On compositions and products of almost continuous functions

by

Tomasz Natkaniec (Bydgoszcz)

Abstract. (MA) Each function f defined on the interval $I = [0, 1]$ into I which takes every value $y \in I$ in each non-degenerate subinterval J of I is a composition of two almost continuous functions. A real function defined on I is a product of two almost continuous functions iff it has a zero in each subinterval in which it changes sign.

I. Introduction. We shall consider the following classes of real functions. $\mathcal{A}$, $\mathcal{C}onn$ and $\mathcal{D}$ are the classes of almost continuous functions, connectivity functions and Darboux functions, respectively. Let A and B be intervals. $\mathcal{D}^*(A, B)$ denotes the class of all functions $f\colon A \to B$ which take on every value $y \in B$ in each non-degenerate subinterval of A, and $\mathcal{D}^*_0(A, B)$ ($\mathcal{D}^{**}(A, B)$) is the class of all $f \in \mathcal{D}^*(A, B)$ such that for every $y \in B$ and every non-degenerate subinterval $J \subset A$ the set $f^{-1}(y) \cap J$ is countable and non-empty (has cardinality continuum). For $A = B = \mathbf{R}$ (where $\mathbf{R}$ denotes the real line) we shall write simply $\mathcal{D}^*$, $\mathcal{D}^*_0$ and $\mathcal{D}^{**}$. Notice that we have the following inclusions:

$$\mathcal{A} \subset \mathcal{C}onn \subset D, \qquad \mathcal{D}^*_0, \mathcal{D}^{**} \subset \mathcal{D}^* \subset \mathcal{D},$$

and all these inclusions are proper.

II. It is obvious that $\mathcal{D}$ is closed under composition, unlike $\mathcal{A}$ [5]. Since $\mathcal{A} \subset \mathcal{D}$, the class of all compositions of two almost continuous functions is included in $\mathcal{D}$. The foregoing also suggests the following question.

PROBLEM. Is each Darboux function a composition of (two) almost continuous functions? ([7], independently [10], and [2] for connectivity functions).

The general answer is unknown to the author. There are, however, some partial results.

J. Ceder [2] proved that under the Martin Axiom there exists a function f defined on $I = [0, 1]$ into I which is not a connectivity function but is a composition of two connectivity functions. The proof of Ceder's theorem yields the following result.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 26A15; Secondary 54C08.

FACT 1. (MA) *Every* $f\in\mathscr{D}_0^*(I, I)$ *is a composition of two almost continuous functions.*

Moreover, the following theorem is proved in [10].

FACT 2. (MA) *Every* $f\in\mathscr{D}^{**}(I, I)$ *is a composition of two almost continuous functions.*

The purpose of the first part of the present paper is to obtain the analogous result for $\mathscr{D}^*(I, I)$.

Now we recall the definition and some information concerning almost continuity. A function $f\colon X\to Y$ is *almost continuous* if for each open set $G\subset X\times Y$ such that $f\subset G$ there exists a continuous $g\colon X\to Y$ such that $g\subset G$. (We make no distinction between a function and its graph.) f is a *connectivity function* if the restriction of f to C is connected whenever C is connected [12].

A closed set $K\subset X\times Y$ is called a *blocking set* for a function $f\colon X\to Y$ iff $K\cap f=\emptyset$ and $K\cap g\neq\emptyset$ for every continuous function $g\colon X\to Y$. Clearly f is almost continuous iff it has no blocking set. Every blocking set of a function defined on an interval J of $\mathbf{R}$ into $\mathbf{R}$ contains an irreducible blocking set whose x-projection is a non-degenerate subinterval of J [5]. Thus if a function f defined on an interval J into an interval K intersects each closed set in $J\times K$ whose x-projection contains a non-degenerate interval, then f is almost continuous.

If A is a planar set, we denote its x-projection by $\operatorname{dom}(A)$.

In the proof of Lemma 1 we shall assume that the union of less than $\mathfrak{c}$ (continuum) nowhere dense sets is a set of the first category. This is a consequence of Martin's Axiom or of the Continuum Hypothesis [11].

