two subcontinua of M. Hence Y is not a continuum of order two of M, contrary to what we have just shown. Thus the supposition that G is not countable leads to a contradiction.

In conclusion I will point out the following interesting fact concerning regular subcontinua of a continuum. Let G be any collection of mutually exclusive subcontinua of a bounded continuum M (in n-space) each of which is a regular subcontinuum of M relative to G. Then if T denotes the point set obtained by adding together all the point sets of the collection G, it is readily seen that each component of $M-T$ is closed, and hence is a bounded continuum. And if E denotes the collection of all continua $[X]$ such that X is either an element of G or a component of $M-T$, then with the aid of a theorem of Menger's[1]) it follows that all the continua of E are regular subcontinua of M relative to E, and hence *with respect to the continua of E as elements, M is a Menger regular curve.*

[1]) K. Menger, loc. cit., Theorem 8.

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On a problem of Menger concerning regular curves[1]).

By

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In his paper *Zur allgemeinen Kurventheorie*[2]) Karl Menger raised the following question: *If M is a regular curve*[3]) *is it true that for every positive number ε the curve M is the sum of a finite number of continua of diameter less than ε such that any two have at most one point in common?*

The purpose of the present paper is to give an example which shows that the answer to Menger's question as stated is in the *negative*, but that for a regular curve M whose ramification points[4]) are not dense on any subcontinuum of M the answer is in the affirmative.

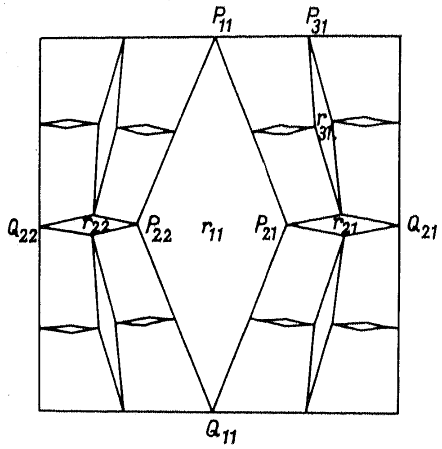
[1]) Presented to the Amer. Math. Soc., Dec. 28, 1928.

[2]) Fundamenta Mathematicae, vol. X (1927), pp. 96—115.

[3]) See Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann. vol. 95 (1925), pp. 287—306. If M is a continuum and for each point P of M and each positive number ε there exists a connected open subset of M containing P and of diameter less than ε whose boundary with respect to M is finite then M is said to be a regular curve. If R is an open subset of M (i. e., no point of R is a limit point of $M-R$), then the boundary of R with respect to M is the set of points $\overline{R}\cdot(M-R)$. See R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920), p. 345.

[4]) A ramification point of a continuous curve M is a point of order greater than 2. See W. Sierpiński, Comptes Rendus, vol. 160, p. 305. A point P of a regular curve M is said to be of order n if n is the smallest integer such that for every positive number ε there exists an open subset of M of diameter less than ε which contains P and whose boundary with respect to M contains at most n points. See Menger, loc. cit., and Urysohn, Comptes Rendus, vol. 175, (1922), p. 481.

Let R_1 be the interior of a square $ABCD$. Let r_{11} be the rhombus such that P_{11} and Q_{11}, the mid points of AB and CD, respectively, are opposite vertices, and the angles of r_{11} at P_{11} and Q_{11} are $\pi/4$. Let M_1 be the point set R minus the points of R in r_{11} plus its interior. Clearly M_1 is the sum of two mutually exclusive domains, R_{21} and R_{22}, whose boundaries have just P_{11} and Q_{11} in common. The domain R_{2i} is bounded by a simple closed curve $A_{2i}B_{2i}C_{2i}D_{2i}$ where $A_{2i}B_{2i}$ is the sum of two sides of the rhombus r_{11} and the arcs $B_{2i}C_{2i}$, $C_{2i}D_{2i}$, and $D_{2i}A_{2i}$ are straight line intervals $(i=1,2)$. Let P_{2i} and Q_{2i} be the mid points of the arcs $A_{2i}B_{2i}$[1]) and $C_{2i}D_{2i}$, respectively, and let r_{2i} be the rhombus with



