_1 + \ldots + \alpha_k + \alpha_{k+1}$, where $\alpha_i > 0$ is a prime component and $\alpha_{i+1} \le \alpha_i$ for $i \le k$, and α_{k+1} is finite. Let $\alpha = \alpha_1 + \ldots + \alpha_k + \alpha_{k+1}$ and $\beta = \beta_1 + \ldots + \beta_l + \beta_{l+1}$ be such representations of α and β, and let $\gamma_1, \gamma_2, \ldots, \gamma_{k+l+2}$ be the elements of $\{\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \beta_1, \ldots, \beta_l, \beta_{l+1}\}$ written in decreasing order. Then the natural sum $\alpha \oplus \beta$ is defined by $\alpha \oplus \beta = \gamma_1 + \ldots + \gamma_{k+l} + (\gamma_{k+l+1} + \gamma_{k+l+2})$. For every ordinal δ, there are only finitely many solutions α, β of the equation $\alpha \oplus \beta = \delta$, and if $\alpha \oplus \beta > \gamma$ for some ordinals α, β, γ, then there exist $\xi \le \alpha$ and $\eta \le \beta$ such that $\xi \oplus \eta = \gamma$. Detailed information about prime components and natural sums can be found in [11] (Chapter XIV.6, 28), and [3] (Chapter IV.14).

The symbol $K_n(\tau)$ denotes the n-dimensional universal Nagata space, i.e., the subspace

$$\{(x_0, x_1, \ldots) \in [J(\tau)]^{\aleph_0} : |\{x_i : d(\mathcal{O}, x_i) \text{ is a positive rational number }\}| \le n\}$$

of $[J(\tau)]^{\aleph_0}$. If $m \le n$, then $K_m(\tau) \subseteq K_n(\tau)$.

Suppose we are given spaces X, Y, a subset $A \subseteq X$, and a mapping $f : X \to Y$. We say that f *separates points of A and closed sets in X* if the following condition is satisfied:

(1.8) if $x \in A$ and $x \notin B = \operatorname{cl} B \subseteq X$, then $f(x) \notin \operatorname{cl} f(B) \subseteq Y$.

2. The lemmas. In this section, we shall formulate three lemmas. The second one is a consequence of a theorem announced in [6] (see Theorem 1), but we will quote [9], where the proof of this theorem can be found.

2.1. Lemma. *Let X be a metrizable space. For every open covering $\{V_m : m \in \mathcal{N}\}$ of X satisfying $V_m \subseteq V_{m+1}$ for all $m \in \mathcal{N}$, there exists an open shrinking $\{W_m : m \in \mathcal{N}\}$ satisfying $W_m \cap W_l = \emptyset$ whenever $|m-l| > 1$.*

Proof. For every $m \in \mathcal{N}$, take a function $f_m : X \to I$ such that

$f_m^{-1}(0) = X - V_m$, and put

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k+2}} f_k(x) \quad \text{for } x \in X.$$

Let $W_m = f^{-1}((\frac{1}{m+3}, \frac{1}{m+1}))$ for $m \in \mathcal{N}$. The family $\{W_m : m \in \mathcal{N}\}$ has the required properties (cf. the proof of Theorem 3.1 in [9]).

2.2. Remark. In the hypothesis of Theorem 2.12 in [9], we can require that $g(F)$ is a closed subset of the space $g(X)$. This follows immediately from the proof presented there (see also [6], Theorem 1).

If under the assumptions of Corollary 2.5 in [9], we additionally fix an m-dimensional closed subspace A of X, then in the hypothesis, we can require that $\operatorname{cl} h(A) \subseteq K_m(\tau)$ for every $h \in \mathcal{P}$. This follows from the proof presented in [9], where Corollary 2.7 instead of Theorem 2.1 should be applied. Thus for every n-dimensional metrizable space X of weight not larger than τ, every m-dimensional closed subspace $A \subseteq X$, and every $x_0 \in X$, there exists an embedding $h : X \to K_n(\tau)$ such that $h(A) \subseteq K_m(\tau)$ and $h(x_0) = \mathcal{O}$.

2.3. Lemma. *Let X be a metrizable space of weight τ, and E and F its closed subsets. If $\dim(F - E) \leq n$, then there exists a mapping $f : X \to K_{n+1}(\tau)$ separating points of $F - E$ and closed sets in X such that*:

(2.1) *$f(F \cup E)$ is a closed subset of $f(X)$,*

(2.2) *$f(F \cup E) \subseteq K_n(\tau) \subseteq K_{n+1}(\tau)$ and $f^{-1}(\mathcal{O}) = E$.*

Proof. By Lemma 2.9 in [9], there exist a metrizable space X' of weight not larger than τ, a point $x' \in X'$ and a continuous mapping $q : X \to X'$ such that $q^{-1}(x') = E$ and $q|X - E$ is a homeomorphic embedding onto $X' - \{x'\}$. Let $F' = q(F) \cup \{x'\}$; then F' is a closed subset of X' and $\dim F' \leq n$.

By Theorem 2.12 in [9], there exist an $(n+1)$-dimensional space Z of weight not larger than τ and a continuous mapping $g : X' \to Z$ separating points of F' and closed subsets in X'; since $x' \in F'$, we have $g^{-1}(g(x')) = \{x'\}$, and by the first part of Remark 2.2, we can assume that $g(F')$ is closed in $g(X')$. By the second part of Remark 2.2, there exists an embedding $h : g(X') \to K_{n+1}(\tau)$ such that $h(g(x')) = \mathcal{O}$ and $h(g(F')) \subseteq K_n(\tau)$. Then the mapping $f = h \circ (g|X') \circ q$ has the required properties.

2.4. Lemma. *Let X, Y be compact or metrizable spaces. If $\{A'_{\beta_1} : \beta_1 \leq \lambda_1\}$ and $\{A''_{\beta_2} : \beta_2 \leq \lambda_2\}$ are closed coverings of X and Y, respectively, by finite-dimensional subsets satisfying* (1.1), (1.2), *then $\{A_\beta : \beta \leq \lambda_1 \oplus \lambda_2\}$, where $A_\beta = \bigcup\{A_{\beta_1} \times A_{\beta_2} : \beta_1 \oplus \beta_2 = \beta,\ \beta_1 \leq \lambda_1,\ \beta_2 \leq \lambda_2\}$, is a closed covering of $X \times Y$ by finite-dimensional subsets satisfying* (1.1), (1.2).

