On f.p.p. and f.*p.p. of some not locally connected continua

by

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Abstract. Let X be a continuum with the fixed point property (f.p.p.) and $f: X \to X$ a continuous mapping. A component C of the fixed point set of f is called *essential* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous mapping $f': X \to X$ with $|f'-f| < \delta$ has a fixed point in the ε-neighborhood $U_\varepsilon(C)$ of C; and X has *f.*p.p.* if the fixed point set of every continuous mapping $f: X \to X$ has at least one essential component. For instance, a compact absolute retract has f.*p.p. We give some examples of not locally connected continua with f.p.p., but without f.*p.p. Also, we exhibit a not locally connected continuum which has both f.p.p. and f.*p.p.

Let X be a continuum. If every mapping $f: X \to X$ (1) has at least one fixed point, X is said to have the *fixed point property* (f.p.p.). In this paper we investigate the existence of essential components of fixed point sets and the property f.*p.p., which are defined as follows: a component C of the fixed point set of f is called *essential* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every mapping $f': X \to X$ with $|f'-f| < \delta$ has a fixed point in the ε-neighborhood $U_\varepsilon(C)$ of C; and X has *f.*p.p.* if it has f.p.p. and the fixed point set of every mapping $f: X \to X$ has at least one essential component.

A retract of a continuum with f.*p.p. has f.*p.p., and the Hilbert cube I^ω has f.*p.p. Hence every absolute retract has f.*p.p. (see [2]). Further, if X and Y are two continua with f.*p.p. and $X \cap Y$ is a single point, then $X \cup Y$ has f.*p.p. (see [1], [5]). The last statement can be extended to the case where the number of continua is countably infinite (see [5]).

The following question is posed in [2]: "Does there exist a space which has f.p.p., but does not have f.*p.p.?" The purpose of this paper is to give an answer to this problem. In Section I, we will construct some examples of continua with f.p.p., but without f.*p.p. None of them is locally connected. Next, in Section II, we will give an example of a not locally connected continuum which has f.*p.p.

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(1) All spaces considered in the paper are separable metric and every mapping is continuous.

Notations and definitions

$|f'-f| = \sup_{x\in X} d(f'(x), f(x))$.

I: the interval $[0,1]$.

∂B: boundary of the 2-disk B.

$\mathrm{Int}B$: the interior of B.

I. Some examples of not locally connected continua with f.p.p., but without f.*p.p. First we state Borsuk's lemma on f.p.p. (see [1] or [4], p. 343).

LEMMA 1 (Borsuk). *Let X and X_n $(n = 1, 2, \ldots)$ be compact metric spaces such that $X \supset X_n$ for every n. Assume that for every $\varepsilon > 0$ there exists f_n: $X \to X_n$ with $|f_n(x)-x| < \varepsilon$. Then, if each X_n $(n = 1, 2, \ldots)$ has f.p.p., so does X.*

The next lemma can be proved similarly to Lemma 1.

LEMMA 2. *Let X and X_n $(n = 1, 2, \ldots)$ be compact metric spaces such that $X_n \supset X$ for every n. Assume that for every $\varepsilon > 0$ there exists f_n: $X_n \to X$ with $|f_n(x)-x| < \varepsilon$. Then, if each X_n $(n = 1, 2, \ldots)$ has f.p.p., so does X.*

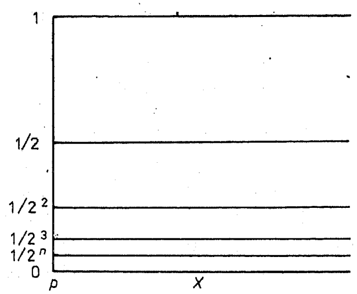

Fig. 1. Y_1 (a comb space)

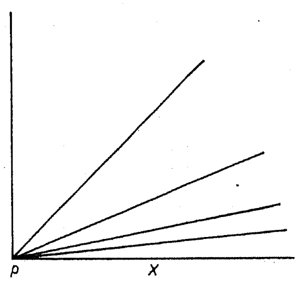

Fig. 2. Y_2

Now we state the main theorems, which give the answer to our problem.

