On a problem of C. Kuratowski concerning upper semi-continuous collections.

By

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In a letter to R. L. Moore dated April 9, 1927, Dr. C. Kuratowski[1]) raised the question as to whether or not there exists an upper semi-continuous collection[2]) X of mutually exclusive continua no one of which is a point such that (1) the sum of the continua of the collection X fills a square plus its interior and (2) if each continuum of the collection X is regarded as a point the space so obtained is in continuous one-to-one correspondence with a square plus its interior. The object of this paper is to answer this question in the affirmative.

I will say that two plane point sets G_1 and G_2 are *equivalent* if and only if there exists a continuous transformation of the planes into itself which throws G_1 into G_2. If M is a point set, $\overline{M}$ will denote the sum of the points of M and the limit points of M. By a *component* of a point set G is meant a maximal connected subset of G.

Let M denote the set of points (x, y) for which $0 < x < 1$ and

[1]) See also C. Kuratowski, Fund. Math. XI, p. 183, footnote [4]). This problem may be stated in the following form: define a continuous function $y = f(x)$ which transforms a square into a square and is such that, for each y_0, the set of all x's such that $f(x) = y_0$ is a continuum (not reducing to a single point).

[2]) Cf. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. Vol. 27 (1925). Let G be a collection of bounded continua and suppose that if g is an element of G and $g_1, g_2, g_3, \ldots$ is a sequence of elements containing points $P_1, P_2, P_3, \ldots$ respectively, such that the sequence $P_1, P_2, P_3, \ldots$ has a sequential limit point in g, then it follows that *every* sequence of points $A_1, A_2, A_3, \ldots$, where for each n A_n belongs to g_n, has a sequential limit point in g. Then the collection G is said to be *upper semi-continuous.*

$y = \sin 1/x(1-x)$. Clearly $\overline{M}$ is a continuum. Let G be a non-dense perfect set on the interval $0 \leqq x \leqq 1$ of the x-axis containing the points $(0, 0)$ and $(1, 0)$ and with complementary segments $s_1, s_2, \ldots$, with respect to the interval $0 \leqq x \leqq 1$. For each point P of G which is not an end point of any complementary segment of G let V_P denote the vertical interval of length 2 with P as center, and let H be the collection of all such intervals. Let M_n be a point set equivalent to M, and whose limit sets are the vertical intervals 2 units in length which have the end points of the segment s_n as mid points, and such that (1) no two points of M_n have the same abscissa, and (2) if $P(x, y)$ is a point of M_n, then $|y| \leqq 1$. Clearly the sum of all the sets $\overline{M}_1, \overline{M}_2, \ldots$, plus all the intervals of H is a continuum which I shall call K. By definition K is the sum of a collection α_K of mutually exclusive continua, the elements of α_K being the intervals of H and the continua of the sequence $\overline{M}_1, \overline{M}_2, \ldots$ The collection α_K is upper semi-continuous and is an arc with respect to its elements. For each continuum N equivalent to K there exists a continuous transformation T_N of S into itself such that $T_N(K) = N$. Let α_N denote the collection of all point sets $T_N(g)$ where g is a continuum of the set α_K.

The truth of the following lemma may be easily established.

Lemma: If J is a simple closed curve $AXBCYDA$ such that the arcs AXB and CYD of J are of diameter greater than 1, then there exists a continuum N equivalent to K, containing AXB and CYD and lying wholly within or on J, and such that the arcs AXB and CYD correspond, under the transformation T_N, to the end elements of α_K, and every element of α_N is of diameter greater than 1.

Theorem I. *If k is any positive number, there exists an upper semi-continuous collection of continua filling the plane, all bounded, all of diameter greater than k, and no one separating the plane.*

Proof: Let γ_i $(i = 1, 2)$ be an arc of diameter greater than 1 which is a subset of M_i. Let T denote a transformation of the plane into itself which translates every point vertically upward through a distance of three units. Let K_1 be the image of K and let β_i $(i = 1, 2)$ be the image of γ_i under the translation T. Let J_i denote the simple closed curve composed of the arcs γ_i and β_i and the two vertical intervals whose end points are the end points of γ_i and β_i. Let N_i denote a continuum equivalent to K such that the end elements of N_i are γ_i and β_i, and such that each element of α_{N_i} is of diameter greater than 1, and all points of N_i except the points on γ_i and β_i are within J_i. Let H_1 denote the sum of the continua $\overline{M}_1$ and $\overline{M}_2$ and all elements of α_K between $\overline{M}_1$ and $\overline{M}_2$; let H_2 be the image of H_1 under the translation T. Let $V_i\,(i=1,2)$ be the sum of the continuum N_i and the elements of α_K and α_{K_1} which contain γ_i and β_i. Let R denote the bounded complementary domain of the continuum $V_1+V_2+H_1+H_2$.