Proof. Since $\dim(X \times Y) \leq \dim X + \dim Y$ for any non-empty compact or metrizable spaces X and Y (see [2], Theorem 3.2.13 or 4.1.21), our lemma follows from the proof of Theorem 5 in [4].

3. The tools. This section does not directly concern dimension theory, but it contains the main tools of the paper. We shall describe a few topological operations, introduce a technical notion important for the sequel, and prove their basic—from our point of view—properties.

Let $|\lambda| \le \tau$ and let T be an arbitrary set of cardinality τ. Suppose we are given a family $\mathcal{X} = \{(X_\alpha, x_\alpha) : \alpha < \lambda\}$ (a family $\mathcal{X} = \{(X_\alpha, x_\alpha, \varrho_\alpha) : \alpha < \lambda\}$) of pointed compact spaces (of pointed metric spaces). For all $\alpha, \xi < \lambda$ and every $t \in T$, set $(X_{\alpha,\xi,t}, x_{\alpha,\xi,t}) = (X_\alpha, x_\alpha)$ (respectively, $(X_{\alpha,\xi,t}, x_{\alpha,\xi,t}, \varrho_{\alpha,\xi,t}) = (X_\alpha, x_\alpha, \varrho_\alpha)$).

1. *The compact case.* We denote by $J_C(\mathcal{X}, \tau)$ the quotient space obtained from the Alexandrov compactification of the topological sum $\bigoplus\{X_{\alpha,\xi,t} : \alpha, \xi < \lambda,\ t \in T\}$—the unique point of the remainder is denoted by x_λ—by identifying the set $\{x_{\alpha,\xi,t} : \alpha, \xi < \lambda,\ t \in T\} \cup \{x_\lambda\}$ to a point $\mathcal{O} \in J_C(\mathcal{X}, \tau)$; the spaces $X_{\alpha,\xi,t}$ can be identified in a natural way with the respective subspaces of $J_C(\mathcal{X}, \tau)$.

2. *The metric case.* We denote by $J_M(\mathcal{X}, \tau)$ the space obtained from $\bigoplus\{X_{\alpha,\xi,t} : \alpha, \xi < \lambda,\ t \in T\}$ by identifying the set $\{x_{\alpha,\xi,t} : \alpha, \xi < \lambda,\ t \in T\}$ to a point $\mathcal{O} \in J_M(\mathcal{X}, \tau)$—the spaces $X_{\alpha,\xi,t}$ can be identified in a natural way with the respective subspaces of $J_M(\mathcal{X}, \tau)$—equipped with the metric ϱ defined by letting $\varrho(x, y) = \varrho_{\alpha,\xi,t}(x, y)$ if $x, y \in X_{\alpha,\xi,t}$ for some $\alpha, \xi < \lambda$, $t \in T$ and $\varrho(x, y) = \varrho(x, \mathcal{O}) + \varrho(y, \mathcal{O})$ otherwise.

The reader can easily check that we have obtained a compact space $J_C(\mathcal{X}, \tau)$ (a metric space $(J_M(\mathcal{X}, \tau), \varrho)$).

The equivalence of metrics ϱ_α and σ_α on X_α for every $\alpha < \lambda$ does not imply that the corresponding metrics ϱ and σ on $J_M(\mathcal{X}, \tau)$ are equivalent. In the sequel, we shall consider the topological space $J_M(\mathcal{X}, \tau)$, where $\mathcal{X}$ is a family of pointed metrizable spaces. One should understand that when defining the space $(J_M(\mathcal{X}, \tau), \varrho)$, we have fixed arbitrary metrics ϱ_α compatible with the topologies on X_α (except in the proof of Corollary 4.4, where we shall additionally assume that the metrics ϱ_α are complete for completely metrizable spaces X_α), and that the topology on $J_M(\mathcal{X}, \tau)$ is induced by the metric ϱ.

In order to give a common proof of both our theorems, we shall sometimes use the symbol $J(\mathcal{X}, \tau)$ instead of $J_C(\mathcal{X}, \tau)$ while proving the generalization of (1.6), and instead of $J_M(\mathcal{X}, \tau)$ while proving the generalization of (1.5).

Then

$$J(\mathcal{X}, \tau) = \bigcup\{X_{\alpha,\xi,t} : \alpha, \xi < \lambda,\ t \in T\}$$

and $\mathcal{O}$ is the unique common point of every pair of subspaces X_{α_1,ξ_1,t_1}, X_{α_2,ξ_2,t_2}.

Let $\omega(0) = 0$ and $\omega(m) = 1 + 2 + \ldots + m$ for every $m \geq 1$. We shall denote by $J^\omega(\mathcal{X},\tau)$ the subspace of $[J(\mathcal{X},\tau)]^{\aleph_0}$ consisting of all points $(x_1,\ldots,x_j,\ldots)$ such that

(3.1) if $\mathcal{O} \neq x_j \in X_{\alpha_j,\xi_j,t_j}$ for a j, then

$$x_k \in \bigcup\{X_{\alpha_k,\xi_k,t_k} : \alpha_k \leq \xi_j,\ \xi_k < \lambda,\ t_k \in T\}$$

for all $k \neq j$,

(3.2) there exists an $m \in \mathcal{N}$ such that $\{j : x_j \neq \mathcal{O}\} \subseteq \{\omega(m)+1,\ldots,\omega(m+2)\}$.

The point $(\mathcal{O},\mathcal{O},\ldots) \in J^\omega(\mathcal{X},\tau)$ will also be denoted by $\mathcal{O}$.

3.1. Proposition. *If the weight of X_α is not greater than τ for every $\alpha < \lambda$, then the weight of $J^\omega(\mathcal{X},\tau)$ is not greater than τ.*

The space $J^\omega(\mathcal{X},\tau)$ is a closed subspace of $[J(\mathcal{X},\tau)]^{\aleph_0}$; thus $J^\omega_C(\mathcal{X},\tau)$ is compact, and $J^\omega_M(\mathcal{X},\tau)$ is metrizable.