THEOREM 1. *Let X be a continuum which satisfies the following conditions:*

(i) *X has f.p.p., and*

(ii) *there exist a point $p \in X$ and a continuous mapping f: $X \to X$ such that $f(p) = p$ and the component C of the fixed point set of f which contains p is not essential.*

Define the subset $Y_1 \subset X \times I$ as follows:

$$Y_1 = (\{p\}\times I)\cup(X\times\{0\})\cup\bigcup_{n=0}^{\infty}(X\times\{1/2^n\}).$$

*Then Y_1 has f.p.p., but does not have f.*p.p.*

Proof. By Lemma 1, we can easily see that Y_1 has f.p.p. Now, define F: $Y_1 \to Y_1$ by $F(x, y) = (f(x), 0)$, where $x \in X$ and $y \in I$. Then F is continuous, and the components of the fixed point set of F are $C \times \{0\}$ and $C_\alpha \times \{0\}$, where $\{C_\alpha\}$ is the collection of all components of the fixed point set of f other than C. We will show that neither $C \times \{0\}$ nor $C_\alpha \times \{0\}$ is essential.

1) Since C is not an essential component of the fixed point set of f, there exists an open set $U \supset C$ such that for any $\delta > 0$ there exists a mapping f': $X \to X$ satisfying

(i) $|f'-f| < \delta$, and

(ii) f' has no fixed point in U.

Define F': $Y_1 \to Y_1$ by $F'(x, y) = (f'(x), 0)$. It is easy to see that F': $Y_1 \to Y_1$ is continuous and satisfies

(i) $|F'-F| < \delta$, and

(ii) F' has no fixed point in $U \times \{0\}$.

Since F' has no fixed point in $Y_1 - (X \times \{0\})$, $C \times \{0\}$ is not an essential component of the fixed point set of F.

2) For any $\delta > 0$ there exists a natural number N such that $1/2^N < \delta$. Define F_N: $Y_1 \to Y_1$ as follows:

$$F_N(x, y) = \begin{cases} (f(x), 0) & \text{for } y \geqslant 1/2^N, \\ (f(x), 1/2^N) & \text{for } y \leqslant 1/2^{N+1}, \\ (p, 1/2^{N-1}-2y) & \text{for } 1/2^{N+1} < y < 1/2^N. \end{cases}$$

It is easy to see that F_N is continuous and its unique fixed point is $(p, 1/(3\cdot 2^{N-1}))$.

Let U_α be an open set such that $U_\alpha \supset C_\alpha \times \{0\}$ and $U_\alpha \cap (\{p\} \times I) = \emptyset$. Then

(i) $|F_N - F| < \delta$, and

(ii) F_N has no fixed point in U_α.

Hence $C_\alpha \times \{0\}$ is not an essential component of the fixed point set of F, which completes the proof.

By using a similar argument to the above, we can show the following:

THEOREM 1'. *Let Y_2 be the quotient space $Y_1/I \times \{p\}$ of Y_1 above. Then Y_2 has f.p.p., but does not have f.*p.p.*

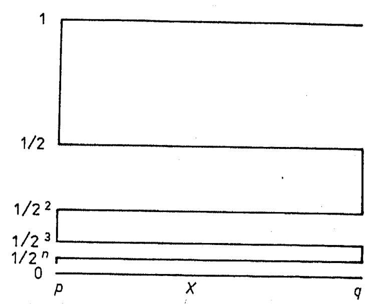

Fig. 3. Y_3

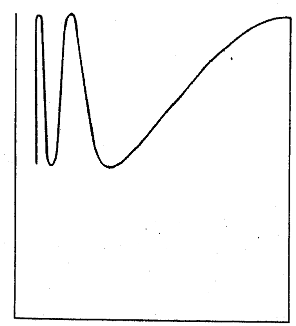
Fig. 4. Y_4

THEOREM 2. *Let X be a continuum which satisfies the following conditions:*

(i) *X has f.p.p., and*

(ii) *there exists a continuous mapping $f\colon X \to X$ such that $f(p) = p$, $f(q) = q$, and $C_p \cap C_q = \emptyset$, where C_p and C_q are the components of the fixed point set of f which contain p and q, respectively.*

