Boundary of a plane domain.

It is seen that this theorem contains as a special case the Ważewski-Menger-Gehman theorem \(^1\) that every acyclic continuous curve is homeomorphic with one in the plane. From the method of proof of our theorem it may be seen that the plane domain, with whose boundary \(M\) is homeomorphic, may be taken as bounded except in the case where \(M\) is acyclic. The necessity of our condition follows from the theorem of R. L. Wilder and the topological invariance of this property. In proving the sufficiency we shall make use of two lemmas:

**Lemma 1.** If \(x\) and \(y\) are two points of a continuous curve \(M\) in which every maximal cyclic curve is a simple closed curve, then the arc-curves \(M(x \pm y)\) [i.e., the sum of all the arcs \(xy\) contained in \(M\)] consists of an arc \(xy\) together with a countable set of arcs \(a_1 b_1, a_2 b_2, \ldots\) such that

1. \((arc a_1 b_1) \cap (arc a_i b_i) = (a_i + b_i), (j \neq i),\)
2. \([arc xy] = \Sigma a_i b_i = \Sigma (a_i + b_i),\)
3. \(a_i, b_i\) and \(a_j, b_j\) of the arc \(xy\) \((i \neq j)\) have at most an end point in common.

**Lemma 2.** A compact continuous curve \(M\) contains a countable set of points \(x_1, y_1, x_2, y_2, \ldots\) such that (1) every point of \(M\) is either an end point of \(M\) or a point of one of the arc-curves \(M(x_i + y_i),\)
(2) for any positive integer \(n\), the set \(\Sigma M(x_i + y_i)\) is a continuous curve \(M_n,\)
(3) \(M_n\) has just one point in common with the arc-curve \(M(x_{i+1} + y_{i+1})\) and this is either the point \(x_{i+1}\) or \(y_{i+1},\)
(4) for any \(\varepsilon > 0\) there exists an integer \(n^*\) such that for \(n > n^*\) the set \(M - M_n\) contains no component of diameter \(> \varepsilon.\)

The first lemma follows easily out of the properties of the arc-curves \(^2\) and the second may be proved in much the same manner as a theorem of R. L. Wilder's of which it is an analogue \(^3\).

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\(^3\) R. L. Wilder, loc. cit., p. 365.

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Continuous curves homeomorphic with the boundary of a plane domain.

By

W. L. Ayres \(^2\) (Vienna).

In a recent conversation Professor C. Kuratowski suggested to me the problem of characterizing by interior properties those continuous curves which may be topologically transformed into the boundary of a plane domain. R. L. Wilder \(^4\) has shown that if a continuous curve \(M\) is the boundary of a plane domain, then no two simple closed curves of \(M\) have more than one point in common. This property is easily seen to be a homeomorphic invariant, and Professor C. Kuratowski suggested the question as to whether every continuous curve having this property is homeomorphic with the boundary of a plane domain. In this note we shall show that such is the case. The condition that no two simple closed curves of \(M\) have more than one point in common may be stated in two other equivalent ways: (a) \(M\) contain no \(\theta\)-curve, (b) every maximal cyclic curve of \(M\) be a simple closed curve. A \(\theta\)-curve is a set consisting of three arcs with common end points and no two having any other point in common. We shall prove the following

**Theorem.** In order that a compact continuous curve \(M\) be homeomorphic with a plane continuous curve which is the boundary of one of its complementary domains it is necessary and sufficient that every maximal cyclic curve of \(M\) be a simple closed curve.

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\(^1\) National Research Fellow in Mathematics (U. S. A.).

Proof of the theorem. Consider the set of points, \( x_1, y_1, x_2, y_2, \ldots \), of \( M \) given by Lemma 2. By Lemma 1,

\[
M(x_1 + y_1) = \text{arc } x_1y_1 + \sum_i \text{arc } a_i b_i.
\]

