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A characterization of representation-finite algebras

by

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Abstract. Let A be a finite-dimensional, basic, connected algebra over an algebraically closed field. Denote by $\Gamma(A)$ the Auslander–Reiten quiver of A. We show that A is representation-finite if and only if $\Gamma(A)$ has at most finitely many vertices lying on oriented cycles and finitely many orbits with respect to the action of the Auslander–Reiten translation.

Let K denote a fixed algebraically closed field and A a finite-dimensional K-algebra (associative, with an identity) which we shall assume to be basic and connected. By an A-*module* is meant a finite-dimensional right A-module. Throughout the paper we shall freely use the terminology and notation introduced in [7]. In particular, we denote by $\operatorname{mod} A$ the category of A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 0$, of $\operatorname{rad}(\operatorname{mod} A)$. From the existence of the Auslander-Reiten sequences in $\operatorname{mod} A$ we know that $\operatorname{rad}(\operatorname{mod} A)$ is generated by irreducible maps as a left and as a right ideal. By $\Gamma(A)$ we denote the Auslander–Reiten quiver of A whose vertices are the isoclasses of indecomposable objects in $\operatorname{mod} A$ and arrows correspond to irreducible maps, and by τ and τ^{-1} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. For the sake of simplicity we identify an A-module with its isomorphism class. The τ-*orbit* of an indecomposable A-module X is the family of non-zero modules of the form $\tau^n X$, $n \in \mathcal{Z}$, where $\mathcal{Z}$ is the set of all integers. An A-module X is called *periodic* if $\tau^n X \simeq X$ for some $n \neq 0$. By a *path* from M to N in $\Gamma(A)$ we mean a sequence of vertices and arrows $M \to M_1 \to \ldots \to M_n \to N$ in $\Gamma(A)$. In this case, M is called a *predecessor* of N and N a *successor* of M. An *oriented cycle* is a non-trivial path from a point to itself. Recall that an algebra A is called *representation-finite* if $\Gamma(A)$ is finite. In [4] the following results were proved:

(a) An algebra A is representation-finite if and only if $\operatorname{rad}^{\infty}(\operatorname{mod} A) = 0$.

(b) If $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ is nilpotent, then A is tame (in the sense of [1] and [8]).

(c) If A is either a tilted algebra or a standard selfinjective algebra then $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ is nilpotent if and only if A is domestic (in the sense of [8]).

Representation-finite algebras are domestic and domestic algebras are tame (see [8]).

We shall prove here the following characterization of representation-finite algebras.

THEOREM. *Let A be an algebra. The following conditions are equivalent.*

(i) *A is representation-finite.*

(ii) *$\Gamma(A)$ admits at most finitely many vertices lying on an oriented cycle and the number of τ-orbits in $\Gamma(A)$ is finite.*

(iii) $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ *is nilpotent and the number of τ-orbits in $\Gamma(A)$ is finite.*

Proof. Obviously (i) implies (ii). Moreover, (iii) implies (i). Indeed, suppose that A is representation-infinite and satisfies (iii). Then the nilpotency of $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ implies that A is tame and then, by the validity of the Brauer–Thrall II conjecture (for a proof, see for example [2]) and [1, 6.7], there are infinitely many pairwise non-isomorphic indecomposable A-modules X with $X \simeq \tau X$, impossible by the second part of (iii).

Therefore, in order to prove the theorem, it is enough to show that (ii) implies the nilpotency of $\mathrm{rad}^\infty(\mathrm{mod}\, A)$.

Observe first that, if $M_0 \to M_1 \to \ldots \to M_n \to M_0$ is an oriented cycle in $\Gamma(A)$, then either all modules M_i are periodic or $\tau^m M_i = 0$ (resp. $\tau^{-m} M_i = 0$) for some $m > 0$ and some i. Indeed, if one of the modules M_i, say M_0, is not periodic and $\tau^m M_i \neq 0$ for all $m \geq 0$ (resp. $m \leq 0$) and all i, $0 \leq i \leq n$, then the modules $\tau^m M_0$, $m \geq 0$ (resp. $m \leq 0$) are pairwise non-isomorphic and lie on oriented cycles $\tau^m M_0 \to \tau^m M_1 \to \ldots \to \tau^m M_n \to \tau^m M_0$, a contradiction to (ii).

Denote by $\Gamma^+(A)$ (resp. $\Gamma^-(A)$) the full translation subquiver of $\Gamma(A)$ formed by all non-periodic indecomposable modules X such that $\tau^n X \neq 0$ (resp. $\tau^{-n} X \neq 0$) for all $n \geq 0$. By the above remark, $\Gamma^+(A)$ and $\Gamma^-(A)$ are quivers without oriented cycles. Then there exist finite sets $\mathcal{X}$ and $\mathcal{Y}$ of indecomposable A-modules such that the following conditions are satisfied:

(1) $\mathcal{X}$ (resp. $\mathcal{Y}$) intersects every τ-orbit in $\Gamma^+(A)$ (resp. $\Gamma^-(A)$).