Define the subset $Y_3 \subset X \times I$ as follows:

$$Y_3 = (X \times \{0\}) \cup \bigcup_{n=0}^{\infty} (X \times \{1/2^n\}) \cup \bigcup_{n=0}^{\infty} (\{p\} \times [1/2^{2n}, 1/2^{2n+1}])$$

$$\cup \bigcup_{n=0}^{\infty} (\{q\} \times [1/2^{2n+1}, 1/2^{2(n+1)}]).$$

*Then Y_3 has f.p.p., but does not have f*p.p.*

Proof. By Lemma 1, we can easily see that Y_3 has f.p.p. Now, define $F\colon Y_3 \to Y_3$ by $F(x, y) = (f(x), 0)$, where $x \in X$ and $y \in I$. Then F is continuous, and the components of the fixed point set of F are $C_p \times \{0\}$, $C_q \times \{0\}$ and $C_\alpha \times \{0\}$, where $\{C_\alpha\}$ is the collection of all components of the fixed point set of f other than C_p and C_q. We will show that neither $C_p \times \{0\}$, $C_q \times \{0\}$ nor $C_\alpha \times \{0\}$ is essential.

1) Let U_p be an open set such that $U_p \supset C_p \times \{0\}$ and $U_p \cap (\{q\} \times I) = \emptyset$, and U_α such that $U_\alpha \supset C_\alpha \times \{0\}$ and $U_\alpha \cap (\{q\} \times I) = \emptyset$. For any $\delta > 0$ there exists a natural number N with $1/2^{2N-1} < \delta$. Define $F_N\colon Y_3 \to Y_3$ as follows:

$$F_N(x, y) = \begin{cases} (f(x), 1/2^{2N}) & \text{for } y \geqslant 1/2^{2N-1}, \\ (f(x), 1/2^{2N-1}) & \text{for } y \leqslant 1/2^{2N}, \\ (q, 3/2^{2N} - y) & \text{for } 1/2^{2N} < y < 1/2^{2N-1}. \end{cases}$$

It is easy to see that F_N is continuous and its unique fixed point is $(q, 3/2^{2N+1})$. Thus $|F_N - F| < \delta$, and F_N has no fixed point in U_p and U_α. Hence neither $C_p \times \{0\}$ nor $C_\alpha \times \{0\}$ is an essential component of the fixed point set of F.

2) Let U_q be an open set such that $U_q \supset C_q \times \{0\}$ and $U_q \cap (\{p\} \times I) = \emptyset$. Define $F'_N\colon Y_3 \to Y_3$ as follows:

$$F'_N(x, y) = \begin{cases} (f(x), 1/2^{2N+1}) & \text{for } y \geqslant 1/2^{2N}, \\ (f(x), 1/2^{2N}) & \text{for } y \leqslant 1/2^{2N+1}, \\ (p, 3/2^{2N+1} - y) & \text{for } 1/2^{2N+1} < y < 1/2^{2N}. \end{cases}$$

It is easy to see that F'_N is continuous and its unique fixed point is $(p, 3/2^{2N+2})$. Thus again $|F'_N - F| < \delta$, and F'_N has no fixed point in U_q. Hence $C_q \times \{0\}$ is not essential either, which completes the proof.

EXAMPLES. By letting X be the interval I in each of Theorems 1, 1′ and 2, we obtain the examples shown in Figures 1, 2, and 3, respectively.

