On continuous images of a compact metric space.

By

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In a recent paper [1]) C. Kuratowski calls a set of points M a *Peano space* if (*a*) it is metric, (*b*) it is the continuous image of the interval $0 \leqq x \leqq 1$. It is the purpose of this note to point out that with the second assumption we may obtain the same properties for M with a much weaker condition than (*a*). Since no properties of the interval are used except that it is compact and metric, we may state our result in the following more general form:

If M is a set of points and for each sequence $[q_i]$ of M there exists a „limit" function such that

(1) *limit $[q_i]$ is a unique subset of M (which may be vacuous)* [2])

(2) *if limit $[q_i] \neq 0$ and $[q_{n_i}]$ is a subsequence of $[q_i]$, then limit $[q_{n_i}] =$ limit $[q_i]$;*

and further

(3) *M is the continuous image of a compact metric space I* [3])

then M is compact and metric [4]).

[1]) *Une caractérisation topologique de la surface de la sphère*, Fund. Math. vol. 13 (1929), pp. 307—318.

[2]) From (2) and (3) it follows easily that if limit $[q_i] \neq 0$, it consists of a single point.

[3]) There exists a correspondence T such that (3a) if $p \varepsilon I$, $T(p) \varepsilon M$, (3b) if $q \varepsilon M$, there exists at least one $p \varepsilon I$ so that $T(p) = q$, (3c) if $p \varepsilon I$, $p_i \varepsilon I$ and $\lim [p_i] = p$ (in the ordinary sense) then limit $[T(p_i)] = T(p)$. For any $N \subset M$, let $T^{-1}(N)$ denote the set of all points $p \varepsilon I$ such that $T(p) \varepsilon N$.

[4]) P. Alexandroff, Ueber stetige Abbildungen kompakter Räume, Math. Ann. 96 (1927), p. 562, proves that a Hausdorff space, which is the continuous image of a compact metric space, is compact and metric. A space satisfying (1) and (2) is more general than a Hausdorff space.

Fréchet [1]) calls a set of elements a space $\mathfrak{L}$ if there exists a limit function satisfying (1) and (2) and a third condition:

(4) *If $q_i = q$ for each i, then limit $[q_i] = q$.*

Thus our conditions (1) and (2) give a more general space than the Fréchet $\mathfrak{L}$ space. We may prove easily that

(5) *M is a space $\mathfrak{L}$ of Fréchet.*

(6) *If $K \subset I$ and $\overline{K} = K$, then $\overline{T(K)} = T(K)$* [2]).

(7) *If $N \subset M$ and $\overline{N} = N$, then $\overline{T^{-1}(N)} = T^{-1}(N)$.*

Let $p \varepsilon T^{-1}(q)$ and $p = p_i$ for each i. Since $\lim [p_i] = p$ (in the ordinary sense) and $T(p_i) = q_i$, we have from (3c) limit $[q_i] = q$. Hence M has property (4) and is a space $\mathfrak{L}$. Suppose $\overline{T(K)} \neq T(K)$. Then there is a $q \varepsilon M - T(K)$ and a sequence $[q_i] \subset T(K)$ such that limit $[q_i] = q$. For each i let $p_i \varepsilon T^{-1}(q_i)$. There is a subsequence $[p_{n_i}]$ of $[p_i]$ which has a limit p. As $\overline{K} = K$, $p \varepsilon K$. From (3c), limit $[T(p_{n_i})] =$ limit $[q_{n_i}] = T(p) \varepsilon T(K)$. From (2), $T(p) = q$ but $q \varepsilon M - T(K)$. Let $[p_i]$ be a sequence of $T^{-1}(N)$ such that $\lim [p_i] = p$. Then $\lim [T(p_i)] = T(p)$. As $p_i \varepsilon T^{-1}(N)$, $T(p_i) \varepsilon N$; and as $\overline{N} = N$, $T(p) \varepsilon N$. Then $p \varepsilon T^{-1}(T(p)) \subset T^{-1}(N)$. Hence $\overline{T^{-1}(N)} = T^{-1}(N)$.