P_{2i} and Q_{2i} as opposite vertices and whose angles at P_{2i} and Q_{2i} are equal to $\pi/4\cdot 2$. Let M_2 be the point set M_1 minus the points of M_1 in r_{21} and r_{22} plus the interiors of r_{21} and r_{22}. Clearly M_2 is the sum of 2^2 mutually exclusive domains R_{31}, R_{32}, R_{33}, and R_{34}. For each i $(i\leqq 4)$ R_{3i} is bounded by a simple closed curve $A_{3i}B_{3i}C_{3i}D_{3i}$ such that $A_{3i}B_{3i}$ is the sum of two sides of the rhombus r_{2j} $(j=1$ or $j=2)$, and each of the arcs $B_{3i}C_{3i}$, $C_{3i}D_{3i}$, and $D_{3i}A_{3i}$ is a straight line interval. Let P_{3i} and Q_{3i} be the midpoints of $A_{3i}B_{3i}$ and $C_{3i}D_{3i}$, respectively, and let r_{3i} be the rhombus with P_{3i} and Q_{3i} as vertices and with angles at P_{3i} and Q_{3i} equal to

[1]) Obviously the point P_{2i} is a vertex of r_{11}.

$\pi/4\cdot 3$ $(i=1,\dots,4)$. Let M_3 be the point set obtained by taking from M_2 all points which belong to $\sum\limits_{i=1}^{4}$ (r_{3i} plus its interior).

Continuing in this manner one can build up an infinite sequence of distinct rhombi, r_{11}; r_{21}, r_{22}; ...; r_{n1}, r_{n2}, ..., $r_{n2^{n-1}}$; ... with the following properties: (1) r_{11} is defined as above, (2) for each n and i $(n>1,\ i\leqq 2^{n-1})$ the interior of r_{ni} lies in some component R_{ni} of M_{n-1}, where M_{n-1} is the point set obtained by taking from R every rhombus r_{mk} plus its interior, where $k\leqq 2^{m-1}$, $m=1,2,\dots(n-1)$, (3) R_{ni} is bounded by a simple closed curve $A_{ni}B_{ni}C_{ni}D_{ni}$ where $A_{ni}B_{ni}$ is the sum of two sides of the rhombus $r_{n-1,k}$ for some value of k and the three arcs $B_{ni}C_{ni}$, $C_{ni}D_{ni}$, and $D_{ni}A_{ni}$ are straight line intervals; P_{ni} and Q_{ni} denoting the mid points of $A_{ni}B_{ni}$ and $C_{ni}D_{ni}$, respectively, the rhombus r_{ni} has P_{ni} and Q_{ni} as opposite vertices and has the angles at P_{ni} and Q_{ni} equal to $\pi/4n$.

Let M be the set of all points common to $\overline{M}_1$, $\overline{M}_2$, ... Obviously M is a regular curve, every point of which is of order greater than 2 (but not greater than 6). It is also readily seen that M is the sum of two continua N_1 and N_2 such that $N_1\cdot N_2=P_{11}+Q_{11}$ and both N_1 and N_2 are strongly equivalent[1]) to M. The curve M contains no cut point, and contains just one pair of points which cuts it.

Suppose $M=\sum\limits_{i=1}^{n}a_i$ $(n>1)$ where a_i is a nondegenerate[2]) continuum and $a_i\cdot a_j$ is vacuous or a single point $(i,j\leqq n,\ i\neq j)$. Let K denote the set of all points common to two continua a_i and a_j $(i\neq j)$. Since N_1 and N_2 have *two* points in common and $N_1+N_2=M$ it follows that N_{i_1} (i_1 equals 1 or 2) contains points distinct from P_{11} and Q_{11} of at least two continua of the set $a_1, a_2, \dots a_n$. If no one of the continua $a_1, a_2, \dots, a_n$ contains both P_{11} and Q_{11} it is easily seen that $a_i\cdot N_{i_1}$ is a continuum, or is vacuous. Let $a_{21}, a_{22}, \dots, a_{2m_1}$ $(2\leqq m_1\leqq n)$ be the sets $a_1\cdot N_{i_1}$, $a_2\cdot N_{i_1}, \dots, a_n\cdot N_{i_1}$, which contain more than one point. If a_i contains both P_{11} and Q_{11} then $a_i\cdot N_{i_1}$ is the sum of at most two continua with no point

[1]) A plane point set N is said to be strongly equivalent to the plane point set M provided there exists a continuous transformation of the plane into itself which throws N into M.

[2]) A point set is said to be *degenerate* if it consists of but a single point.

in common. In this case call these two continua a_{21} and a_{22}, and let $a_{23}, \dots, a_{2m_1}$ be the other continua $a_1 . N_{i_1}, \dots, a_n . N_{i_1}$. Let K_1 be the set of all points $a_{2i} . a_{2j}$ $(i, j \leqq m_1, i \neq j)$. Obviously K_1 is a subset of K.