Remark 1. From Lemma 2 we can also derive that Y_1 and Y_2 have f.p.p. For instance, this lemma can be applied to Y_1 by taking

$$X_n = (\{p\} \times I) \cup (X \times [1/2^n, 0]) \cup \bigcup_{k=0}^{n-1} (X \times \{1/2^k\}),$$

$$f_n(x, y) = \begin{cases} (x, y) & \text{for } y \geqslant 1/2^{n-1}, \\ (x, 0) & \text{for } y \leqslant 1/2^n, \\ (p, 2y - 1/2^{n-1}) & \text{for } 1/2^n < y < 1/2^{n-1}. \end{cases}$$

Remark 2 (f*p.p. and Borsuk's lemma). Using Borsuk's lemma, we have proved that each of Y_1, Y_2 and Y_3 has f.p.p., but by Theorems 1, 1′ and 2, none of them has f*p.p. Hence, in Borsuk's lemma, f.p.p. cannot be replaced by f*p.p. A similar argument is true for Lemma 2.

THEOREM 2′. *Let*

$$C = \{(x, y) \mid y = (1/2)\sin(1/x) + 1/2,\ 0 < x \leqslant 2/\pi\},$$
$$I_1 = \{(0, y) \mid -1 \leqslant y \leqslant 1\},$$
$$I_2 = \{(x, -1) \mid 0 \leqslant x \leqslant 2/\pi\},$$
$$I_3 = \{(2/\pi, y) \mid -1 \leqslant y \leqslant 1\}.$$

*Let $Y_4 = C \cup I_1 \cup I_2 \cup I_3$. Then Y_4 has f.p.p., but does not have f*p.p.*

Remark. It is well known that Y_4 has f.p.p. The space Y_4 is called the *Polish* (or *Warsaw*) *circle*.

Proof. Define $F\colon Y_4 \to Y_4$ by $F(x, y) = (0, y^2)$. Then by a similar argument to Theorem 2, we can see that neither of the two fixed points $(0,0)$ and $(0,1)$ of F is essential.

II. A not locally connected continuum with f*p.p.

LEMMA 3. *Let X and Y be compact metric spaces such that $X \cap Y = \{p\}$. Assume that X has f*p.p. If $f\colon X \cup Y \to X \cup Y$ satisfies $f(p) \in X - p$, then its fixed point set has an essential component.*

Proof. Apply the same argument as in the proof of Theorem 1 in [5].

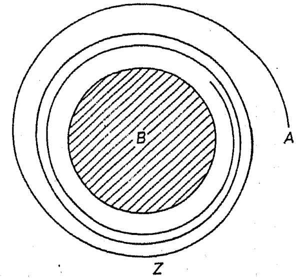

Fig. 5

Now we construct a not locally connected continuum which has both f.p.p. and f.*p.p.

THEOREM 3. *Let A and B be given in polar coordinates by*

$$A = \{(r, \theta) \mid r = 2\pi/\theta + 1,\ \theta \geqslant 2\pi\}, \qquad B = \{(r, \theta) \mid r \leqslant 1\}.$$

*Let $Z = A \cup B$. Then Z has f.*p.p.*

Proof. First we note that Z has f.p.p. by [3]. Now we prove that Z has f.*p.p., i.e. the fixed point set of every mapping $f\colon Z \to Z$ has an essential component.

Case 1: $f(A) \subset A$.

Case 1(a): There exists a point $p = (2\pi/\theta_0 + 1, \theta_0) \in A$ such that $f(p) \in A - A_p$, where $A_p = \{(r, \theta) \mid r = 2\pi/\theta + 1,\ \theta \geqslant \theta_0\}$. It then follows from Lemma 3 that the fixed point set of f has an essential component.

Case 1(b): $f(p) \in A_p$ for every $p \in A$. In this case, $f(\partial B) = \partial B$ and $f(B) \subset B$. Further, it is easy to see that the degree of $f|_{\partial B}\colon \partial B \to \partial B$ is not 0. Then $f(B) = B$. Hence every mapping $f'\colon Z \to Z$ with $|f' - f| < 1$ satisfies $f'(B) \subset B$. Therefore the fixed point set of f has an essential component.

