Continuous curves homeomorphic with the boundary of a plane domain.

By

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In a recent conversation Professor C. Kuratowski suggested to me the problem of characterizing by interior properties those continuous curves which may be topologically transformed into the boundary of a plane domain. R. L. Wilder[2]) has shown that if a continuous curve M is the boundary of a plane domain, then no two simple closed curves of M have more than one point in common. This property is easily seen to be a homeomorphic invariant, and Professor C. Kuratowski suggested the question as to whether every continuous curve having this property is homeomorphic with the boundary of a plane domain. In this note we shall show that such is the case. The condition that no two simple closed curves of M have more than one point in common may be stated in two other equivalent ways: (a) M contain no θ-curve, (b) every maximal cyclic curve of M be a simple closed curve. A θ-curve is a set consisting of three arcs with common end points and no two having any other point in common. We shall prove the following

Theorem. *In order that a compact continuous curve M be homeomorphic with a plane continuous curve which is the boundary of one of its complementary domains it is necessary and sufficient that every maximal cyclic curve of M be a simple closed curve.*

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[2]) *Concerning continuous curves*, Fund. Math., vol. 7 (1925), p. 354.

It is seen that this theorem contains as a special case the Ważewski-Menger-Gehman theorem[1]) that every acyclic continuous curve is homeomorphic with one in the plane. From the method of proof of our theorem it may be seen that the plane domain, with whose boundary M is homeomorphic, may be taken as bounded except in the case where M is acyclic. The necessity of our condition follows from the theorem of R. L. Wilder and the topological invariance of this property. In proving the sufficiency we shall make use of two lemmas:

Lemma 1. *If x and y are two points of a continuous curve M in which every maximal cyclic curve is a simple closed curve, then the arc-curve $M(x+y)$ [i. e. the sum of all the arcs xy contained in M] consists of an arc xy together with a countable set of arcs a_1b_1, a_2b_2, $a_3b_3, \ldots$ such that*

1) $(\text{arc } a_ib_i) \,.\, (\text{arc } a_jb_j) \subset (a_i+b_i)(a_j+b_j)$, $(j \neq i)$,

2) $[\text{arc } xy] \,.\, [\sum_i \text{arc } a_ib_i] = \sum_i (a_i+b_i)$,

3) *the subarcs a_ib_i and a_jb_j of the arc xy $(i \neq j)$ have at most an end point in common.*

Lemma 2. *A compact continuous curve M contains a countable set of points $x_1, y_1, x_2, y_2, \ldots$ such that* (1) *every point of M is either an end point of M or a point of one of the arc-curves $M(x_i+y_i)$,* (2) *for any positive integer n, the set $\sum_{i=1}^{n} M(x_i+y_i)$ is a continuous curve M_n,* (3) *M_n has just one point in common with the arc-curve $M(x_{n+1}+y_{n+1})$ and this is either the point x_{n+1} or y_{n+1},* (4) *for any $\varepsilon > 0$ there exists an integer n^* such that for $n > n^*$ the set $M - M_n$ contains no component of diameter $> \varepsilon$.*

The first lemma follows easily out of the properties of the arc-curves[2]) and the second may be proved in much the same manner as a theorem of R. L. Wilder's of which it is an analogue[3]).

[1]) T. Ważewski, *Sur les courbes de Jordan ne renfermant aucune courbe fermée de Jordan*, Ann. Soc. Pol. de Math. vol. 2 (1923), pp. 49—170; K. Menger, *Über reguläre Baumkurven*, Math. Ann., vol. 96 (1926), pp. 572—582; H. M. Gehman, *Concerning acyclic continuous curves*, Trans. Amer. Math. Soc., vol. 29 (1927), pp. 553—568.

[2]) W. L. Ayres, *Concerning the arc-curves and basic sets of a continuous curve*, Trans. Amer. Math. Soc., vol. 30 (1928), pp. 567—578.

[3]) R. L. Wilder, *loc. cit.*, p. 365.