If X_α is strongly countable-dimensional for every $\alpha < \lambda$, then $J^\omega_C(\mathcal{X},\tau)$ is strongly countable-dimensional.

Proof. The first part of our proposition follows from the existence of a continuous mapping of the space $\bigoplus\{X_\alpha : \alpha < \lambda\}$ onto $J(\mathcal{X},\tau)$, the inequality $|\lambda| \leq \tau$, and the inclusion $J^\omega(\mathcal{X},\tau) \subseteq [J(\mathcal{X},\tau)]^{\aleph_0}$.

If a point $(x_1,\ldots,x_j,\ldots) \in [J(\mathcal{X},\tau)]^{\aleph_0}$ does not satisfy (3.1), then there are distinct $j,k = 1,2,\ldots$ such that $\mathcal{O} \neq x_j \in X_{\alpha_j,\xi_j,t_j}$ and

$$x_k \notin \bigcup\{X_{\alpha_k,\xi_k,t_k} : \alpha_k \leq \xi_j,\ \xi_k < \lambda,\ t_k \in T\}.$$

Since $J(\mathcal{X},\tau) = \bigcup\{X_{\alpha,\xi,t} : \alpha,\xi < \lambda,\ t \in T\}$ and $\mathcal{O} \in X_{\alpha,\xi,t}$ for every $\alpha,\xi < \lambda$, $t \in T$, we conclude that

$$\mathcal{O} \neq x_k \in X_{\alpha_k,\xi_k,t_k} \quad \text{for some } \alpha_k > \xi_j,\ \xi_k < \lambda,\ t_k \in T;$$

then the set

$$U = \{(y_1,\ldots,y_j,\ldots) \in [J(\mathcal{X},\tau)]^{\aleph_0} : \mathcal{O} \neq y_j \in X_{\alpha_j,\xi_j,t_j},\ \mathcal{O} \neq y_k \in X_{\alpha_k,\xi_k,t_k}\}$$

is a neighbourhood of $(x_1,\ldots,x_j,\ldots)$ and no point of U satisfies (3.1).

Thus the set C of all points of $[J(\mathcal{X},\tau)]^{\aleph_0}$ satisfying (3.1) is closed.

The set D of all points of $[J(\mathcal{X},\tau)]^{\aleph_0}$ satisfying (3.2) can be represented as the union of the family $\{J_m : m \in \mathcal{N}\}$, where J_m is the subspace of $[J(\mathcal{X},\tau)]^{\aleph_0}$ consisting of all points $(x_1,\ldots,x_j,\ldots)$ such that

$$\{j : x_j \neq \mathcal{O}\} \subseteq \{\omega(m)+1,\ldots,\omega(m+2)\},$$

for $m \in \mathcal{N}$. Since J_m is closed in $[J(\mathcal{X},\tau)]^{\aleph_0}$ for every $m \in \mathcal{N}$, the point $(\mathcal{O},\mathcal{O},\ldots)$ satisfies (3.2), and every point of $[J(\mathcal{X},\tau)]^{\aleph_0}$ distinct from $(\mathcal{O},\mathcal{O},\ldots)$ has a neighbourhood U such that $U \cap J_m = \emptyset$ for all but finitely many $m \in \mathcal{N}$, so that the set D is closed in $[J(\mathcal{X},\tau)]^{\aleph_0}$.

Thus $J^\omega(X,\tau) = C \cap D$ is a closed subspace of $[J(\mathcal{X},\tau)]^{\aleph_0}$ and the second part of the proposition is established.

Since $J^\omega_C(X,\tau)$ is a closed subspace of D, and $D = \bigcup\{J_m : m \in \mathcal{N}\}$, it suffices to show that J_m is a strongly countable-dimensional space for each $m \in \mathcal{N}$; but every J_m is homeomorphic to a product of finitely many copies of $J_C(X,\tau)$, so, by Theorem 3.2.13 in [2], it suffices to show that $J_C(X,\tau)$ is strongly countable-dimensional.

Let $X_\alpha = \bigcup\{X_{\alpha,n} : n \in \mathcal{N}\}$, where $X_{\alpha,n}$ is compact and $\dim X_{\alpha,n} \le n$ for every $n \in \mathcal{N}$; put $X_{\alpha,\xi,t,n} = X_{\alpha,n}$ for every $\alpha,\xi < \lambda$, $t \in T$, $n \in \mathcal{N}$, and

$$X_n = \{\mathcal{O}\} \cup \bigcup\{X_{\alpha,\xi,t,n} : \alpha,\xi < \lambda,\ t \in T\}$$

for $n \in \mathcal{N}$. Then X_n is closed subspace of $J_C(X,\tau)$ and $J_C(X,\tau) = \bigcup\{X_n : n \in \mathcal{N}\}$. From the definition of the covering dimension it follows directly that $\dim X_n \le n$ for every $n \in \mathcal{N}$. Hence the third part of the proposition is also established.

Suppose we are given a space X, its subsets F and U, closed and open, respectively, and a space Y, its subspace A, and a point $y \in A$. We say that the sixtuple F, U, X, A, y, Y has *property* (S) if for every open subset $V \subseteq U$, there exists a mapping $f : X \to Y$ separating points of $F \cap V$ and closed sets in X such that $f^{-1}(y) = X - V$ and $f(F) \subseteq A$.

3.2. PROPOSITION. *Let F be a closed subset and U an open subset of a space X. Consider a sequence $\emptyset = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_k = U$ of open subsets of X and a sequence $(Y_1,y_1),\ldots,(Y_k,y_k)$ of pointed spaces and set $F_{i+1} = F - U_i$ for $i = 0,\ldots,k-1$ and $(Y,y) = (Y_1 \times \ldots \times Y_k,(y_1,\ldots,y_k))$. If the sixtuple F_i, U_i, X, Y_i, y_i, Y_i has property* (S) *for $i = 1,\ldots,k$, then so does F, U, X, Y, y, Y.*

P r o o f. Let $V \subseteq U$ be an open subset; define $V_i = V \cap U_i$ for $i = 1,\ldots,k$. Since $F_i, U_i, X, Y_i, y_i, Y_i$ has property (S), there exists a mapping $f_i : X \to Y_i$ separating points of $F_i \cap V_i$ and closed sets in X such that $f_i^{-1}(y_i) = X - V_i$. Let $f = f_1 \triangle \ldots \triangle f_k : X \to Y = Y_1 \times \ldots \times Y_k$; then

$$\begin{aligned} f^{-1}(y) &= f_1^{-1}(y_1) \cap \ldots \cap f_k^{-1}(y_k) \\ &= X - \bigcup_{i=1}^k V_i = X - \bigcup_{i=1}^k (V \cap U_i) = X - V \end{aligned}$$

and f separates points of $(V_1 \cap F_1) \cup \ldots \cup (V_k \cap F_k)$ and closed sets in X.