LEMMA 1. (MA) *Let J be an open subinterval of $I=[0, 1]$. If $f\in\mathscr{D}^*(J, I)$ satisfies either*

(*) *for each $y\in I$, $f^{-1}(y)$ is of the first category,*

or

(**) *there exists $y\in I$ for which $f^{-1}(y)$ is of the second category at every $x\in J$,*

then f is the composition of two almost continuous functions: g in $\mathscr{D}^*(J, J)$ and h in $\mathscr{D}^*(J, I)$.

Proof. We shall employ some ideas from [2]. Let $\{K_\alpha\colon \alpha<\mathfrak{c}\}$ and $\{K'_\alpha\colon \alpha<\mathfrak{c}\}$ be well-ordered families of all closed subsets of $J\times J$ and $J\times I$ such that $\operatorname{dom}(K_\alpha)$ and $\operatorname{dom}(K'_\alpha)$ are non-degenerate intervals for each $\alpha<\mathfrak{c}$. For every ordinal $\alpha<\mathfrak{c}$ we shall choose (by induction) sequences of points $(x_\alpha, y_\alpha)\in K_\alpha$, $(v_\alpha, w_\alpha)\in K'_\alpha$, sets A_α, B_α, C_α, $D_\alpha\subset J$ and functions g_α, h_α, k_α, t_α for which the following conditions will be satisfied:

(1) If $L_\alpha=K_\alpha\cap\bigcup_{\beta<\alpha}(g_\beta\cup k_\beta)\neq\emptyset$ then $(x_\alpha, y_\alpha)\in L_\alpha$ and $A_\alpha=B_\alpha=\emptyset=g_\alpha=h_\alpha$. If $L_\alpha=\emptyset$ then:

(a) $(x_\alpha, y_\alpha)\in K_\alpha$, $x_\alpha\notin\bigcup_{\beta<\alpha}(A_\beta\cup C_\beta)$, $y_\alpha\notin\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)$.

(b) $A_\alpha\cap\bigcup_{\beta<\alpha}(A_\beta\cup C_\beta)=\emptyset$, A_α is a countable dense subset of J, $x_\alpha\in A_\alpha$ and $A_\alpha\subset f^{-1}(f(x_\alpha))$.

(c) B_α is a countable dense subset of J, $y_\alpha\in B_\alpha$ and $B_\alpha\cap\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)=\emptyset$.

(d) Let $(A_{\alpha,n})_{n=0}^{\infty}$ be a partition of A_α into countably many disjoint sets each of which is dense in J, let $x_\alpha\in A_{\alpha,0}$ and let $(b_{\alpha,n})_{n=0}^{\infty}$ be the sequence of all elements of B_α with $b_{\alpha,0}=y_\alpha$. We define $g_\alpha\colon A_\alpha\to B_\alpha$ by $g_\alpha(x)=b_{\alpha,n}$ for $x\in A_{\alpha,n}$, $n=0, 1, \ldots$, and a function h_α on B_α by $h_\alpha(x)=f(x_\alpha)$ for each $x\in B_\alpha$.

(2) If $M_\alpha=K'_\alpha\cap\left(\bigcup_{\beta<\alpha}(h_\beta\cup t_\beta)\cup h_\alpha\right)\neq\emptyset$ then $(v_\alpha, w_\alpha)\in M_\alpha$ and $C_\alpha=D_\alpha=\emptyset=k_\alpha=t_\alpha$. If $M_\alpha=\emptyset$ then:

(e) $(v_\alpha, w_\alpha)\in K'_\alpha$, $v_\alpha\notin\bigcup_{\beta\leqslant\alpha}B_\beta\cup\bigcup_{\beta<\alpha}D_\beta$.

(f) C_α is a countable dense subset of J, $C_\alpha\subset f^{-1}(w_\alpha)$ and $C_\alpha\cap\left(\bigcup_{\beta\leqslant\alpha}A_\beta\cup\bigcup_{\beta<\alpha}C_\beta\right)=\emptyset$.

(g) D_α is a countable dense subset of J, $D_\alpha\cap\left(\bigcup_{\beta\leqslant\alpha}B_\beta\cup\bigcup_{\beta<\alpha}D_\beta\right)=\emptyset$ and $v_\alpha\in D_\alpha$.

