aries. Let T denote the collection of all continua t_y for each continuum y of the collection Y. It can easily be shown that T is an upper semi-continuous collection of mutually exclusive continua. Let Z denote the collection of all continua of the collection T, and all points outside the simple closed curve J. There exists [1]) a continuous one-to-one correspondence between the continua of Z and the points of a Euclidean plane. It readily follows that the subset of the plane which corresponds to J plus its interior is a simple closed curve plus its interior. If G is any simple closed curve plus its interior there exists a continuous transformation of the plane into itself which maps J plus its interior on G. Clearly the collection of continua corresponding to T under such a transformation satisfies the conclusion of theorem II.

c) From the viewpoint of analysis situs a hemisphere is equivalent to a simple closed curve plus its interior. Hence the truth of the last part of theorem II readily follows from the truth of the second part.

[1]) See R. L. Moore, loc. cit.

University of Texas, Nov. 2, 1928.

Continuous curves and arc-sums [1]).

By

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Menger [2]) has suggested the problem of characterizing a continuous curve which is the sum of a countable number of simple continuous arcs. In this paper two theorems will be proved along the line of this problem. The first of these reduces the problem for a continuous curve M in general to the same problem concerning the maximal cyclic curves of M; and together these two theorems give a considerable amount of new information, and at the same time yield as corollaries most of the known results connected with this problem.

By a *continuous curve* is meant any connected im kleinen continuum. A continuous curve C is *cyclicly connected* [3]) if and only if every two of its points lie together on some simple closed curve in C. A *maximal cyclic curve* [3]) of a continuous curve M is a subcontinuous curve of M which is saturated with respect to the property of being cyclicly connected. The theorems below hold true in any locally compact, metric, and separable space.

[1]) Presented to the American Mathematical Society, December 28, 1928.

[2]) K. Menger, *Über reguläre Baumkurven*, Math. Ann., vol. 96 (1926), pp. 572—582, see footnote to p. 578. Menger states the problem only for *regular* curves, a special type of continuous curve.

[3]) Cf. my papers *Cyclicly connected continuous curves*, Proc. Ntl. Acad. of Sci., vol. 13 (1927), pp. 31—38, and *Concerning the structure of a continuous curve*, Amer. Journal Math., vol. 50 (1928), pp. 167—194. Extensions of most of the results in the former paper to n dimensions have been made by W. L. Ayres; cf. his forthcoming paper *Concerning continuous curves in space of n dimensions.*

Theorem 1. *In order that a continuous curve M should be the sum of a countable number of arcs it is necessary and sufficient that* (1) *the end points*[1]) *of M be countable and* (2) *each maximal cyclic curve of M be the sum of a countable number of arcs.*

Proof. The condition is sufficient. Let $C_1, C_2, C_3, \ldots$ be the maximal cyclic curves of M. By hypothesis, for each i, C_i is the sum of a countable number of arcs $\sum\limits_n T_{ni}$ of M. Hence $N = C_1 + C_2 + + C_3 + \ldots = \sum\limits_i \sum\limits_n T_{ni}$ is the sum of a countable number of arcs in M. Let H be the set of all end points of M. Since H is countable, the set G of all pairs (A, B) of points of H is countable. And if for each pair (A, B) in G, T_{ab} denotes an arc in M from A to B, then $H \subset \sum\limits_{(A,B) \text{ in } G} T_{ab}$. Hence H is a subset of the sum of a countable number of arcs in M; and if K denotes the set of all cut points of M, by a theorem of the author's [2]), K is a subset of the sum $\sum t_i$ of a countable number of arcs t_i in M. But now [3]) $K + H + N = M$. Hence $M = \sum t_i + \sum\limits_{(A,B) \text{ in } G} T_{ab} + \sum\limits_i \sum\limits_n T_{ni}$, and therefore M is the sum of a countable number of arcs.

The condition is also necessary. For suppose $M = \sum A_i B_i$, where $A_i B_i$ is an arc in M from A_i to B_i. If H denotes the set of all end points of M, then clearly $H \subset \sum (A_i + B_i)$. Hence H is countable[4]). Let C be any maximal cyclic curve of M. Now [5]) for each i, $C \cdot A_i B_i$ is either connected or vacuous, and therefore is either vacuous, an arc, or a point. For every i such that $C \cdot A_i B_i$ is a point, let T_i be an arc in C having $C \cdot A_i B_i$ as one end point; for every other i, let $T_i = 0$. Then for each i, $C \cdot A_i B_i + T_i$ is either vacuous or an arc in C, and $C = \sum (C \cdot A_i B_i + T_i)$. Thus C is the sum of a countable number of arcs.

[1]) An end point of a continuous curve M is a point of M not interior to any arc in M; or, what is equivalent (cf. H. M. Gehman, Trans. Amer. Math. Soc., vol. 30 (1928), pp. 63—84), it is a point of Menger order 1 of M (cf. Menger loc. cit.).

[2]) *Concerning the cut points of continua*, Trans. Amer. Math. Soc., vol. 30 (1928), pp. 597—609, Theorem 14.