Since

$$\begin{aligned}(V_1 \cap F_1) \cup \ldots \cup (V_k \cap F_k) \\ &= (V \cap U_1 \cap (F - U_0)) \cup \ldots \cup (V \cap U_k \cap (F - U_{k-1})) \\ &= V \cap F \cap [(U_1 - U_0) \cup \ldots \cup (U_k - U_{k-1})] = V \cap F,\end{aligned}$$

the proof is complete.

3.3. THEOREM. *Let X be a metrizable space of weight τ, F and U its closed and open subset*, *respectively. Consider an open covering $\{U_\alpha : \alpha < \lambda\}$ of U and a family $\mathcal{X} = \{(X_\alpha, x_\alpha) : \alpha < \lambda\}$ of pointed compact or metrizable spaces. If for every $\alpha < \lambda$ the sixtuple F, U_α, X, X_α, x_α, X_α has property* (S), *then so does F, U, X, $J^\omega(\mathcal{X}, \tau)$, $\mathcal{O}$, $J^\omega(\mathcal{X}, \tau)$.*

Proof. Take an open subset $V \subseteq U$ and put $V_\alpha = V \cap U_\alpha$ for $\alpha < \lambda$. Let $\mathcal{G}$ be a locally finite open refinement of $\{V_\alpha : \alpha < \lambda\}$, and $\mathcal{H}$ an open covering each of whose elements intersects only finitely many members of $\mathcal{G}$. Choose a refinement $\mathcal{V} = \bigcup\{\mathcal{V}_m : m = 1, 2, ...\}$ of the covering $\{G \cap H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ of V, where each family $\mathcal{V}_m$ is discrete in X. Let $V_m = \bigcup\{\bigcup \mathcal{V}_n : n = 1, \ldots, m\}$ for $m = 1, 2, \ldots$ By Lemma 2.1, there exists an open shrinking $\{W_m : m = 1, 2, \ldots\}$ of $\{V_m : m = 1, 2, \ldots\}$ such that $W_l \cap W_m = \emptyset$ whenever $|m - l| > 1$. For $j \in \mathcal{N}$, take an $m \in \mathcal{N}$ and an $n = 1, \ldots, m + 1$ such that $j = \omega(m) + n$, and define $\mathcal{W}_j = \{W_{m+1} \cap V : V \in \mathcal{V}_n\}$ and $\mathcal{W} = \bigcup\{\mathcal{W}_j : j = 1, 2, \ldots\}$.

The families defined above have the following properties:

(3.3) the family $\mathcal{W}_j$ is discrete in X and its elements are open subsets of V for every $j = 1, 2, \ldots,$

(3.4) the family $\mathcal{W}$ is a covering of V,

(3.5) for every $W \in \mathcal{W}$, there exist $\alpha, \xi < \lambda$ such that $W \subseteq U_\alpha$ and if $W \cap W' \neq \emptyset$ for a $W' \in \mathcal{W}$, then $W' \subseteq U_{\alpha'}$ for some $\alpha' \leq \xi$,

(3.6) if $W \cap W' \neq \emptyset$ for some $W \in \mathcal{W}_j$ and $W' \in \mathcal{W}_k$, then

$$j, k \in \{\omega(m) + 1, \ldots, \omega(m + 2)\} \qquad \text{for some } m \in \mathcal{N};$$

we denote by $\alpha(W)$ and $\xi(W)$ the smallest α and ξ satisfying (3.5).

Fix a $j \in \mathcal{N}$. Since the weight of X is equal to τ, we have $|\mathcal{W}_j| \leq \tau$, and therefore for our set T of cardinality τ there exists an injection $\theta : \mathcal{W}_j \to T$. For every $W \in \mathcal{W}_j$, there exists, by the assumptions of our theorem, a mapping $f_W : X \to X_{\alpha(W),\xi(W),\theta(W)}$ separating points of $W \cap F$ and closed sets in X such that $f_W^{-1}(x_{\alpha(W),\xi(W),\theta(W)}) = X - W$.

It follows from (3.3) that the family of mappings $\{f_W : W \in \mathcal{W}_j\}$ yields a mapping $f_j : X \to J(\mathcal{X}, \tau)$ separating points of $(\bigcup \mathcal{W}_j) \cap F$ and closed sets in X such that $f_j^{-1}(\mathcal{O}) = X - (\bigcup \mathcal{W}_j)$.

Let $f = \triangle\{f_j : j = 1, 2, \ldots\}$; then $f^{-1}(\mathcal{O}) = X - V$ by (3.4), and f separates points of $F \cap V$ and closed sets in X. It follows from (3.5) that every point $f(x) \in [J(\mathcal{X}, \tau)]^{\aleph_0}$ satisfies (3.1), and by (3.6), it also satisfies (3.2). Thus $f(X) \subseteq J^\omega(\mathcal{X}, \tau)$, and the proof of our theorem is complete.

4. The spaces $M_\alpha(\tau)$ and $Z_\alpha(\tau)$, their basic properties. For every α such that $|\alpha| \le \tau$, we define by induction a metrizable space $M_\alpha(\tau)$ and a compact space $Z_\alpha(\tau)$. In this section, we establish the properties of these spaces announced in the abstract, except for the inequalities $D(M_\alpha(\tau)) \le \alpha$, $D(Z_\alpha(\tau)) \le \alpha$ that are proved in Section 5. In Section 2 (see Lemma 2.3) and in Section 3, we prepared the tools that will now allow us to carry out in the general case the (suitably modified) argument used by L. Luxemburg in the special case of separable spaces (see [8], the proofs of Theorems 1.3 and 1.4).