(h) Let $(C_{\alpha,n})_{n=0}^{\infty}$ be a decomposition of C_α into countably many disjoint sets each of which is dense in J and let $(d_{\alpha,n})_{n=0}^{\infty}$ be the sequence of all points of D_α with $d_{\alpha,0}=v_\alpha$. We define $k_\alpha\colon C_\alpha\to D_\alpha$ by $k_\alpha(x)=d_{\alpha,n}$ for $x\in C_{\alpha,n}$, $n=0, 1, \ldots$, and $t_\alpha\colon D_\alpha\to\{w_\alpha\}$ by $t_\alpha(x)=w_\alpha$ for $x\in D_\alpha$.

Such a choice is possible. Indeed, assume that $L_\alpha=\emptyset$. First notice that $K_\alpha\cap(J\times\{\lambda\})$ is nowhere dense in $J\times\{\lambda\}$ for each $\lambda\in\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)$. Indeed, otherwise there exists a non-empty subinterval $J_1\subset J$ such that $J_1\times\{\lambda\}\subset K_\alpha$. Let $\beta=\min\{\gamma\colon \lambda\in B_\gamma\cup D_\gamma\}$. If $\lambda\in B_\beta$ ($\lambda\in D_\beta$) then it follows from the density of $\operatorname{dom}(g_\beta)$ ($\operatorname{dom}(k_\beta)$) that $g_\beta\cap K_\alpha\neq\emptyset$ ($k_\beta\cap K_\alpha\neq\emptyset$) and consequently $L_\alpha\neq\emptyset$, a contradiction. Thus $\operatorname{dom}(K_\alpha\cap(I\times\{\lambda\}))$ is nowhere dense in $\operatorname{dom}(K_\alpha)$ and using MA we find that

$$E_\alpha=\operatorname{dom}(K_\alpha)\setminus\bigcup_{\beta<\alpha}\left(\operatorname{dom}(K_\alpha\cap(I\times(B_\beta\cup D_\beta)))\cup A_\beta\cup C_\beta\right)$$

is residual in $\operatorname{dom}(K_\alpha)$.

Now we choose $x_\alpha\in E_\alpha$ as follows. We consider two cases. If f satisfies (*) then

$$F_\alpha=E_\alpha\setminus\bigcup_{\beta<\alpha}\left(f^{-1}(f(x_\beta))\cup f^{-1}(w_\beta)\right)\neq\emptyset$$

and we can choose $x_\alpha\in F_\alpha$. If f satisfies (**) then there exists $y\in I$ for which $f^{-1}(y)$ is of the second category at each $x\in J$. Let $G=J\cap f^{-1}(y)$. This set is of the second category in $\operatorname{dom}(K_\alpha)$ and therefore $G\cap E_\alpha\neq\emptyset$. We choose $x_\alpha\in G\cap E_\alpha$. Since $\operatorname{card}\left(\bigcup_{\beta<\alpha}(A_\beta\cup C_\beta)\right)<\mathfrak{c}$, using MA we conclude that $H_\alpha=G\setminus\bigcup_{\beta<\alpha}(A_\beta\cup C_\beta)$ is dense in J and we can choose a countable set $A_\alpha\subset H_\alpha$ dense in J with $x_\alpha\in A_\alpha$. Then $A_\alpha\subset f^{-1}(f(x_\alpha))$. Fix $y_\alpha\in J$ with $(x_\alpha, y_\alpha)\in K_\alpha$. Since $x_\alpha\in E_\alpha$, $y_\alpha\notin\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)$. Since $\operatorname{card}\left(\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)\right)<\mathfrak{c}$, there exists a countable subset $B_\alpha\subset J\setminus\bigcup_{\beta<\alpha}(B_\beta\cup D_\beta)$ dense in J which contains y_α. Then (x_α, y_α) and A_α, B_α satisfy (1).