Suppose we have defined a collection of continua $H_1, H_2, \dots, H_n$, $V_1, V_2, \dots, V_n$ which has the following properties:

Property 1. For each k $(2<k\leqq n)$ the continuum H_k (V_k) is a subset of the point set $R+V_1+V_2$ $(R+H_1+H_2)$.

Property 2. For each i not greater than n the continuum H_i (V_i) is the sum of the elements of an upper semi-continuous collection F_{H_i} (F_{V_i}) such that (1) each element of F_{H_i} (F_{V_i}) is of diameter greater than 1 and is either a simple continuous arc or a continuum equivalent to $\overline{M}$, (2) F_{H_i} (F_{V_i}) is a simple continuous arc with respect to its elements, and (3) the end elements of F_{H_i} (F_{V_i}) are elements of F_{V_1} and F_{V_2} $(F_{H_1}$ and $F_{H_2})$.

Property 3. For each pair of values of i and j $(i\leqq n,\ j\leqq n)$ the common part of H_i and V_j is an element of F_{H_i} and an element of F_{V_j}. If $i\neq j$ neither H_i and H_j nor V_i and V_j have a point in common.

Property 4. a) For each bounded complementary domain D of the continuum X_{k-1} $(X_{k-1}=V_1+V_2+\sum_1^{k-1}H_i,\ k>2)$ there exist two positive integers i_D and j_D $(i_D<j_D<k)$ such that the boundary of D is a subset of $V_1+V_2+H_{i_D}+H_{j_D}$. If $k\leqq n$ and D_{k-1} and D are complementary domains of X_{k-1} such that $\overline{D}_{k-1}$ contains H_k, then $i_{D_{k-1}}+j_{D_{k-1}}\leqq i_D+j_D$, and each point P of D_{k-1} at a distance greater than $1/(k-1)$ from every point of $H_{i_{D_{k-1}}}$ is separated from this continuum in D_{k-1} by the continuum H_k.

b) For each bounded complementary domain D of the continuum Y_{k-1} $(Y_{k-1}=H_1+H_2+\sum_1^{k-1}V_i,\ k>2)$ there exist two positive integers i_D and j_D $(i_D<j_D<k)$ such that the boundary of D is a subset of the point set $H_1+H_2+V_{i_D}+V_{j_D}$. If $k\leqq n$ and D_{k-1} and D are complementary domains of Y_{k-1} such that $\overline{D}_{k-1}$ contains V_k, then $i_{D_{k-1}}+j_{D_{k-1}}\leqq i_D+j_D$, and each point P of D_{k-1} at a distance greater than $1/(k-1)$ from every point of $V_{i_{D_{k-1}}}$ is separated from this continuum in the domain D_{k-1} by the continuum V_k.

Property 5. For every i not greater than n every component of $H_i-H_i\sum_1^n V_k$, and every component of $V_i-V_i\sum_1^n H_k$ is equivalent to $H_1-(\overline{M}_1+\overline{M}_2)$.