Suppose P_{21} and Q_{21} are the points of N_{i_1} which correspond to P_{11} and Q_{11} under a continuous transformation which throws N_{i_1} into M, and let N_{21} and N_{22} be the continua such that $N_{21} + N_{22} = N_{i_1}$, and $N_{21} . N_{22} = P_{21} + Q_{21}$ and both N_{21} and N_{22} are strongly equivalent to M. Then one can readily see that either for i equal to 1 or 2 (for convenience suppose 1) $N_{21} = \sum_{j=1}^{m_2} a_{3j}$ where (1) $m_2 > 1$, (2) a_{3k} is a nondegenerate continuum, (3) $a_{3k} . a_{3s}$ is either vacuous or is a single point $(k, s \leqq m_2, k \neq s)$ and (4) K_2, the set of all points $a_{3k} . a_{3s}$ $(k, s \leqq m_2, k \neq s)$ is a subset of K_1.

Continuing this process one obtains a curve N_{kl} strongly equivalent to M which contains only one point of K (since the diameter of N_{ki} approaches zero as k increases indefinitely), but which is the sum of two or more nondegenerate continua $a_{k1}, a_{k2}, \dots, a_{km_k}$ such that $a_{ki} . a_{kj}$ is vacuous or is a single point, and the set of all such points belongs to K. But N_{ki} contains no cut point, and contains only one point of K. Hence $N_{ki} - N_{ki} . K$ is connected, and we have reached a contradiction.

Theorem I[1]. *If M is a compact regular curve and the ramification points of M form a null dimensional*[2] *set then for each positive number ε M is the sum of a finite number of continua of*

[1] I have proved the following theorem: If M is a compact regular curve and (1) only a countable number of points of M are of order greater than 2 or (2) only a finite number of points of M are of order greater than 3, then for each positive number ε M is the sum of a finite number of continua of diameter less than ε, the common part of any two of which is at most one point. In a discussion of this theorem and its proof, G. T. Whyburn raised the question as to whether or not the proposition herewith stated as theorem I holds true. The proof here given of the truth of this proposition is a slight modification of my original proof of the theorem stated above.

[2] Both Urysohn and Menger have shown that a set K is null dimensional if and only if for every point P of K and every positive number ε there exists a *closed* subset F_{P_ε} of M of diameter less than ε such that $M - F_{P_\varepsilon} = M_P + M_0$, where M_P contains P and is of diameter less than ε, and M_{P_ε} and M_0 are mutually separated sets. See Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math. vol. 7 (1925) and Menger, loc. cit.

diameter less than ε, the common part of any two of which is vacuous or a single point.

Proof. Let ε_1 be $\varepsilon/4$. By a theorem of Menger[1] $M = \sum_{i=1}^{k} a_i$, where a_i is a continuum of diameter less than ε_1 and $a_i . a_j$ is finite $(i, j \leqq k; i \neq j)$. Let F be the finite point set $\Sigma a_i . a_j$ $(i, j \leqq k; i \neq j$ and suppose $F = P_1 + P_2 + \dots + P_n$. Let ε_2 be the smallest of the numbers $\delta(P_i, P_j)$[2] $(i, j \leqq n; i \neq j)$. Clearly $\varepsilon_2 \leqq \varepsilon_1$. Since the ramification points of M form a null dimensional set it follows that for each i for which the point P_i is a ramification point of M there exists a closed subset F_i of M of diameter less than $\varepsilon_2/4$, every point of which is of order 1 or 2, which separates P_i from P_j in M $(i, j \leqq n; i \neq j)$. For each point Q of F_i there exists a domain D_Q of diameter less than $\varepsilon_2/4$ such that (1) D_Q contains Q, (2) the boundary of D_Q contains not more than 2 points of M, and (3) $M . \overline{D}_Q$ is a continuum. Let $D_{i1}, D_{i2}, \dots, D_{ik_i}$ denote a finite set of such domains covering F_i. In case P_i is of order 1 or 2 let D_{i1} be a domain containing P_i of diameter less than $\varepsilon_2/4$ and with properties (1), (2) and (3) as stated above. Clearly D_{ij} and D_{km} have no point in common if $i \neq k$.