Similarly we can prove that there exist (v_α, w_α), C_α, D_α for which (2) holds. Assume that $M_\alpha=\emptyset$. As above we prove that $\operatorname{dom}(K'_\alpha\cap(J\times\{\lambda\}))$ is nowhere dense in $\operatorname{dom}(K'_\alpha)$ for each $\lambda\in\{f(x_\gamma), w_\beta\colon \beta<\alpha, \gamma\leqslant\alpha\}$. Thus

$$H_\alpha=\operatorname{dom}(K'_\alpha)\setminus\left[\bigcup_{\beta\leqslant\alpha}\operatorname{dom}(K'_\alpha\cap(I\times\{f(x_\beta)\}))\cup\bigcup_{\beta<\alpha}\operatorname{dom}(K'_\alpha\cap(I\times\{w_\beta\}))\right]$$

is residual in $\mathrm{dom}(K'_\alpha)$. Let $(v_\alpha, w_\alpha)\in K'_\alpha$ and $v_\alpha\in H_\alpha$. Then $w_\alpha \neq f(x_\beta)$ for $\beta \leqslant \alpha$ and $w_\alpha \neq w_\beta$ for $\beta < \alpha$. Choose a countable set $D_\alpha \subset J\setminus(\bigcup_{\beta\leqslant\alpha} B_\beta \cup \bigcup_{\beta<\alpha} D_\beta)$ dense in J such that $v_\alpha\in D_\alpha$. Since $f\in\mathcal{D}^*(J, I)$, $f^{-1}(w_\alpha)$ is dense in J, and we choose a countable set $C_\alpha \subset f^{-1}(w_\alpha)$ dense in J. Since $f^{-1}(w_\alpha)\cap(\bigcup_{\beta\leqslant\alpha} A_\beta \cup \bigcup_{\beta<\alpha} C_\beta) = \emptyset$, C_α is disjoint from each A_β, $\beta\leqslant\alpha$, and each C_β, $\beta<\alpha$. This finishes the choice of C_α, D_α, (v_α, w_α) satisfying (2).

Now observe that for each $x\in J$ we have either $f(x) = w_\alpha$ for some $\alpha < \mathfrak{c}$ or $f(x) = f(x_\alpha)$ for some $\alpha<\mathfrak{c}$. In fact, for a fixed $x\in J$ there exists $\alpha<\mathfrak{c}$ such that $K'_\alpha \subset J\times\{f(x)\}$. Then either $f(x) = f(x_\beta)$ for some $\beta\leqslant\alpha$ or $f(x) = w_\beta$ for some $\beta\leqslant\alpha$. Let $y(z) = v_\alpha$ if $f(z) = w_\alpha$ and $y(z) = y_\alpha$ if $f(z) = f(x_\alpha)$. Now define

$$g(x) = \begin{cases} g_\alpha(x) & \text{for } x\in\mathrm{dom}(g_\alpha), \\ k_\alpha(x) & \text{for } x\in\mathrm{dom}(k_\alpha), \end{cases}\quad \alpha<\mathfrak{c}, \qquad g(x) = y(x) \text{ for } x\notin\bigcup_{\alpha<\mathfrak{c}}(\mathrm{dom}(g_\alpha)\cup\mathrm{dom}(k_\alpha)),$$

$$h(x) = \begin{cases} h_\alpha(x) & \text{for } x\in\mathrm{dom}(h_\alpha), \\ t_\alpha(x) & \text{for } x\in\mathrm{dom}(t_\alpha), \end{cases}\quad \alpha<\mathfrak{c}, \qquad h(x) = 0 \text{ for } x\notin\bigcup_{\alpha<\mathfrak{c}}(\mathrm{dom}(h_\alpha)\cup\mathrm{dom}(t_\alpha)).$$

Then $f = h\circ g$. Since g intersects each blocking set in $J\times J$, it is almost continuous. If J_1 is a non-degenerate closed subinterval of J and if $y\in J$, then $g\cap(J_1\times\{y\})\neq\emptyset$. Thus $g\in\mathcal{D}^*(J, J)$, and g is almost continuous. Similarly, h is almost continuous and $h\in\mathcal{D}^*(J, I)$. This finishes the proof.

LEMMA 2. *Let $f\colon I\to\mathbf{R}$ be almost continuous. If $g\colon I\to\mathbf{R}$ is continuous then $f+g$ is almost continuous* [10], [4].

LEMMA 3. *Assume that C is a closed nowhere dense subset of I, $\{0, 1\}\subset C$ and let $(I_n)_{n=1}^\infty$ be the sequence of all components of the complement of C. If $f\colon I\to\mathbf{R}$ satisfies $f(x) = 0$ for each $x\in C$, $0\in C^-(f|(I\setminus C), x)$ for $x\in C\setminus\{0\}$, $0\in C^+(f|(I\setminus C), x)$ for $x\in C\setminus\{1\}$ and $f|I_n$ is almost continuous for each $n\in\mathbf{N}$, then f is almost continuous.*

The symbols $C^+(f, x)$ and $C^-(f, x)$ denote the unilateral cluster sets of f at x.