Proof of the theorem. Consider the set of points, $x_1, y_1, x_2, y_2, \ldots,$ of M given by Lemma 2. By Lemma 1,

$$M(x_1+y_1) = \text{arc}\, x_1 y_1 + \sum_i \text{arc}\, a_{1i} b_{1i}.$$

Let I_1 be the interval from $(1,0)$ to $(0,0)$ in a plane E_2, and let Φ be a homeomorphism between the arc x_1y_1 and the interval I_1. In the plane E_2 on each of the subintervals $\Phi(a_{1i}), \Phi(b_{1i})$ as base construct an equilateral triangle Δ_{1i}. We define a homeomorphism between the arc $a_{1i}b_{1i}$ and that arc of Δ_{1i} from $\Phi(a_{1i})$ to $\Phi(b_{1i})$ that does not belong to I_1 so that a_{1i} and $\Phi(a_{1i})$ and b_{1i} and $\Phi(b_{1i})$ correspond, and call this correspondence Φ for the arc $a_{1i}b_{1i}$. Then Φ is a homeomorphism so that

$$\Phi(M(x_1+y_1)) = I_1 + \sum_i \Delta_{1i} = N_1.$$

$M(x_2+y_2)$ has either x_2 or y_2 in common with M_1 and we may suppose that it is x_2. Let D_2 be a triangle in E_2 having $\Phi(x_2)$ as one vertex, of diameter $<\frac{1}{4}$, whose interior contains no point of N_1 and which lies except for $\Phi(x_2)$ in the unbounded complementary domain of N_1. Let I_2 be an interval which lies within D_2 except for one end point $\Phi(x_2)$. We have

$$M(x_2+y_2) = \text{arc}\, x_2 y_2 + \sum_i \text{arc}\, a_{2i} b_{2i}.$$

Take any homeomorphism between the arc x_2y_2 and the interval I_2 so that x_2 and $\Phi(x_2)$ correspond and call this Φ. On each subinterval $\Phi(a_{2i}), \Phi(b_{2i})$ of I_2 as a base construct an isosceles triangle Δ_{2i} with altitude less than $d(\Phi(a_{2i}), \Phi(b_{2i}))$ and lying wholly within D_2. As above we define Φ for the arcs $a_{2i}b_{2i}$. Then

$$\Phi(M_2) = N_1 + I_2 + \sum_i \Delta_{2i} = N_2.$$

Continue this process. $M(x_n+y_n) \cdot M_{n-1} = x_n$ or y_n, say x_n. Let D_n be a triangle such that $d(D_n) < 1/2^n$, $(N_{n-1} + \sum_{i=2}^{n-1} D_i) \cdot (D_n + \text{interior } D_n) = \Phi(x_n)$ and D_n lies inside those triangles $D_i (i<n)$ which $\Phi(x_n)$ is inside and only those. Let I_n be an interval which lies inside D_n except for one end point $\Phi(x_n)$. We have

$$M(x_n+y_n) = \text{arc}\, x_n y_n + \sum_i \text{arc}\, a_{ni} b_{ni}.$$

Let Φ be a homeomorphism between the arc x_ny_n and the interval I_n as before so that x_n and the point previously defined as $\Phi(x_n)$ correspond. On each subinterval $\Phi(a_{ni}), \Phi(b_{ni})$ as a base construct an isosceles triangle Δ_{ni} lying within D_n and having altitude less than $d(\Phi(a_{ni}), \Phi(b_{ni}))$. Define Φ for the arcs $a_{ni}b_{ni}$ as before. Continue this process indefinitely unless for some n,

$$M = \sum_{i=1}^{n} M(x_i+y_i).$$

In this case it is obvious that M is homeomorphic with N_n. In the other case we may show, following the methods due to H. M. Gehman [1]), that the correspondence Φ between $\sum_{i=1}^{\infty} M(x_i+y_i)$ and $\sum_{i=1}^{\infty} N_i$ is uniformly continuous. Hence Φ may be extended to a $1-1$ bicontinuous correspondence between $\overline{\sum_i M(x_i+y_i)} = M$ and $\overline{\sum_i N} = N$.

From the construction we see that N is a continuous curve and is the boundary of the unbounded complementary domain of N. If it is desired to have N the boundary of one of its bounded complementary domains it is only necessary to invert the plane with center inside one of the triangles Δ_{ni}.

[1]) loc. cit., pp. 553—556.