Now for each $q \varepsilon M$, let K_{nq} be the set of all $p \varepsilon I$ such that

$$\varrho(p, T^{-1}(q)) \geq 1/2^n \quad (n = 1, 2, \ldots) \tag{8}$$

Evidently $\overline{K}_{nq} = K_{nq}$ and from (6) the set $U_n(q) = M - T(K_{nq})$ is an open set and $q \varepsilon U_n(q)$. From (3c) there exists an open set $V_{nq} \supset T^{-1}(q)$ such that $T(V_{nq}) \subset U_n(q)$. From the Borel theorem there exists a finite subset $V_{n1}, V_{n2}, \ldots, V_{nk_n}$ of $[V_{nq}]$ such that

$$\sum_{i=1}^{n_k} V_{ni} \supset I \quad (n = 1, 2, \ldots). \tag{9}$$

If $V_{ni} = V_{nq}$, let U_{ni} denote $U_n(q)$. Then

$$M \subset \sum T(V_{ni}) \subset \sum U_{ni} \quad \text{for each } n. \tag{10}$$

[1]) M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Cir. Math. Palermo, vol. 22 (1906), pp. 1—72.

[2]) The point q of M is said to be a limit point of the set $N \subset M$ if there exists a sequence $[q_i] \subset N$ such that $q \neq q_i \neq q_j$ $(i \neq j)$ and limit $[q_i] = q$. From this we may define closed and open sets in M. The symbol $\overline{N}$ is used as usual to denote N plus all its limit points.

We shall now show that the countable set $[U_{ni}]$ $(i=1, 2, ..., k_n;$ $n=1, 2, \ldots)$ may be used to define limit point of a point-set of M, i. e. that M is perfectly separable. Let q' be a limit point of a set $N \subset M$, i. e. there exists a sequence $[q_i] \subset N$ such that

$$(11) \qquad q' \neq q_i \neq q_j \ (i \neq j) \quad \text{and} \quad (12) \ \text{limit}\ [q_i] = q',$$

and suppose there exists a U_{ni} such that $q' \,\varepsilon\, U_{ni}$ and $U_{ni} \cdot (N - q') = 0$. Let $L = M - U_{ni}$. Then with (7), we have

$$(13) \quad \overline{T^{-1}(L)} = T^{-1}(L), \quad T^{-1}(L) \cdot T^{-1}(q') = 0, \quad T^{-1}(L) \supset [p_i],$$

where $p_i \,\varepsilon\, T^{-1}(q_i)$ for each i. From (11) and (3a), $p_i \neq p_j$ $(i \neq j)$. There is a subsequence $[p_{n_i}]$ such that $\lim [p_{n_i}] = p' \,\varepsilon\, T^{-1}(L)$. From (2) and (3c), $T(p') = q'$, but $T(p') \,\varepsilon\, L$ while q' non-ε L. Now suppose conversely that for every $U_{ni} \supset q'$,

$$(14) \qquad U_{ni} \cdot (N - q') \neq 0.$$

We shall show that q' is a limit point of N. Consider the sets $T^{-1}(q')$ and $T^{-1}(N - q')$. Suppose there is a sequence $[p_i] \subset \subset T^{-1}(N - q')$ such that $\lim [p_i] = p' \,\varepsilon\, T^{-1}(q')$. Evidently $T(\Sigma p_i)$ contains infinitely many distinct points. Thus there is a subsequence $[p_{n_i}]$ of $[p_i]$ such that $q' \neq T(p_{n_i}) \neq T(p_{n_j})$ $(i \neq j)$ and $\text{limit}\,[T(p_{n_i})] = T(p') = q'$. In this case then q' is a limit point of N. Now suppose on the contrary that

$$(15) \qquad \overline{T^{-1}(q')} \cdot \overline{T^{-1}(N - q')} = 0.$$

For each n there is a $V_{ni_n} \supset T^{-1}(q')$. If for any n, $K_{ni_n} \supset \supset T^{-1}(N - q')$, then

$$(16) \qquad U_{ni_n} = M - T(K_{ni_n}) \supset q', \quad U_{ni_n} \cdot (N - q') = 0$$

contrary to (14). Hence by definition of K_{ni_n}, there exists a $r_n \,\varepsilon\, T^{-1}(q)$, $s_n \,\varepsilon\, T^{-1}(N - q')$ and two points u_n and v_n of I such that

$$(17) \qquad \varrho(r_n, u_n) < 1/2^n, \quad \varrho(s_n, v_n) < 1/2^n, \quad T(u_n) = T(v_n).$$

Then there exist $r \,\varepsilon\, T^{-1}(q')$ and $s \,\varepsilon\, I$ and a sequence of integers $m_1, m_2, \ldots$ such that

$$\lim [u_{m_i}] = \lim [r_{m_i}] = r \quad \text{and} \quad \lim [v_{m_i}] = \lim [s_{m_i}] = s.$$

From (3c) we have

$$\text{limit}\ [T(u_{m_i})] = T(r) = q',$$

$$\text{limit}\ [T(v_{m_i})] = \text{limit}\,[T(s_{m_i})] = T(s).$$

But from (17)

$$q' = \text{limit}\,[T(u_{m_i})] = \text{limit}\,[T(v_{m_i})] = T(s).$$

Hence we have

$$\lim [s_{m_i}] = s \,\varepsilon\, T^{-1}(q')$$

contrary to (15). Thus the set M is perfectly separable.