Proof. Let G be open in $I\times\mathbf{R}$ such that $f\subset G$. For each $x\in C$ we define open intervals U_x in I, V_x in $\mathbf{R}$ such that $(x, 0)\in U_x\times V_x\subset \bar{U}_x\times\bar{V}_x\subset G$ and the end points of U_x (other than 0 and 1) do not belong to C. Since C is compact, there exists a finite covering $U_{x_1}, \ldots, U_{x_m}$ of C. Let $V = \bigcap_{i=1}^m V_{x_i}$. Now let $U_0, \ldots, U_n$ be the components of $\bigcup_{i=1}^m \bar{U}_{x_i}$ such that if $x\in U_i$, $y\in U_j$ for $i<j$, then $x<y$. Then $C\subset U_0\cup\ldots\cup U_n$ and $\bar{U}_i\cap\bar{U}_j = \emptyset$ for $i\neq j$. Moreover, $0\in U_0$, $1\in U_n$ and the other end points of the U_i $(0\leqslant i\leqslant n)$ do not belong to C. For each $i = 0, \ldots, n$ we choose $[a_{2i}, a_{2i+1}]\subset U_i$ such that $C\cap U_i = C\cap[a_{2i}, a_{2i+1}]$ and $a_{2i}, a_{2i+1}\in C$. (It can happen that $a_{2i} = a_{2i+1}$.) Additionally, set $a_0 = 0$ and $a_{2n+1} = 1$. Then for each $i = 0, \ldots, n-1$, (a_{2i+1}, a_{2i+2}) is a component of $I\setminus C$. Thus $f|(a_{2i+1}, a_{2i+2})$ is almost continuous. Since $0\in C^+(f, a_{2i+1})\cap C^-(f, a_{2i+2})$, $f|[a_{2i+1}, a_{2i+2}]$ is almost continuous ([4], Lemma 3.4). Then there exists a continuous function $g_i\colon [a_{2i+1}, a_{2i+2}]\to\mathbf{R}$ such that $g_i\subset G$ and $g_i(a_{2i+1}) = g_i(a_{2i+2}) = 0$ ([4], Lemma 3.3). The function $g\colon I\to\mathbf{R}$ defined as $g = \bigcup_{i=0}^{n-1} g_i\cup(I\setminus\bigcup_{i=0}^n(a_{2i}, a_{2i+1}))\times\{0\}$ is continuous and contained in G, which completes the proof.

LEMMA 4. *Assume that J is a closed interval (or $J = \mathbf{R}$), C is a closed nowhere dense subset of I, $\{0, 1\}\subset C$ and $(I_n)_{n=1}^\infty$ is the sequence of all components of $I\setminus C$. If $f\colon I\to J$ is such that $f|I_n$ is almost continuous for each $n = 1, 2, \ldots$ and if $C^+(f, c_n) = C^-(f, d_n) = J$ where $I_n = (c_n, d_n)$, $n = 1, 2, \ldots$, then f is almost continuous.*