Case 2: $f(A) \subset B$. In this case we have $f(B) \subset B$.

Case 2(a): $f(Z) \cap \operatorname{Int} B \neq \emptyset$. Let $x_0 \in Z$ be such that $f(x_0) \in \operatorname{Int} B$ and let $d_0 = d(f(x_0), \partial B)$. Hence, every mapping $f'\colon Z \to Z$ with $|f' - f| < d_0$ satisfies $f'(B) \subset B$. Therefore the fixed point set of f has an essential component.

Case 2(b): $f(Z) \subset \partial B$. For convenience, we define another metric d^* on A as a half-line as follows:

$$d^*(x, y) = |x - y|^* = |\theta_0 - \theta_1| \quad \text{for } x = (r_0, \theta_0),\ y = (r_1, \theta_1),$$

and let $V_\varrho(C)$ be the ϱ-neighborhood of C in A with metric d^*.

Let $g = f|_{\partial B}\colon \partial B \to \partial B$. First, we note that the degree of g is 0 and hence there exists $x \in \partial B$ with $f(x) \neq x$. Using polar coordinates, define the projection $P\colon A \to \partial B$ by $P(r, \theta) = (1, \theta)$. Let

$$I_n = \{(r, \theta) \mid r = 2\pi/\theta + 1,\ 2n\pi \leqslant \theta < 2(n+1)\pi\}.$$

Choose a point a with $g(a) = a$ and let $P^{-1}(a) = \{a_n\}$, where $a_n \in I_n$. Since the degree of g is 0, there exists $g_n\colon A \to A$, the lift of g with $g_n(a_n) = a_n$. Note that $\lim_{n\to\infty} g_n|_{A_n} = g$, where A_n is the closure of $V_\varrho(g_n(A))$. Let C_n be any component of the fixed point set of $g_n|_{A_n}\colon A_n \to A_n$. Let $x_n \in A_n$ and $x = P(x_n) \in \partial B$. Since $g_n(x_n) \neq x_n$ for x_n with $g(x) \neq x$, $P(C_n) = C$ is a component of the fixed point set of g. Note that every $g'_n\colon A \to A$ with $|g'_n - g_n|^* < \varrho$ ($\varrho < \pi$) is the lift of some $g'\colon \partial B \to \partial B$ with $|g' - g|^* < \varrho$. Hence, if C_n is an essential component of the fixed point set of $g_n|_{A_n}$, $P(C_n) = C$ is an essential component of the fixed set of g, and vice versa.

Let C^* be an essential component of the fixed point set of $f|_B\colon B \to B$. Then C^* is also an essential component of the fixed point set of $g = f|_{\partial B}\colon \partial B \to \partial B$. Therefore, the component C^*_n of the fixed point set of $g_n|_{A_n}$ such that $P(C^*_n) = C^*$ is essential.

We will show C^* is an essential component of the fixed point set of $f\colon Z \to Z$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $f'\colon Z \to Z$ with $|f' - f| < \delta$ has a fixed point in $U_\varepsilon(C^*)$.

Case 2(b)(i): $f'(Z) \subset B$. Since C^* is an essential component of the fixed point set of $f|_B\colon B \to B$, for any $\varepsilon > 0$ there exists $\delta_B > 0$ such that every $f'|_B\colon B \to B$ with $|f'|_B - f|_B| < \delta_B$ has a fixed point in $U_\varepsilon(C^*)$.

