Admettons maintenant que l'ensemble N est mesurable (L). Tout ensemble mesurable de mesure positive contenant un sous-ensemble parfait, il résulte de I que N devrait être de mesure nulle. Or, dans ce cas, le complémentaire Q de N contiendrait un sous-enemble parfait P de mesure positive, et le point $f(P)$ de P appartiendrait à Q, donc pas à N, contrairement à la définition. L'hypothèse que II n'est pas vrai implique donc une contradiction [1]).

Notre proposition est ainsi démontrée.

[1]) On pourrait démontrer sans peine que l'ensemble Q est non mesurable et de deuxième catégorie dans tout intervalle.

Concerning functions of sets.

By

Stanisław Ulam (Lwów).

On considering the properties of functions defined on every subset of a certain space [1]) arises the problem of the existence of a function which satisfies the condition of „sustractivity" i. e., that

$$F(A-B)=F(A)-F(B),$$

but does *not* satisfy the „infinite additivity" i. e. the condition:

$$F(A_1+A_2+\ldots+A_n+\ldots)=F(A_1)+F(A_2)+\ldots+F(A_n)+\ldots$$

(the values of the function F are sets!).

I shall prove here the existence of such a function. In the proof Zermelo's axiom of choice will play an essential part.

The space (denoted by 1) on which the function is defined is the set of all natural numbers. We shall call two sets of natural numbers M and N „almost identical" if the set $(M-N)+(N-M)$ is finite or vacuous.

M „almost contains" N will mean, that M contains in the usual sense a set N_1 „almost identical" with N.

It is easy to conclude, that if A is „almost identical" with B and B with C, A is „almost identical" with C. Every two finite sets are „almost identical".

The class of all sets of natural numbers X may be ordered with the aid of Zermelo's axiom into a transfinite sequence $\mathfrak{A}$.

We shall place now the sets of the sequence $\mathfrak{A}$ into certain classes: $K_0, K_1, \ldots, K_\alpha \ldots$ in the following way:

[1]) See e. g. A. Tarski Ann. Soc. Pol. Math. VI. pp. 127 et 132.

(I) The class K_0 consists of all finite and the vacuous set.

(II) The class K_1 contains 1°: the first infinite set X of the sequence $\mathfrak{A}$, 2°: all the sets Y which „almost contain" X.

(It is obvious that $1 \varepsilon K_1$ i. e. the set of *all* natural numbers belongs to K_1),

(III) If η is *even* (all limit ordinal numbers are here to be considered as even) and > 0, K_η contains 1°: the first set X in the sequence $\mathfrak{A}$ which belongs to no K_ξ with $\xi < \eta$, 2°: all the sets „almost identical" with X, 3°: the vacuous set.

(IV) If η is *odd* > 1, K_η consists of all the sets which almost contain" the differences $A - B$, where A belongs to K_α, B to $K_{\eta-1}$ α is odd and $< \eta$

I shall prove now, that no set X can belong to two classes, one of which has an even index, the second an odd one.

For this purpose I shall prove that:

1) If α and η are odd and $\alpha < \eta$, then $K_\alpha \subset K_\eta$

2) If $X \varepsilon K_\alpha$ and $Y \varepsilon K_\alpha$ and α is odd, then $X . Y \neq 0$ and $X . Y \varepsilon K_\alpha$.

Ad 1). Let $X \varepsilon K_\alpha$. The set X is obviously a difference X—0. 0 belongs to $K_{\eta-1}$ It follows from (IV) that the set X belongs also to K_η.

Ad 2). I shall prove this lemma by transfinite induction. The lemma holds obviously true for $\alpha = 1$.

Suppose it is true for all classes K_α, $\alpha < \eta$; we shall conclude that it is true for K_η.

Every set X of the class K_η „almost contains" a set of the form $X_1 - C$, where X_1 belongs to a class K_α with an odd index $\alpha < \eta$, and $C \varepsilon K_{\eta-1}$. Since K_α contains all sets „almost identical" with X_1, we may suppose that $X \supset X_1 - C$. Similarly $Y \supset Y_1 - D$, where Y_1 and D have an analogous meaning.

We shall prove, that $X_1 . Y_1 - (C + D) \neq 0$

By hypothesis: $X_1 . Y_1 \varepsilon K_\alpha$, on the other hand $(C + D) \varepsilon K_{\eta-1}$ for it is (by III) „almost identical" with either C or D. Now no set from the class $K_{\eta-1}$ can contain a set from the class K_α, $\alpha < \eta - 1$. [In this case it would belong by definition also to the class K_α!]

Thus $X . Y \supset X_1 . Y_1 - (C + D) \neq 0$, and as $X_1 . Y_1 \varepsilon K_\alpha$ and $(C + D) \varepsilon K_{\eta-1}$ it follows by (IV) that $X . Y \varepsilon K_\eta$.

The lemma 2) is proved. We shall conclude, that *no set X can figure in two classes, one with an even index, the second with an odd one.*

Suppose $X \varepsilon K_\xi . K_\eta$, ξ even, η odd. By (IV) $(1 - X) \varepsilon K_{\xi+1}$ (for $1 \varepsilon K_1$)

Thus, the two sets: X and $1 - X$ belong to the classes K_η and $K_{\xi+1}$ resp. Now, by lemma 1) we have either $K_\eta \subset K_{\xi+1}$ or $K_{\xi+1} \subset K_\eta$. Hence the sets X and $1 - X$ belong simultaneously either to $K_{\xi+1}$ or K_η.

Since:

$$(1 - X) . X = 0,$$

we have a contradiction with lemma 2).

We may now define our function $F(X)$ as follows: let $X \varepsilon K_\alpha$; if α is odd: $F(X) = 1$ (= the whole space), if α is even: $F(X) = 0$.

In virtue of this definition, we have by lemma 2)

a) If $F(X) = 1 = F(Y)$, $F(X . Y) = 1$.

b) By (I): $F(0) = 0$.

c) By (II) and (IV): if $X \subset Y$ and $F(X) = 1$, then $F(Y) = 1$.

We shall prove, that:

$$F(X - Y) = F(X) - F(Y).$$

In this purpose we shall show at first:

d) $F(1 - X) = 1 - F(X)$.

If $F(X) = 0$, $X \varepsilon K_\alpha$, α is even. By (IV): $(1 - X) \varepsilon K_{\alpha+1}$ hence $F(1 - X) = 1 = 1 - F(X)$.

If $F(X) = 1$, $F(1 - X) = 0$, for otherwise the sets X and $1 - X$ both belong to a class with an odd index (by lemma 1) which is a contradiction to lemma 2), since $X . (1 - X) = 0$.

Now we shall prove that:

e) $F(X . Y) = F(X) . F(Y)$.

If $F(X) = 0$ we have by d) $F(1 - X) = 1$ hence by c) $F(XY) = 0$ Thus to prove e) it remains to consider the case where $F(X) = 1 = F(Y)$. Now, by a) in this case we have $F(XY) = 1 = F(X) . F(Y)$.

By d) and e) the „sustractivity" is proved. On the other hand. the function does not satisfy the condition:

$$F(A_1 + A_2 + \ldots A_n + \ldots) = F(A_1) + F(A_2) + \ldots F(A_n) + \ldots$$

This is seen if we define A_n as the set composed of the one element n: $A_n = \{n\}$.