We shall now show that M is a Hausdorff space, where any U_{ni} is defined to be a neighborhood of a point if it contains it. Two of the conditions are seen immediately to be satisfied. Now suppose $q \,\varepsilon\, U_{mi} \cdot U_{nj}$. Then $q \,\varepsilon\, M - (T(K_{mi}) + T(K_{nj}))$. Hence

$$(18) \qquad T^{-1}(q) \cdot (T^{-1}(T(K_{mi}) + T(K_{nj}))) = 0.$$

and from (6) and (7) these two sets are closed. Under these conditions we showed in the above paragraph that there exists a

$$K_{rk} \supset T^{-1}(T(K_{mi}) + T(K_n\ ))$$

and so that

$$V_{rk} \supset T^{-1}(q).$$

Hence

$$q \subset U_{rk} \subset U_{mi} \cdot U_{nj},$$

which is a third of the Hausdorff conditions. In place of the fourth condition we will prove that M is normal, a stronger property, that is, if A and B are subsets of M such that

$$(19) \qquad \overline{A} = A, \quad \overline{B} = B, \quad A \cdot B = 0,$$

there exists open sets U and V of M such that

$$(20) \qquad A \subset U, \quad B \subset V, \quad U \cdot V = 0.$$

The sets $T^{-1}(A)$ and $T^{-1}(B)$ are closed and thus there exists a number n such that

$$0 < n < \varrho(T^{-1}(A), T^{-1}(B)).$$

Let R be the set of all points p such that $\varrho(p, T^{-1}(B)) \leqq n$. Let $S = \overline{I - R}$.

Then

$$\overline{R} = R, \quad R \,.\, T^{-1}(A) = 0, \quad S \,.\, T^{-1}(B) = 0.$$

Let $U = M - T(R)$ and $V = M - T(S)$. We have $U \supset A$, $V \supset B$ and

$$U \,.\, V = (M - T(R)) \,.\, (M - T(S)) = M - (T(R) + T(S)) =$$
$$= M - (T(R) + T(\overline{I - R})) = M - T(I) = 0.$$

Thus M is a perfectly separable, normal Hausdorff space and is metric by a theorem due to P. Urysohn [1]).

[1]) P. Urysohn, Math. Ann., vol. 94 (1925), p. 309.

Über den Begriff der Definitheit in der Axiomatik.

Von

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In meinen „Untersuchungen über die Grundlagen der Mengenlehre" Math. Ann. 65. 1908) habe ich auf S. 263 den Terminus „definit" eingeführt für solche Aussagen $E(x)$ „über deren Gültigkeit oder Ungültigkeit die Grundbeziehungen des Bereiches vermöge der Axiome und der allgemeingültigen logischen Gesetze ohne Willkür entscheiden". Dieser Begriff scheint nicht allgemein verstanden worden zu sein; manche finden ihn „unklar", und die meisten neueren Autoren auf dem Gebiete der mengentheoretischen Axiomatik versuchen ohne ihn auszukommen. Die verschiedenen Versuche, den Begriff der „Definitheit" zu vermeiden, lassen sich in die folgenden Gruppen einteilen:

1) Man läßt ihn einfach *weg*, hält ihn für *überflüssig* oder überläßt ihn der allgemein Logik bzw. der Logistik, da seine Bedeutung eben nicht spezifisch mengentheoretischer Natur ist. Dieser einfache Behelf, einer Schwierigkeit dadurch aus dem Wege zu gehen, daß man sie ignoriert oder einer anderen Wissenschaft zuschiebt, wäre gewiß auch für mich das Bequemste gewesen und hätte mir manche Angriffe und Kritiken erspart. Aber ist es schon an und für sich nicht die Aufgabe des wissenschaftlichen Forschers, sich die Arbeit bequem zu machen, so gilt dies noch am wenigsten für Untersuchungen, die es mit dem „Grundlagen" einer Wissenschaft zu tun haben. Es kam mir darauf an, die Hauptsätze der allgemeinen Mengenlehre aus möglichst geringen Vorraussetzungen und mit möglichst beschränkten Hilfsmitteln herzuleiten, und ich erkannte, daß der uneingeschränkte Gebrauch der „Satzfunktionen" hier entbehrlich, ja wegen gewisser „Antinomien" vielleicht sogar