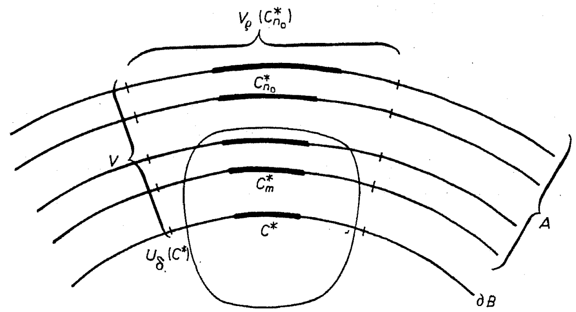

Fig. 6

Case 2(b)(ii): $f'(Z) \subset A$. There exists n_0 such that for every $n \geqslant n_0$, C^*_n is contained in $U_{\varepsilon/2}(C^*)$. Let $\varrho_0 > 0$ be a real number such that $(1/n_0 + 1)\varrho_0 < \varepsilon/2$. Then $V_{\varrho_0}(C^*_n) \subset U_\varepsilon(C^*)$ $(n \geqslant n_0)$. Since C^*_n is an essential component of the fixed point set of $g_n|_{A_n}\colon A_n \to A_n$ and g_n is the lift of g, for every n there exists $\varrho > 0$ such that every $g'_n\colon A \to A$ with $|g'_n|_{A_n} - g_n|_{A_n}|^* < \varrho$ has a fixed point in $V_{\varrho_0}(C^*_n)$. Let $V = \bigcup_{n=n_0}^{\infty} V_\varrho(C^*_n)$ and let $\delta > 0$ be a real number such that $U_\delta(C^*) \cap A \subset V$. By the continuity of f, there exists N such that $|f(x) - f(P(x))| < \delta/4$ for every $n \geqslant N$ and $x \in A_n$.

Now let $\delta_A = \min\{\delta/4, \varkappa\}$ and let f' be a mapping with $|f' - f| < \delta_A$, where $\varkappa = \sup\{d(x, \partial B) \mid x \in A_N\}$. Since $f(C^*_n) \subset \partial B$, we have

$$d(f'(C^*_n), \partial B) \leqslant d(f'(C^*_n), f(C^*_n)) \leqslant |f' - f| < \varkappa.$$

Furthermore, for every $n \geqslant N$ and $x \in A_n$, we have

$$|f'(x) - f(P(x))| \leqslant |f'(x) - f(x)| + |f(x) - f(P(x))| < \delta/4 + \delta/4 = \delta/2.$$

Hence, $d(f'(x), f(C^*)) = d(f'(x), C^*) < \delta/2$ for $x \in C^*_n$. Then, $f'(C^*_n) \subset U_\delta(C^*)$ $(n \geqslant N)$. Since the degree of $Pf'|_{\partial B}$ is 0, there exists $m \geqslant N$ such that $f'(C^*_n) \subset V_\varrho(C^*_n)$ for every $n \geqslant N$. Hence $f'(C^*_m) \subset U_\delta(C^*) \subset V$ and $f'(C^*_m) \subset V_\varrho(C^*_m)$. Since $|f(P(x)) - g_m(x)| < 2|f' - f| < \delta/2$ for $x \in A_m$, we have

$$|f'(x) - g_m(x)| \leqslant |f'(x) - f(x)| + |f(x) - f(P(x))| + |f(P(x)) - g_m(x)|$$
$$< \delta/4 + \delta/4 + \delta/2 = \delta.$$

By the continuity of f', we have $|f'|_{A_m} - g_m|_{A_m}|^* < \varrho$. Hence $f'|_{A_m}\colon A_m \to A_m$ has a fixed point in $V_{\varrho_0}(C_m^*)$. Thus, f' has a fixed point in $V_{\varrho_0}(C_m^*) \subset U_\varepsilon(C^*)$.

From Cases 2(b)(i) and (ii) it follows that every mapping $f'\colon Z \to Z$ with $|f'-f| < \min\{\delta_A, \delta_B\}$ has a fixed point in $U_\varepsilon(C^*)$. This completes the proof.

Remark. While Z has f.*p.p., the cone over Z does not have f.p.p. (see [3]).