The closed point set $M - \sum_{l=1}^{n} \sum_{m=1}^{k_l} M . D_{ml}$ is[3] the sum of a finite number of mutually exclusive continua of diameter less than $2\varepsilon_1$, $b_1, b_2, \dots, b_r$. Let $c_1, c_2, \dots, c_s$ denote the set of all continua $M . \overline{D}_{ij}$ $(j \leqq k_i, i \leqq n)$. If c_i and c_j have at least two points in common their sum has but two boundary points with respect to M. Hence it follows that there exists a set of continua $c'_1, c'_2, c'_3, \dots, c'_{s'}$ each two having at most one point in common and such that (1) $\sum_{i=1}^{s'} c'_i = \sum_{i=1}^{s} c_i$ and (2) c'_i has at most two boundary points with respect to M $(i \leqq s')$. If for some integer j there exists an integer i such that b_j and c_i have two points in common, then let d_j be the sum of b_j and c'_i. Otherwise let d_j be the same as b_j. Let $d_{r+1}, \dots, d_t$ be the continua $c'_1, c'_2, \dots, c'_{s'}$ which do not belong to $\sum_{i=1}^{r} d_i$. Since d_i is

[1] Loc. cit.

[2] The symbol $\delta(P, Q)$ should be read "the diameter of the point set $P + Q$".

[3] This follows from the fact that its boundary with respect to M is finite.

either an element of the set $b_1, b_2, \ldots, b_r, c_1', \ldots, c_{s'}'$, or is the sum of two such elements, and furthermore $\delta(b_i) < 2\varepsilon_1$ and $\delta(c_i) < \varepsilon_2$ where $\varepsilon_2 \leqq \varepsilon_1 < \varepsilon/4$, it follows that $\delta(d_i) < \varepsilon$. Since in addition $M = \sum_{i=1}^{t} d_i$ and $d_i . d_j$ is at most one point $(i, j \leqq t; i \neq j)$ it follows that the theorem is established.

In his paper *On regular points of continua and regular curves of at most order n* [1]) G. T. Whyburn proves the following

Theorem B. *If for the closed subset N of a continuum M there exists an integer n such that the set K of points of N which are points of order $\leqq n$ of M and which locally separate* [2]) *N in M is dense in N, then the points of M of order $\leqq n/2+1$ are dense in N.*

This theorem will be used to help establish the following

Corollary to theorem I. *If M is a compact regular curve and the set of points of M of order greater than 3 is not dense on any subcontinuum of M then for each positive number ε M is the sum of a finite number of continua of diameter less than ε, the common part of any two of which is at most one point.*

Proof. Let H denote the set of points of M of order greater than 3, and let N denote any subcontinuum of M. Since by hypothesis H is not dense on N it follows that N contains a continuum N_1 which contains only points of order not greater than 3 of M. Now it is easily proved that every subcontinuum K of a regular curve M contains a dense set of points each of which locally separates K in M. Therefore by theorem B the continuum N_1 contains points of M of order 2 or 1. Hence R the set of points of M of order greater than 2, is punctiform (i. e., contains no continuum). And since by a theorem of Menger's [3]) R is an F_σ (i. e., the sum of a countable number of closed sets), it follows by a theorem of Mazurkiewicz [1]) that R is homeomorphic with some linear set. Therefore, since R is punctiform, it is [2]) null dimensional [3]).

I have therefore shown that the set of points of M of order greater than 2 is null dimensional, whence the conclusion of the corollary follows from theorem I.

[1]) Bull. of the Amer. Math. Soc., vol. 35 (1929).

[2]) The point P of a continuum M will be said to *locally separate* a given subset N of M in M if there exists a compact neighborhood G of P such that if R is any neighborhood of P lying in G, then $M . \overline{R} - P$ is separated between some two points of $N . R$, i. e., $M . \overline{R} - P = M_1 + M_2$, where M_1 and M_2 are mutually separated and $M_1 . N . R \neq 0 \neq M_2 . N . R$. See Whyburn, loc. cit.

[3]) Loc. cit.

[1]) S. Mazurkiewicz, Bulletin de l'Académie des Sciences de Cracovie, 1913.

[2]) Cf. W. Sierpiński, *Sur les ensembles connexes et non connexes*, Fund. Math. vol. 2 (1921), p. 89.

[3]) The last two sentences are copied from Whyburn's proof of his theorem 5, loc. cit., which reads as follows: *In any regular curve M of at most order n, the set H of points of order $>n/2+1$ is null dimensional.* In view of the above argument it is clear that a slight addition to Whyburn's argument proves the following more general theorem: *If the points of a regular curve M of order $>n$ are not dense on any subcontinuum of M then the set H of points of M of order $>n/2+1$ is null dimensional.*

The University of Texas, March 30, 1929.