Admettons maintenant que l'ensemble $N$ est mesurable $(L)$. Tout ensemble mesurable de mesure positive contenant un sous-ensemble parfait, il résulte de $I$ que $N$ devrait être de mesure nulle. Or, dans ce cas, le complémentaire $Q$ de $N$ contiendra un sous-ensemble parfait $P$ de mesure positive, et le point $f(P)$ de $P$ appartiendrait à $Q$, donc pas à $N$, contrairement à la définition. L'hypothèse que $II$ n'est pas vrai implique donc une contradiction $^9$.

Notre proposition est ainsi démontrée.

$^9$ On pourrait démontrer sans peine que l'ensemble $Q$ est non mesurable et de deuxième catégorie dans tout intervalle.

Concerning functions of sets.

By

Stanislaw Ulam (Lwów).

On considering the properties of functions defined on every subset of a certain space $^1$ arises the problem of the existence of a function which satisfies the condition of "subactivity" i. e., that

$$F(A - B) = F(A) - F(B),$$

but does not satisfy the "infinite additivity" i. e. the condition:

$$F(A_1 + A_2 + \ldots + A_s + \ldots) = F(A_1) + F(A_2) + \ldots + F(A_s) + \ldots$$

(the values of the function $F$ are sets!)

I shall prove here the existence of such a function. In the proof Zermelo's axiom of choice will play an essential part.

The space (denoted by 1) on which the function is defined is the set of all natural numbers. We shall call two sets of natural numbers $M$ and $N$ "almost identical" if the set $(M - N) + (N - M)$ is finite or vacuous.

$M$ "almost contains" $N$ will mean, that $M$ contains in the usual sense a set $N$, "almost identical" with $N$.

It is easy to conclude, that if $A$ is "almost identical" with $B$ and $B$ with $C$, $A$ is "almost identical" with $C$. Every two finite sets are "almost identical".

The class of all sets of natural numbers $X$ may be ordered with the aid of Zermelo's axiom into a transfinite sequence $\mathcal{C}$.

We shall place now the sets of the sequence $\mathcal{C}$ into certain classes: $K_1, K_2, \ldots, K_\alpha$, in the following way:

Concerning functions of sets.

Suppose \( X \in K_\xi, K_\eta, \xi \text{ even, } \eta \text{ odd.} \) By (IV) \((1 - X)eK_{\xi+1}\) (for \(1 \in K_1\)).

Thus, the two sets: \(X\) and \(1 - X\) belong to the classes \(K_\eta\) and \(K_{\xi+1}\) resp. Now, by lemma 1) we have either \(K_\eta \subset K_{\xi+1}\) or \(K_{\xi+1} \subset K_\eta\).

Hence the sets \(X\) and \(1 - X\) belong simultaneously either to \(K_{\xi+1}\) or \(K_\eta\).

Since: \((1 - X) \cdot X = 0,\)

we have a contradiction with lemma 2).

We may now define our function \(F(X)\) as follows: let \(X \in K_\alpha;\)

if \(\alpha\) is odd: \(F(X) = 1\) (= the whole space), if \(\alpha\) is even: \(F(X) = 0.\)

In virtue of this definition, we have by lemma 2)

a) If \(F(X) = 1 \Rightarrow F(Y), F(X) = 1.\)

b) By (I) \(F(0) = 0.\)

c) By (II) and (IV): if \(X \subset Y\) and \(F(X) = 1\), then \(F(Y) = 1.\)

We shall prove, that:

\(F(X - Y) = F(X) - F(Y).\)

In this purpose we shall show at first:

d) \(F(1 - X) = 1 - F(X).\)

If \(F(X) = 0, X \in K_\alpha, \alpha \text{ is even.} \) By (IV): \((1 - X) \in K_{\alpha+1}\) hence \(F(1 - X) = 1 = 1 - F(X).\)

If \(F(X) = 1, F(1 - X) = 0.\) for otherwise the sets \(X\) and \(1 - X\) both belong to a class with an odd index (by lemma 1) which is a contradiction to lemma 2), since \(X \cdot (1 - X) = 0.\)

Now we shall prove that:

e) \(F(X, Y) = F(X). F(Y).\)

If \(F(X) = 0\) we have by d) \((1 - X) = 1 \Rightarrow F(Y) \neq 0\) Thus to prove e) it remains to consider the case where \(F(X) = 1 = F(Y)\). Now, by a) in this case we have \(F(X, Y) = 1 = F(X), F(Y).\)

By d) and e) the \(\alpha\)-subtraction is proved. On the other hand, the function does not satisfy the condition:

\(F(A_1 + A_2 + \ldots + A_n) = F(A_1) + F(A_2) + \ldots + F(A_n) + \ldots\)

This is seen if we define \(A_n\) as the set composed of the one element \(n: A_n = \{n\}.\)