[3]) Cf. *Cyclicly connected continuous curves*, loc. cit., Theorems 6 and 7.

[4]) Menger (loc. cit.) has already pointed out that the countability of H is necessary in order that M be the sum of a countable number of arcs.

[5]) Cf. *Concerning the structure of a continuous curve*, loc. cit., Theorem 30.

Corollary 1 a. *If each maximal cyclic curve of a continuous curve M contains only a finite number of simple closed curves (or is a Baum im kleinen), then M is the sum of a countable number of arcs if and only if the end points of M are countable.*

Corollary 1a follows at once from Theorem 1 with the aid of Menger's theorem (loc. cit.) that a Baum im kleinen curve having only a countable number of end points is the sum of a countable number of arcs.

Corollary 1 b. *In order that the boundary M of a complementary domain of a plane continuous curve M should be the sum of a countable number of arcs it is necessary and sufficient that the end points of M should be countable.*

Corollary 1 b is an immediate consequence of Theorem 1 and the fact that each maximal cyclic curve of the boundary of a plane continuous curve is a simple closed curve.

Theorem 2. *The set K of all the im kleinen cut points*[1]) *of any continuous curve M is a subset of the sum of a countable number of arcs of M.*

Proof. By a theorem of the author's M contains a countable collection $[M_i]$ of continuous curves such that each point of K is a cut point of some curve of this collection. By another theorem of the author's [2]), for each i, the set K_i of all the cut points of M_i is a subset of the sum $\sum\limits_n T_{ni}$ of a countable number of arcs T_{ni} of M_i. Thus $K \subset \sum\limits_i \sum\limits_n T_{ni}$, and our theorem is proved.

Corollary 2 a. *If every point of a continuous curve M is an im kleinen cut point, or if all save possible a countable number of points of M are im kleinen cut points, then M is the sum of a countable number of arcs.*

Corollary 2 b. *If every point of each maximal cyclic curve C of a continuous curve M is an im kleinen cut point of C, or if all save a countable number of points of C are im kleinen cut points, then M*

[1]) An im kleinen cut point of a continuous curve M is any point of M which is a cut point of some connected open subset of M. For references and theorems concerning this notion see my paper *On points of continuous curves defined by certain im kleinen properties*, offered to Mathematische Annalen.

[2]) This theorem was proved incidentally in proving the theorem (cf. Theorem 8 in the paper just referred to above) that the set K is an F_σ.

[3]) *Concerning the cut points of continua*, loc. cit.

is the sum of a countable number of arcs if and only if the end points of M are countable.

Remarks. The theorem (Cf. Menger, loc. cit.) that an acyclic continuous curve (Baumkurve) M is the sum of a countable number of arcs if and only if its end points are countable is a corollary to Theorem 1, because, obviously, no such curve can have any maximal cyclic curves. The theorem that a Baum im kleinen curve M is the sum of a countable number of arcs if and only if its end points are countable is a corollary to Theorem 2, because *every point of a Baum im kleinen curve is either an end point or an im kleinen cut point.*

University of Texas, Jan. 20, 1929.

Sur les points accessibles des continus indécomposables.

Par

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1. M. Kuratowski a posé le problème suivant

Existe-t-il dans tout continu indécomposable plan un ensemble $\mathfrak{P}$, qui ne contient aucun point accessible.

Un continu est *indécomposable*, s'il n'est pas la somme de deux continus différant de lui. Un ensemble est *un ensemble* $\mathfrak{P}$ d'un continu indécomposable C s'il est 1) sous-ensemble vrai de C, 2) semi-continu, 3) saturé par rapport aux propriétés 1) et 2)[1]. Un point z d'un ensemble plan A est *accessible*, s'il existe un arc simple J à extremité z tel que $A \times J = z$.

Je vais résoudre le problème cité par l'affirmative dans le cas du continu borné. Je vais démontrer à cet effet le théorème suivant:

Théorème. *C étant un continu indécomposable, plan et borné — l'ensemble somme des ensembles $\mathfrak{P}$ qui contiennent un point accessible est de première catégorie par rapport à C.*

2. Lemme. *Soit A un continu irréductible entre les points a_1, a_2, d'un espace métrique et compact, B, D deux sous-ensembles fermés de A, tels que:*

$$(2,1) \qquad a_1 + a_2 \subset D \qquad B - D \neq 0 \qquad B + D = A$$

[1]) Le terme „ensemble $\mathfrak{P}$" ($\mathfrak{P}$-Menge) a été introduit par M. Kuratowski dans une publication récente (*Math. Ann.* 98, p. 399) pour remplacer le terme „composant" (utilisé par Janiszewski, Knaster, Kuratowski, Mazurkiewicz, *Fund. Math.* passim); ce dernier terme pouvant donner lieu à des confusions avec les „Komponenten" dans le sens de M. Hausdorff.

Cf. le terme „nerve" de M. L. E. J. Brouwer, Proceed. Amsterdam 14. (1911), p. 144.