We shall distinguish inductively a point $\mathcal{O}$ in every $M_\alpha(\tau)$ and $Z_\alpha(\tau)$.

Let $M_n(\tau) = K_n(\tau)$ and let $Z_n(\tau)$ be an n-dimensional compactification of weight τ of $K_n(\tau)$ (see [2], Theorem 3.3.3) for $n = 0, 1, \ldots$; we have already distinguished the point $\mathcal{O} \in M_n(\tau) \subseteq Z_n(\tau)$.

Let $\alpha = \lambda + n$; then $\lambda = \lambda_1 + \ldots + \lambda_k$, where λ_j is a prime component for $j = 1, \ldots, k$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k$.

We first define some auxiliary pointed spaces $M'_\lambda(\tau)$ and $Z'_\lambda(\tau)$. If $k = 1$, then let $Z'_\lambda(\tau) = J^\omega_C(\mathcal{X}, \tau)$, where $\mathcal{X} = \{(Z_\alpha(\tau), \mathcal{O}) : \alpha < \lambda\}$, and $M'_\lambda(\tau) = J^\omega_M(\mathcal{X}, \tau)$, where $\mathcal{X} = \{(M_\alpha(\tau), \mathcal{O}) : \alpha < \lambda\}$. If $k > 1$, then let $Z'_\lambda(\tau) = Z_{\lambda_1}(\tau) \times \ldots \times Z_{\lambda_k}(\tau)$ and $M'_\lambda(\tau) = M_{\lambda_1}(\tau) \times \ldots \times M_{\lambda_k}(\tau)$. In the first case, we have already distinguished the points $\mathcal{O}$ (see the definition of $J^\omega(\mathcal{X}, \tau)$ in Section 3); in the second case, take $(\mathcal{O}, \ldots, \mathcal{O})$ as $\mathcal{O}$.

Now, let $M_\alpha(\tau) = \{(x, y) \in M'_\lambda(\tau) \times K_{n+1}(\tau) : \text{if } x = \mathcal{O}, \text{then } y \in K_n(\tau)\}$, and $Z_\alpha(\tau) = Z'_\lambda(\tau) \times Z_n(\tau)$; in both cases take $(\mathcal{O}, \mathcal{O})$ as $\mathcal{O}$.

Using Proposition 3.1 and induction on α we obtain the following proposition.

4.1. Proposition. *Let $|\alpha| \le \tau$. The space $M_\alpha(\tau)$ is metrizable and the space $Z_\alpha(\tau)$ is compact. The space $Z_\alpha(\tau)$ is strongly countable-dimensional. The weight of $Z_\alpha(\tau)$ and of $M_\alpha(\tau)$ is equal to τ.*

4.2. Theorem. *Let X be a metrizable space of weight τ, F and U its subsets, closed and open, respectively. If $D(F \cap U) \le \lambda + n$, where λ is* 0 *or a limit ordinal, then the sixtuples F, U, X, $M_{\lambda+n}(\tau)$, $\mathcal{O}$, $M_{\lambda+n+1}(\tau)$ and F, U, X, $Z_{\lambda+n+1}(\tau)$, $\mathcal{O}$, $Z_{\lambda+n+1}(\tau)$ have property* (S). *If $D(F \cap U) <\le \lambda$, then the sixtuples F, U, X, $M'_\lambda(\tau)$, $\mathcal{O}$, $M'_\lambda(\tau)$ and F, U, X, $Z'_\lambda(\tau)$, $\mathcal{O}$, $Z'_\lambda(\tau)$ have property* (S).

Proof. We use induction on λ. From Lemma 2.3 it follows that

(4.1) for every metrizable space X of weight τ and its subsets F and U, closed and open, respectively, such that $\dim(F \cap U) \le n$, the sixtuple F, U, X, $K_n(\tau)$, $\mathcal{O}$, $K_{n+1}(\tau)$ has property (S).

Thus the theorem holds for $\lambda = 0$.

Let λ be a limit ordinal. Since $D(F \cap U) \le \lambda + n$, there exists a closed covering $\{A_\beta : \beta \le \lambda\}$ of the space $F \cap U$ by finite-dimensional subsets satisfying (1.1)–(1.4). Let $W = U - A_\lambda$; then W is open and $D(F \cap W) <\le \lambda$. We are going to prove that

(4.2) the sixtuples F, W, X, $M'_\lambda(\tau)$, $\mathcal{O}$, $M'_\lambda(\tau)$ and F, W, X, $Z'_\lambda(\tau)$, $\mathcal{O}$, $Z'_\lambda(\tau)$ have property (S).

Case 1: λ is a prime component. Let $W_\gamma = W - \bigcup\{A_\beta : \gamma \le \beta < \lambda\}$ for $\gamma < \lambda$. By (1.1), the sets W_γ are open; obviously, $W = \bigcup\{W_\gamma : \gamma < \lambda\}$. If γ is a limit ordinal, then $D(F \cap W_\gamma) <\le \gamma$; otherwise it follows from (1.4) that $D(F \cap W_\gamma) < \gamma$. Hence, by the inductive assumption, the sixtuples F, W_γ, X, $M_\gamma(\tau)$, $\mathcal{O}$, $M_\gamma(\tau)$ and F, W_γ, X, $Z_\gamma(\tau)$, $\mathcal{O}$, $Z_\gamma(\tau)$ have property (S) for $\gamma < \lambda$. Thus, by Theorem 3.3, so do F, W, X, $M'_\lambda(\tau)$, $\mathcal{O}$, $M'_\lambda(\tau)$ and F, W, X, $Z'_\lambda(\tau)$, $\mathcal{O}$, $Z'_\lambda(\tau)$.