Let D_n denote a bounded complementary domain of X_n $(X_n=V_1+V_2+\sum_1^n H_i)$ such that if D is any other bounded complementary domain of X_n then $i_{D_n}+j_{D_n}\leqq i_D+j_D$. Let $e=i_{D_n}$. There exists[1]) a simple closed curve J_n enclosing H_e but not containing or enclosing any point of any other continuum H_j for $j\leqq n$, and furthermore such that every point within J_n is at a distance less than $1/n$ from some point of H_e. For every t $(t\leqq n)$ the simple continuous arc of elements F_{V_t} contains an element $\overline{M}_{tn}$ such that (1) $\overline{M}_{tn}$ is equivalent to $\overline{M}$ and (2) if Q_{tn} denotes the element common to F_{V_t} and F_{H_e}, then the continuum $\overline{M}_{tn}$ and all elements of F_{V_t} between $\overline{M}_{tn}$ and Q_{tn} belong to D_n and to the interior of J_n. The continuum $\overline{M}_{tn}$ contains an arc γ_{tn} of diameter greater than 1 which, under a continuous transformation of $\overline{M}_{tn}$ into $\overline{M}$ goes into a subset of M. Let S be the set of $n-1$ components of $D_n-D_n\sum_1^n V_k$. For each domain G of the set S there exist just two integers r_G and s_G $(r_G<s_G\leqq n)$ such that the arcs $\gamma_{r_G n}$ and $\gamma_{s_G n}$ are on the boundary of G. Clearly there exist in $\overline{G}$ and within J_n two mutually exclusive arcs which together with $\gamma_{r_G n}$ and $\gamma_{s_G n}$ form a simple closed curve J_G lying, except for the arcs $\gamma_{r_G n}$ and $\gamma_{s_G n}$ wholly in G. Let N_G denote a continuum equivalent to K, such that every element of α_{N_G} except $\gamma_{r_G n}$ and $\gamma_{s_G n}$ is a point set of diameter greater than 1 lying wholly within J_G. Let H_{n+1} be the sum of all the continua N_G for each domain G of the set S, plus the point set $(\overline{M}_{1n}+\overline{M}_{2n}+\dots+\overline{M}_{nn})$.

[1]) See R. L. Moore, *Concerning the separation of point sets by curves*, Proc. Nat. Ac. Sc. Vol. 11 (1925) p. 469, theorem 1.

In an analogous manner I can define V_{n+1}. Then the collection of continua $H_1, H_2, \ldots, H_n, H_{n+1}, V_1, V_2, \ldots, V_n, V_{n+1}$ has properties one to five.

Now the collection H_1, H_2, V_1, V_2 has properties one to five. Hence I have shown the existence of an infinite collection of continua, $H_1, H_2, \ldots; V_1, V_2, \ldots$ such that for every positive integer n ($n \geqq 2$) the subcollection $H_1, H_2, \ldots, H_n, V_1, V_2, \ldots, V_n$ has properties one to five. From property 4 it readily follows that if P is any point of R not belonging to the continuum H_n (V_n) then there exists an integer k (t) such that the continuum H_k (V_t) separates P from H_n (V_n) in the domain R.

Let P denote any point whatsoever of R. Consider every continuum S which does not contain P and which is for some n a subset of the continuum $\sum_{i=1}^{n}(H_i + V_i)$. For each such continuum S let G_{SP} denote the complementary domain of S which contains P. Let T_P denote the common part of all the domains G_{SP} for every continuum S. It is easily seen that T_P is the common part of a countable infinity of domains $G_1, G_2, \ldots$ of which G_n contains $\bar{G}_{n+1}$. Hence T_P is a continuum. Since for every n the boundary of G_n contains a subset of diameter greater than 1 the domain G_n itself is of diameter greater than 1. Hence the continuum T_P is of diameter greater than or equal to 1.

Obviously if P and Q are points, either T_P and T_Q are identical or they have no point in common. Let X denote the collection of all continua T_P for every point P of R. Let h denote a continuum of the collection X containing the point A, and $h_1, h_2, \ldots$ a sequence of such continua, containing the points $A_1, A_2, \ldots$ respectively, such that A is the sequential limit point of the sequence $A_1, A_2, \ldots$ Let g be any continuum of the collection X except h. Since h and g are distinct, by definition of h there exists a continuum S which does not contain any point of h, and such that complementary domain of S which does contain h does not contain k. However this domain must contain all but a finite number of the continua $h_1, h_2, \ldots$ Therefore the continuum g cannot contain a limit point of any sequence of points $Q_1, Q_2, \ldots$ where Q_n belongs to h_n. Hence the collection X is upper semi-continuous.

I have therefore shown the existence of an upper semi-continuous collection of continua filling up the domain R, each continuum of this collection being of diameter greater than or equal to 1, and no one of them separating the plane. Now if k is any positive number there exists a continuous one-to-one correspondence between the points of the domain R and the whole plane which is such that if x and y are two points of R, the distances between their images is greater than k times their distance in the domain R. Obviously the image of a continuum in R is a continuum in the plane, and the collection corresponding to X satisfies the conclusion of theorem I.