Let \( I_1 \) be the interval from \((1, 0)\) to \((0, 0)\) in a plane \( E_1 \), and let \( \Phi \) be a homeomorphism between the arc \( x_1 \), \( y_1 \), and the interval \( I_1 \). In the plane \( E_1 \) on each of the subintervals \( \Phi(a_{1i}), \Phi(b_{1i}) \) as base construct an equilateral triangle \( \Delta_{1i} \). We define a homeomorphism between the arc \( a_i b_i \), and that are of \( \Delta_{1i} \) from \( \Phi(a_{1i}) \) to \( \Phi(b_{1i}) \) that does not belong to \( I_1 \) so that \( a_i \), and \( \Phi(a_{1i}) \) and \( b_i \), and \( \Phi(b_{1i}) \) correspond, and call this correspondence \( \Phi \) for the arc \( a_i b_i \). Then \( \Phi \) is a homeomorphism so that

\[
\Phi(M(x_1 + y_1)) = I_1 + \sum_i \Delta_{1i} = N_1.
\]

\( M(x_2 + y_2) \) has either \( x_2 \) or \( y_2 \) in common with \( M_1 \), and we may suppose that it is \( x_2 \). Let \( D_2 \) be a triangle in \( E_2 \) having \( \Phi(x_2) \) as one vertex, of diameter \(< \frac{1}{2} \), whose interior contains no point of \( N_1 \), and which lies except for \( \Phi(x_2) \) in the unbounded complementary domain of \( N_1 \). Let \( I_2 \) be an interval which lies within \( D_2 \) except for one end point \( \Phi(x_2) \). We have

\[
M(x_2 + y_2) = \text{arc } x_2y_2 + \sum_i \text{arc } a_i b_i.
\]

Take any homeomorphism between the arc \( x_2y_2 \) and the interval \( I_2 \) so that \( x_2 \) and \( \Phi(x_2) \) correspond and call this \( \Phi \). On each subinterval \( \Phi(a_{2i}), \Phi(b_{2i}) \) of \( I_2 \) as a base construct an isosceles triangle \( \Delta_{2i} \) with altitude less than \( d(\Phi(a_{2i}), \Phi(b_{2i})) \) and lying wholly within \( D_2 \). As above we define \( \Phi \) for the arcs \( a_i b_i \). Then

\[
\Phi(M_2) = N_1 + I_2 + \sum_i \Delta_{2i} = N_2.
\]

Continue this process. \( M(x_n + y_n) = M_{n-1} \) or \( y_n \), say \( x_n \). Let \( D_n \) be a triangle such that \( d(D_n) < 1/2^n \), \((N_{n-1} + \sum_2^n D_i \cdot D_n + \text{interior } D_n) = \Phi(x_n) \) and \( D_n \) lies inside those triangles \( D_i(i < n) \) which \( \Phi(x_n) \) is inside and only those. Let \( I_n \) be an interval which lies

inside \( D_n \) except for one end point \( \Phi(x_n) \). We have

\[
M(x_n + y_n) = \text{arc } x_ny_n + \sum_i \text{arc } a_i b_i.
\]

Let \( \Phi \) be a homeomorphism between the arc \( x_n \), \( y_n \), and the interval \( I_n \), as before so that \( x_n \) and the point previously defined as \( \Phi(x_n) \) correspond. On each subinterval \( \Phi(a_{ni}), \Phi(b_{ni}) \) as a base construct an isosceles triangle \( \Delta_{ni} \) lying within \( D_n \) and having altitude less than \( d(\Phi(a_{ni}), \Phi(b_{ni})) \). Define \( \Phi \) for the arcs \( a_i b_i \) as before. Continue this process indefinitely unless for some \( n \),

\[
M = \sum_1^n M(x_i + y_i).
\]

In this case it is obvious that \( M \) is homeomorphic with \( N \). In the other case we may show, following the methods due to H. M. Gehman \(^1\), that the correspondence \( \Phi \) between \( \sum M(x_i + y_i) \)

and \( \sum N_i \) is uniformly continuous. Hence \( \Phi \) may be extended to a \( 1 \rightarrow 1 \) bicontinuous correspondence between \( \sum M(x_i + y_i) = M \)

and \( \sum N = N \).

From the construction we see that \( N \) is a continuous curve and is the boundary of the unbounded complementary domain of \( N \). If it is desired to have \( N \) the boundary of one of its bounded complementary domains it is only necessary to invert the plane with center inside one of the triangles \( \Delta_n \).

\(^1\) loc. cit., pp. 558—556.