(2) Every path in $\Gamma(A)$ with source and target in $\mathcal{X}$ (resp. $\mathcal{Y}$) has all vertices in $\mathcal{X}$ (resp. $\mathcal{Y}$).

(3) There is no oriented cycle in $\Gamma(A)$ consisting of modules from $\mathcal{X}$ (resp. $\mathcal{Y}$).

(4) Every predecessor (resp. successor) of some module of $\mathcal{X}$ (resp. $\mathcal{Y}$) belongs to $\Gamma^+(A)$ (resp. $\Gamma^-(A)$).

Denote by $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) the full translation subquiver of $\Gamma(A)$ formed by all proper predecessors (resp. proper successors) of modules of $\mathcal{X}$ (resp. $\mathcal{Y}$). We may assume that $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint. Observe that $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) is a disjoint union of translation quivers of the form $(-N)\Delta$ (resp. $N\Delta$) for some quiver Δ without oriented cycles. Let $\mathcal{D}$ be the family of all indecomposable A-modules which are neither in $\mathcal{C}^+$ nor in $\mathcal{C}^-$. Then $\mathcal{D}$ is (up to isomorphism) finite; denote by d the maximum of dimensions of modules from $\mathcal{D}$.

Now let M and N be two indecomposable A-modules and $f : M \to N$ a non-zero map in $\operatorname{rad}^\infty(\operatorname{mod} A)$. Assume that M does not belong to $\mathcal{C}^-$. We claim that f factors through a direct sum of modules of $\mathcal{Y}$. Since $f \in \operatorname{rad}^\infty(M, N)$, there exists, for each $t > 0$, a chain

$$M \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_2 \to \ldots \to M_{t-1} \xrightarrow{g_t} M_t$$

of irreducible maps in mod A and a morphism $h_t \in \operatorname{rad}^\infty(M_t, N)$ such that $f = h_t g_t \ldots g_1$. Then there exists $p \geq 0$ such that, for $t \geq p$, M_t does not contain direct summands from $\mathcal{C}^+$. Applying the lemma of Harada and Sai [3] (for a proof we refer to [6, 2.2]) we conclude that, for $t \geq p + 2^d$, M_t is a direct sum of modules of $\mathcal{C}^-$. Observe that, for $s = p+2^d$, $g_s \ldots g_1$ is a linear combination of paths in $\Gamma(A)$ from M to indecomposable direct summands of M_s, hence lying in $\mathcal{C}^-$, which must factor through modules of $\mathcal{Y}$. Similarly, we show that, if N is not in $\mathcal{C}^+$, then f factors through a direct sum of modules of $\mathcal{X}$. Let now $m = 2^d+1$. We shall show that $(\operatorname{rad}^\infty(\operatorname{mod} A))^{2m} = 0$. It is enough to show that for each chain of morphisms

$$Z_0 \xrightarrow{u_1} Z_1 \xrightarrow{u_2} Z_2 \to \ldots \to Z_{2m-1} \xrightarrow{u_{2m}} Z_{2m}\,,$$

where all Z_i are indecomposable A-modules and where the u_i belong to $\operatorname{rad}^\infty(\operatorname{mod} A)$, the composition $u = u_{2m} \ldots u_1$ is zero. Since we assume that $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, it follows that either u_i factors through a direct sum of modules of $\mathcal{X}$ or u_{i+1} factors through a direct sum of modules of $\mathcal{Y}$. Consequently, for each j, $1 \leq j \leq m$, we have $u_{2j}u_{2j-1} = \beta_j\alpha_j$, where $\alpha_j \in \operatorname{rad}(Z_{2j-2}, V_j)$, $\beta_j \in \operatorname{rad}(V_j, Z_{2j})$ and V_j is a direct sum of indecomposable modules of $\mathcal{X} \cup \mathcal{Y}$. Let $\gamma_j = \alpha_{j+1}\beta_j$ for $j = 1, \ldots, m-1$. Applying again the lemma of Harada and Sai, we conclude that $\gamma_{m-1} \ldots \gamma_1 = 0$. Hence $u = \beta_m\gamma_{m-1} \ldots \gamma_1\alpha_1 = 0$, which finishes the proof of the theorem.

The following corollary is an immediate consequence of the theorem.

Corollary 1. *Let A be an algebra such that $\Gamma(A)$ has no oriented cycles. Then A is representation-finite if and only if the number of τ-orbits in $\Gamma(A)$ is finite.*

Observe that wild hereditary algebras are representation-infinite and their Auslander–Reiten quiver does not contain oriented cycles (see [5]). Recall also that an algebra A is called *representation-directed* provided every indecomposable A-module is *directing*, that is, it does not belong to an oriented cycle $M_0 \to M_1 \to \ldots \to M_n \to M_0$ of non-zero non-isomorphisms between indecomposable A-modules M_i. It was recently shown in [9] that a connected Auslander–Reiten component consisting of directing modules has only finitely many τ-orbits and the number of such components is finite. Hence the above corollary also implies the following characterization of representation-directed algebras due to Ringel (see [7, 2.4]).

COROLLARY 2. *An algebra A is representation-directed if and only if A is representation-finite and $\Gamma(A)$ has no oriented cycles.*

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