Case 2: $\lambda = \lambda_1 + \ldots + \lambda_k$, where $k > 1$. Let

$$W_i = W - \bigcup\{A_\beta : \lambda_1 + \ldots + \lambda_i \le \beta \le \lambda\}$$

for $i = 1, \ldots, k$. By (1.1), the sets W_i are open; obviously, $\emptyset = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_k = W$. Put $F_{i+1} = F - W_i$ for $i = 0, \ldots, k-1$; then

$$F_{i+1} \cap W_{i+1} = \Big(\bigcup\{A_\beta : \lambda_0 + \ldots + \lambda_i \le \beta \le \lambda\}\Big) - \Big(\bigcup\{A_\beta : \lambda_0 + \ldots + \lambda_{i+1} \le \beta \le \lambda\}\Big),$$

where $\lambda_0 = 0$, and one can easily check that $D(F_{i+1} \cap W_{i+1}) <\le \gamma_{i+1}$ for $i = 0, \ldots, k-1$. By the inductive assumption, the sixtuples F, W_i, X, $M_{\lambda_i}(\tau)$, $\mathcal{O}$, $M_{\lambda_i}(\tau)$ and F_i, W_i, X, $Z_{\lambda_i}(\tau)$, $\mathcal{O}$, $Z_{\lambda_i}(\tau)$ have property (S) for $i = 1, \ldots, k$. Hence, by Proposition 3.2, so do F, W, X, $M'_\lambda(\tau)$, $\mathcal{O}$, $M'_\lambda(\tau)$ and F, W, X, $Z'_\lambda(\tau)$, $\mathcal{O}$, $Z'_\lambda(\tau)$.

If $D(F \cap U) <\le \lambda$, then $A_\lambda = \emptyset$, and the proof of the first part of our theorem is complete.

If $D(F \cap U) \le \lambda + n$, then $\dim A_\lambda \le n$ by (1.3), and the second part of our theorem follows from (4.1) and (4.2).

4.3. Corollary. *Let $|\alpha| \le \tau$. If X is a metrizable space of weight τ such that $D(X) \le \alpha$, then X is homeomorphic to a subspace of $M_\alpha(\tau)$ and to a subspace of $Z_{\alpha+1}(\tau)$.*

Proof. It suffices to apply Theorem 4.2 to $F = U = X$.

Denote by τ^+ the smallest cardinal greater than τ.

4.4. Corollary. *For every cardinal τ, there exist strongly countable-dimensional spaces Z_τ and M_τ of weight τ^+, compact and completely metrizable, respectively, such that each strongly countable-dimensional completely metrizable space of weight τ is homeomorphic to a subspace of Z_τ and to a subspace of M_τ.*

Proof. For every $n \in \mathcal{N}$ the space $K_n(\tau)$ is completely metrizable as a G_δ subset of $[J(\tau)]^{\aleph_0}$; the space $(J_M(\mathcal{X},\tau),\varrho)$ is complete whenever the spaces $(X_\alpha,\varrho_\alpha)$ are complete for $\alpha<\lambda$ (see the remark following the definition of $J(\mathcal{X},\tau)$), and therefore, by Proposition 3.1, the space $J^\omega_M(\mathcal{X},\tau)$ is completely metrizable for every α such that $|\alpha|\le\tau$.

Since $D(X)<\Delta$ and $|D(X)|\le\tau$ for every strongly countable-dimensional completely metrizable space X of weight τ, it suffices to take the Alexandroff compactification of $\bigoplus\{Z_\alpha(\tau):|\alpha|\le\tau\}$ as Z_τ, and $\bigoplus\{M_\alpha(\tau):|\alpha|\le\tau\}$ as M_τ.

Note that Corollary 4.4 is obvious under GCH.

5. The D-dimension of $M_\alpha(\tau)$ and $Z_\alpha(\tau)$. In this section, we evaluate $D(M_\alpha(\tau))$ and $D(Z_\alpha(\tau))$ for every α such that $|\alpha|\le\tau$.

Let T be our set such that $|T|=\tau$. Fix an $l\ge1$ and a λ such that $|\lambda|\le\tau$. We denote by $J(\mathcal{X},\tau)_l$ the subspace of $J(\mathcal{X},\tau)^l$ consisting of all points $(x_1,\ldots,x_j,\ldots,x_l)$ satisfying (3.1).

5.1. Lemma. *Let λ be a prime component and $\mathcal{X}=\{(X_\alpha,x_\alpha):\alpha<\lambda\}$ a family of pointed metrizable or compact spaces of weight τ. If $D(X_\alpha)\le\alpha$ for every $\alpha<\lambda$, then $J(\mathcal{X},\tau)_l$ has the following property*:

(5.1) *there exists its closed covering $\{A_\beta:\beta\le\lambda\}$ by finite-dimensional subsets satisfying* (1.1), (1.2), (1.4) *and such that $A_\lambda=\{\mathcal{O}\}$.*

Proof. Since $J(\mathcal{X},\tau)=\bigcup\{X_{\alpha,\xi,t}:\alpha,\xi<\lambda,\ t\in T\}$ (see Section 3), by (3.1),

$$J(\mathcal{X},\tau)_l=\bigcup\{R^{\xi_1,\ldots,\xi_l}_{\alpha_1,\ldots,\alpha_l}:\alpha_1,\xi_1,\ldots,\alpha_l,\xi_l<\lambda\ \text{ and }\ \max_{i\ne j}\alpha_i\le\xi_j\ \text{for every } j=1,\ldots,l\},$$

where $R^{\xi_1,\ldots,\xi_l}_{\alpha_1,\ldots,\alpha_l}=\bigcup\{X_{\alpha_1,\xi_1,t_1}\times\ldots\times X_{\alpha_1,\xi_1,t_1}:t_1,\ldots,t_l\in T\}$.

Take a closed covering $\{A^\alpha_\beta:\beta\le\lambda(\alpha)\}$ of X_α by finite-dimensional subsets satisfying (1.1)–(1.4), where $\lambda(\alpha)$ is the largest limit ordinal less than or equal to α, and put $B^\alpha_\beta=A^\alpha_\beta$ for $\beta<\lambda(\alpha)$, $B^\alpha_\beta=\emptyset$ for $\lambda(\alpha)\le\beta<\alpha$, $B^\alpha_\alpha=A^\alpha_{\lambda(\alpha)}$.