Theorem II. *If G denotes a) the Euclidean plane, b) a simple closed curve plus its interior, or (3) a sphere, there exists, an upper semi-continuous collection T of mutually exclusive continua such that (1) each point of G belongs to some continuum of the collection T, (2) there exists a positive number k such that each continuum of the collection T is of diameter greater than k, and (3) if the continua of the collection T are regarded as points the space so obtained is in continuous one-to-one correspondence with the point set G.*

Proof: a) If G is a plane, let T be any collection of continua satisfying the conclusion of theorem I. Then [1]) the collection T satisfies the conclusion of theorem II.

b) Consider the domain R of theorem I. If P is a point of R let I_P denote the set of all positive integers i for which $(H_i + V_i)$ does not contain P, and let T_P denote that component of $\bar{R} - \sum_{I_P}(H_i + V_i)$ which contains P. Clearly T_P is a continuum not separating the plane and of diameter not less than 1. By a method similar to that given in the proof of theorem I it can be shown that the collection Y of all continua T_P for each point P of $\bar{R}$ is an upper semi-continuous collection of mutually exclusive continua. There exists a simple closed curve J such that (1) J incloses the domain R, and (2) if D is a bounded complementary domain of $J + \bar{R}$, there exists a continuum y_D of the collection Y such that the boundary of D contains no point of $\bar{R} - y_D$. For each continuum y of Y let t_y be the continuum $y + (\bar{D}_{1y} + \bar{D}_{2y} + \ldots)$ where the sequence $D_{1y}, D_{2y}, \ldots$ consists of those bounded complementary domains of $J + \bar{R}$ which have a point of y on their bound-

[1]) See R. L. Moore, *Concerning upper semi-continuous collections of continua*, pp. 416—428.

aries. Let T denote the collection of all continua t_y for each continuum y of the collection Y. It can easily be shown that T is an upper semi-continuous collection of mutually exclusive continua. Let Z denote the collection of all continua of the collection T, and all points outside the simple closed curve J. There exists [1]) a continuous one-to-one correspondence between the continua of Z and the points of a Euclidean plane. It readily follows that the subset of the plane which corresponds to J plus its interior is a simple closed curve plus its interior. If G is any simple closed curve plus its interior there exists a continuous transformation of the plane into itself which maps J plus its interior on G. Clearly the collection of continua corresponding to T under such a transformation satisfies the conclusion of theorem II.

c) From the viewpoint of analysis situs a hemisphere is equivalent to a simple closed curve plus its interior. Hence the truth of the last part of theorem II readily follows from the truth of the second part.

[1]) See R. L. Moore, loc. cit.

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Continuous curves and arc-sums [1]).

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Menger [2]) has suggested the problem of characterizing a continuous curve which is the sum of a countable number of simple continuous arcs. In this paper two theorems will be proved along the line of this problem. The first of these reduces the problem for a continuous curve M in general to the same problem concerning the maximal cyclic curves of M; and together these two theorems give a considerable amount of new information, and at the same time yield as corollaries most of the known results connected with this problem.

By a *continuous curve* is meant any connected im kleinen continuum. A continuous curve C is *cyclicly connected* [3]) if and only if every two of its points lie together on some simple closed curve in C. A *maximal cyclic curve* [3]) of a continuous curve M is a subcontinuous curve of M which is saturated with respect to the property of being cyclicly connected. The theorems below hold true in any locally compact, metric, and separable space.

[1]) Presented to the American Mathematical Society, December 28, 1928.

[2]) K. Menger, *Über reguläre Baumkurven*, Math. Ann., vol. 96 (1926), pp. 572—582, see footnote to p. 578. Menger states the problem only for *regular* curves, a special type of continuous curve.

[3]) Cf. my papers *Cyclicly connected continuous curves*, Proc. Ntl. Acad. of Sci., vol. 13 (1927), pp. 31—38, and *Concerning the structure of a continuous curve*, Amer. Journal Math., vol. 50 (1928), pp. 167—194. Extensions of most of the results in the former paper to n dimensions have been made by W. L. Ayres; cf. his forthcoming paper *Concerning continuous curves in space of n dimensions*.