Addendum. We can construct an example of a locally connected continuum which has f.p.p. but does not have f.*p.p. Define

$$B_n = \{x \mid (x - 1/2^n)^2 + y^2 \leqslant 1/(3 \cdot 2^n)^2\},$$

$$Y_5 = (\{(0,0)\} \times I) \cup \bigcup_{n=0}^{\infty} (\partial B_n \times I) \cup \bigcup_{n=0}^{\infty} (B_n \times \{0\}).$$

By a similar argument to that of Theorem 1, we can prove that Y_5 has f.p.p. but does not have f.*p.p. Another similar example corresponding to Theorem 2 can also be easily constructed.

References

[1] K. Borsuk, *Einige Sätze über stetige Streckenbilder*, Fund. Math. 18 (1932), 198–214.

[2] S. Kinoshita, *On essential components of the set of fixed points*, Osaka J. Math. 4 (1952), 19–22.

[3] R. J. Knill, *Cones, products and fixed points*, Fund. Math. 60 (1967), 35–46.

[4] K. Kuratowski, *Topology II*, Academic Press, New York 1968.

[5] Y. Yonezawa and S. Kinoshita, *A property of essential components of the fixed point set*, Kwansei Gakuin Univ. Annual Stud. 37 (1988), 147–151.

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Torsion free types

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Abstract. It is shown how the known classification of nonsingular, injective modules M into Types I, II, and III as well as corresponding direct sum decompositions $M = M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}}$ are merely special cases of a more general phenomenon. There is a functor Ξ from rings R to complete Boolean lattices (equivalently Boolean rings) $\Xi(R)$, where each point of $\Xi(R)$ is a class of similar nonsingular modules. Types I, II, III, continuous, discrete, and certain other classes of modules correspond to unique elements of $\Xi(R)$. Appropriate finite sets of disjoint classes of modules induce direct sum decompositions of Ξ as a direct sum of subfunctors. The latter give rise to corresponding direct sum decompositions of nonsingular injective modules M, such as $M = M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}}$.

Introduction. This article will show how the classification of certain torsion free modules into Types I, II, and III ([MN], [K], [B], and [GB]) is a special case of a classification scheme developed in [D]. If A is a unital right R-module and ZA its singular submodule, then the second singular submodule $ZA \subseteq Z_2A \subseteq A$ is defined by $Z[A/ZA] = (Z_2A)/ZA$. A module is *torsion free* if $ZA = Z_2A = 0$, and *torsion* if it equals its *torsion submodule* $Z_2A = A$. This is a continuation of [D] where the following was shown. There exists a contravariant functor Ξ applicable to any associative ring R with identity. The result is a complete Boolean lattice $\Xi(R)$. The functor Ξ classifies or partitions the class of all torsion free right R-modules $\{A, B, \ldots\}$ into equivalence classes $\Xi(R) = \{[A], [B], \ldots\}$ where $A \in [A]$, and $[A]$ consists of a class of modules that are similar, or are like A. An appropriate ring homomorphism $R \to S$ induces a lattice (equivalently Boolean ring) homomorphism $\Xi(S) \to \Xi(R)$.

Goodearl and Boyle ([GB]) extended the Murray–von Neumann–Kaplansky ([MN], [K], and [B]) classification of operator algebras, W^*-algebras, and Baer $*$-rings into Types $\mathrm{I} = \mathrm{I}_f \cup \mathrm{I}_\infty$, $\mathrm{II} = \mathrm{II}_f \cup \mathrm{II}_\infty$, III, abelian, directly finite, and purely infinite to all torsion free injective modules. Here this latter theory is extended to all torsion free modules over any ring by defining M to belong to any of the latter classes if and only if its injective hull EM does (e.g. $M \in \mathrm{III}$ iff $EM \in \mathrm{III}$). In order to obtain necessary and sufficient conditions for M (as opposed to EM) to be of Types I, II, III, abelian etc. (4.2, 4.4, and 4.5), the usual definitions are reformulated without reference to idempotents (3.3, 3.4).

It is shown that there exist unique largest elements [I], [II], [III] $\in \Xi(R)$ which determine Type I, II, and III modules (Corollary 3.16). More specifically, [III] consists