Further, let

$$A^{\alpha,\xi,t}_\beta=B^\alpha_\beta\qquad\text{for } \alpha,\xi<\lambda,\ t\in T,\ \beta\le\alpha,$$

$$A_{\beta}^{\alpha,\xi} = \{\mathcal{O}\} \cup \bigcup\{A_{\beta}^{\alpha,\xi,t} : t \in T\} \quad \text{for } \alpha,\xi < \lambda,\ \beta \le \alpha;$$

and $A_{\beta}^{\alpha,\xi} = \emptyset$ for $\alpha,\xi < \lambda$, $\alpha < \beta \le \lambda$. Then the family $\{A_{\beta}^{\alpha,\xi} : \beta \le \lambda\}$ is a closed covering of the subspace $R_{\alpha,\xi} = \bigcup\{X_{\alpha,\xi,t} : t \in T\}$ of $J(\mathcal{X},\tau)$ by finite-dimensional subsets satisfying (1.1), (1.2), (1.4).

Define

$$A_{\beta} = \bigcup\{A_{\beta_1}^{\alpha_1,\xi_1} \times \ldots \times A_{\beta_l}^{\alpha_l,\xi_l} : \beta_1 \oplus \ldots \oplus \beta_l = \beta,\ \alpha_1,\xi_1,\ldots,\alpha_l,\xi_l < \lambda,\ \max_{i\ne j} \alpha_i \le \xi_j \text{ for every } j = 1,\ldots,l\}$$

for $\beta < \lambda$ and let $A_{\lambda} = \{\mathcal{O}\}$.

Obviously, A_{β} is a subset of $J(\mathcal{X},\tau)_l$ for every $\beta \le \lambda$. The $\beta = \lambda$ is the greatest number $\beta \le \lambda$ such that $\mathcal{O} \in A_{\beta}$. Let $\mathcal{O} \ne (x_1,\ldots,x_l) \in J(\mathcal{X},\tau)_l \subseteq [J(\mathcal{X},\tau)]^l$. Consider a $k = 1,\ldots,l$. If $x_k \ne \mathcal{O}$, then there are a unique $\alpha_k < \lambda$ and a unique $\xi_k < \lambda$ such that $x_k \in R_{\alpha_k,\xi_k}$, and a greatest $\beta_k \le \alpha_k$ such that $x_k \in A_{\beta_k}^{\alpha_k,\xi_k}$; since $(x_1,\ldots,x_l)$ satisfies (3.1), $\alpha_k \le \xi_i$ for each $k \ne i = 1,\ldots,l$ such that $x_i \ne \mathcal{O}$. If $x_k = \mathcal{O}$, then put $\alpha_k = \beta_k = \min\{\xi_i : i = 1,\ldots,l \text{ and } \mathcal{O} \ne x_i \in R_{\alpha_i,\xi_i}\}$, $\xi_k = \max\{\alpha_i : i = 1,\ldots,l\}$.

Since $\mathcal{O} \in A_{\beta}^{\alpha,\xi}$ for every $\alpha,\xi < \lambda$ and $\beta \le \alpha$, we have $(x_1,\ldots,x_l) \in A_{\beta_1}^{\alpha_1,\xi_1} \times \ldots \times A_{\beta_l}^{\alpha_l,\xi_l}$, and since $\max_{i\ne j} \alpha_i \le \xi_j$ for every $j = 1,\ldots,l$, it follows that $A_{\beta_1}^{\alpha_1,\xi_1} \times \ldots \times A_{\beta_l}^{\alpha_l,\xi_l} \subseteq A_{\beta}$ for $\beta = \beta_1 \oplus \ldots \oplus \beta_l$ (we have $\beta < \lambda$, because $\beta \le \alpha_i < \lambda$ for every $i = 1,\ldots,l$ and λ is a prime component).

It follows from the definition of A_{β}'s that the chosen β is the greatest β such that $(x_1,\ldots,x_l) \in A_{\beta}$. Thus, we have shown that $\{A_{\beta} : \beta \le \lambda\}$ is a covering of $J(\mathcal{X},\tau)_l$ and (1.2) is satisfied.

We shall now prove that A_{β} is closed in $[J(\mathcal{X},\tau)]^l$ for every $\beta < \lambda$ (for $\beta = \lambda$ this is obvious).

Since the equation $\beta_1 \oplus \ldots \oplus \beta_l = \beta$ has only finitely many solutions $\beta_1,\ldots,\beta_l$, it suffices to prove that the set

$$A_{\beta_1,\ldots,\beta_l} = \bigcup\{A_{\beta_1}^{\alpha_1,\xi_1} \times \ldots \times A_{\beta_l}^{\alpha_l,\xi_l} : \alpha_1,\xi_1,\ldots,\alpha_l,\xi_l < \lambda \text{ and } \max_{i\ne j} \alpha_i \le \xi_j \text{ for every } j = 1,\ldots,l\}$$

is closed. Take an $(x_1,\ldots,x_l) \notin A_{\beta_1,\ldots,\beta_l}$. If there exists an $i \in \{1,\ldots,l\}$ such that $x_i \notin A_{\beta_i}^{\alpha_i,\xi_i}$ for every $\alpha_i,\xi_i < \lambda$, then

$$U = \Big\{(y_1,\ldots,y_l) \in [J(\mathcal{X},\tau)]^l : y_i \notin \bigcup\{A_{\beta_i}^{\alpha_i,\xi_i} : \alpha_i,\xi_i < \lambda\}\Big\}$$

is a neighbourhood of $(x_1,\ldots,x_l)$ and $U \cap A_{\beta_1,\ldots,\beta_l} = \emptyset$.

Thus assume that for every $i = 1,\ldots,l$, $x_i \in A_{\beta_i}^{\alpha_i,\xi_i}$ for some $\alpha_i,\xi_i < \lambda$. Put $\alpha_i = \beta_i$ whenever $x_i = \mathcal{O}$, and if $x_i \ne \mathcal{O}$, then let α_i be the unique ordinal less than λ such that $x_i \in R_{\alpha_i,\xi_i}$ for some $\xi_i < \lambda$. Further, let ξ_i be the unique ordinal less than λ such that $x_i \in R_{\alpha_i,\xi_i}$ whenever $x_i \neq \mathcal{O}$, and let $\xi_i = \max\{\alpha_j : i \neq j = 1, \ldots, l\}$ whenever $x_i = \mathcal{O}$.

Then $x_i \in A_{\beta_i}^{\alpha_i,\xi_i}$ for $i = 1, \ldots, l$. Since $(x_1, \ldots, x_l) \notin A_{\beta_1,\ldots,\beta_l}$, we have $\xi_j < \alpha_i$ for some distinct $i, j = 1, \ldots, l$; this is possible only for j such that $x_j \neq \mathcal{O}$. If also $x_i \neq \mathcal{O}$, then

$$U = \{(y_1, \ldots, y_l) \in [J(\mathcal{X}, \tau)]^l : \mathcal{O} \neq y_i \in R_{\alpha_i,\xi_i},\ \mathcal{O} \neq y_j \in R_{\alpha_j,\xi_j}\}$$

is a neighbourhood of $(x_1, \ldots, x_l)$ and $U \cap A_{\beta_1,\ldots,\beta_l} \subseteq U \cap J(\mathcal{X}, \tau)_l = \emptyset$. If $x_i = \mathcal{O}$, then $\beta_i = \alpha_i > \xi_j$, so that

$$U = \{(y_1, \ldots, y_l) \in [J(\mathcal{X}, \tau)]^l : \mathcal{O} \neq y_j \in R_{\alpha_j,\xi_j}\}$$

is a neighbourhood of $(x_1, \ldots, x_l)$ satisfying $U \cap A_{\beta_1,\ldots,\beta_l} = \emptyset$.

Thus A_β is a closed subset of $[J(\mathcal{X}, \tau)]^l$ for every $\beta \leq \lambda$.

In order to prove (1.1), observe that

$$\begin{aligned}
&\bigcup\{A_\beta : \delta \leq \beta \leq \lambda\} \\
&= \bigcup\Big\{\bigcup\Big\{\Big[\bigcup\{A_{\beta_1}^{\alpha_1,\xi_1} : \delta_1 \leq \beta_1 \leq \lambda_1\}\Big] \times \ldots \times \Big[\bigcup\{A_{\beta_l}^{\alpha_l,\xi_l} : \delta_l \leq \beta_l \leq \lambda_l\}\Big] : \\
&\qquad \alpha_1, \xi_1, \ldots, \alpha_l, \xi_l < \lambda,\ \max_{i \neq j} \alpha_i < \xi_j \text{ for } j = 1, \ldots, l\Big\} : \delta_1 \oplus \ldots \oplus \delta_l = \delta\Big\},
\end{aligned}$$

because if $\beta = \beta_1 \oplus \ldots \oplus \beta_l \geq \delta$, then there are $\delta_1 \leq \beta_1, \ldots, \delta_l \leq \beta_l$ such that $\delta_1 \oplus \ldots \oplus \delta_l = \delta$.

Since $\bigcup\{A_{\beta_i}^{\alpha_i,\xi_i} : \delta_i \leq \beta_i \leq \alpha_i\}$ is a closed subset of $J(\mathcal{X}, \tau)$ for every $\alpha_i, \xi_i < \lambda$ and $\delta_i \leq \alpha_i$, by the above equality, the proof of the closedness of $\bigcup\{A_\beta : \delta \leq \beta \leq \lambda\}$ can be carried out exactly as the proof of the closedness of A_β.

Let

$$C_\beta = \bigcup\{A_{\beta_1}^{\alpha_1,\xi_1} \times \ldots \times A_{\beta_l}^{\alpha_l,\xi_l} : \alpha_1, \xi_1, \ldots, \alpha_l, \xi_l < \lambda,\ \beta = \beta_1 \oplus \ldots \oplus \beta_l\}.$$

It follows from Lemma 2.4 that $\dim C_\beta \leq m$ for every $\beta = \gamma + m < \lambda$, where γ is 0 or a limit ordinal. Since A_β is a closed subset of C_β, $\dim A_\beta \leq m$. Thus the family $\{A_\beta : \beta \leq \lambda\}$ has the property described in (1.4); in particular, the sets A_β are finite-dimensional.

5.2. Lemma. *Let λ be a prime component such that $|\lambda| \leq \tau$ and $\mathcal{X} = \{(X_\alpha, x_\alpha) : \alpha < \lambda\}$ a family of pointed metrizable or compact spaces of weight τ. If $D(X_\alpha) \leq \alpha$ for every $\alpha < \lambda$, then $J^\omega(\mathcal{X}, \tau)$ has property* (5.1).

Proof. For $m = 0, 1, \ldots$, let J_m be the subspace of $J^\omega(\chi, \tau)$ consisting of all points $(x_1, \ldots, x_j, \ldots)$ such that $\{j : x_j \neq \mathcal{O}\} \subseteq \{\omega(m)+1, \ldots, \omega(m+2)\}$. Obviously, J_m's are closed subsets of $J^\omega(\mathcal{X}, \tau)$ and $J^\omega(\mathcal{X}, \tau) = \bigcup\{J_m : m = 0, 1, \ldots\}$, and by Lemma 5.1, J_m's have property (5.1). Take a covering $\{A_\beta^m : \beta \leq \lambda\}$ of J_m satisfying (5.1) for $m = 0, 1, \ldots$ For $\beta = \gamma + m$, where

$\gamma < \lambda$ is 0 or a limit ordinal, put

$$A_\beta = A^0_{\gamma+m} \cup \ldots \cup A^m_{\gamma+0};$$

let $A_\lambda = \{\mathcal{O}\}$. Then $\{A_\beta : \beta \le \lambda\}$ is a covering of $J^\omega(\mathcal{X}, \tau)$ satisfying (5.1).

5.3. THEOREM. *For every α such that $|\alpha| \le \tau$, $D(M_\alpha(\tau)) \le \alpha$ and $D(Z_\alpha(\tau)) \le \alpha$.*

Proof. We apply induction on α and we simultaneously prove that for every λ,

(5.2) $Z'_\lambda(\tau)$ and $M'_\lambda(\tau)$ have property (5.1).

For all finite α, the theorem follows directly from the definition of $M_\alpha(\tau)$ and $Z_\alpha(\tau)$. Let α be infinite. If $\alpha = \lambda$ is a prime component, then (5.2) follows from Lemma 5.2 and the inductive assumption. If $\alpha = \lambda$ is a limit ordinal, but not a prime component, then (5.2) follows from Lemma 2.4 and the inductive assumption. For $\alpha = \lambda + n$, our theorem is an immediate consequence of (5.2) for λ.

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