Proof. Let G be open in $I\times J$ such that $f\subset G$. For each $x\in C$ we define open intervals $U_x\subset I$, $V_x\subset J$ such that $(x, f(x))\in U_x\times V_x\subset\bar{U}_x\times\bar{V}_x\subset G$ and the end points of U_x (other than 0 and 1) do not belong to C. Let $\{x_1, \ldots, x_n\}$ be a subset of C such that $C\subset U_{x_1}\cup\ldots\cup U_{x_n}$. We can assume that if $\bar{U}_{x_i}\cap\bar{U}_{x_j}\neq\emptyset$ then $\bar{U}_{x_i}\subset\bar{U}_{x_j}$ or $\bar{U}_{x_j}\subset\bar{U}_{x_i}$ (we can partition some U_{x_i} if necessary). Now we choose a sequence $U_0, \ldots, U_k$ of sets from the covering $\{\bar{U}_{x_i}\colon 1\leqslant i\leqslant n\}$ such that $U_i = [c_i, d_i]$, $c_0 = 0$, $d_k = 1$, $d_i < c_j$ for $i<j$ and $[d_i, c_{i+1}]$ is contained in some component $I_{n_i} = (a_{n_i}, b_{n_i})$ of $I\setminus C$. Then $a_{n_i}\in U_i$ and $b_{n_i}\in U_{i+1}$. For every i we choose $k(i)\leqslant n$ such that $U_i\subset U_{x_{k(i)}}$. Let $V_i = V_{x_{k(i)}}$ for $i = 0, \ldots, k$. Since $V_i\subset C^+(f, a_{n_i})$ and $V_{i+1}\subset C^-(f, b_{n_i})$, there exist t_i, v_i such that $t_i\in U_i$, $t_i > a_{n_i}$, $v_i\in U_{i+1}$, $v_i < b_{n_i}$, $f(t_i)\in V_i$ and $f(v_i)\in V_{i+1}$. Let

$$G_i = (G\cap(I_{n_i}\times J))\setminus(\{t_i\}\times(J\setminus V_i)\cup\{v_i\}\times(J\setminus V_{i+1})).$$

This is an open neighbourhood of $f|I_{n_i}$ and it follows from the almost continuity of $f|I_{n_i}$ that there exists a continuous function $g_i\colon I_{n_i}\to J$ contained in G_i. Then $g_i(t_i)\in V_i$ and $g_i(v_i)\in V_{i+1}$. We define $g\colon I\to J$ as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x\in\{0, 1\}, \\ g_i(x) & \text{for } x\in[t_i, v_i],\ i = 0, \ldots, k-1, \\ \text{linear on each interval: } [0, t_0], [v_k, 1] \text{ and} \\ \qquad [v_i, t_{i+1}],\ i = 1, \ldots, k-1. \end{cases}$$

This function is continuous and contained in G, which finishes the proof.

THEOREM 1. (MA) *Each $f\in\mathcal{D}^*(I, I)$ is a composition of two almost continuous functions.*

Proof. Let $f\in\mathcal{D}^*(I, I)$ and let $A = A_1\cup A_2$, where A_1 is the union of all open intervals $J\subset I$ such that for each $y\in I$ the set $f^{-1}(y)\cap J$ is of the first category and A_2 is the union of all open intervals $J\subset I$ for which there exists $y\in I$ such that $f^{-1}(y)\cap J$ is of the second category at each $x\in J$. Observe that A is open and dense in I. Indeed, assume that a non-empty open interval $K\subset I$ is disjoint from A_1. Then there exists $y\in I$ for which $K\cap f^{-1}(y)$ is of the second category. Let $Z = \{x\in K\cap f^{-1}(y)\colon K\cap f^{-1}(y)$ is of the second category at $x\}$. This set is of the second category and therefore it is dense in some non-empty subinterval $J\subset K$ (see e.g. [8], pp. 51–52). It is obvious that $f^{-1}(y)\cap J$ is of the second category at each $x\in J$ and consequently, $K\cap A_2\neq\emptyset$. Thus $C = I\setminus A$ is closed and nowhere dense. Let $(I_n)_{n=1}^\infty$ be the sequence of all components of A. Then each $f|I_n$ satisfies either (*) or (**) of Lemma 1. It follows from that lemma that for each $n\in\mathbf{N}$ there exist almost continuous functions $g_n\in\mathcal{D}^*(I_n, I_n)$ and $h_n\in\mathcal{D}^*(I_n, I)$ such that $f|I_n = h_n\circ g_n$.

Now we define $g, h\colon I \to I$ as follows:

$$g(x) = \begin{cases} x & \text{for } x \in C, \\ g_n(x) & \text{for } x \in I_n,\ n = 1, 2, \ldots, \end{cases}$$

$$h(x) = \begin{cases} f(x) & \text{for } x \in C, \\ h_n(x) & \text{for } x \in I_n,\ n = 1, 2, \ldots \end{cases}$$

Of course, $f = h \circ g$. By Lemma 3, $g_1 = g - \mathrm{id}$ is almost continuous. The almost continuity of $g = g_1 + \mathrm{id}$ and h follows from Lemmas 2 and 4, respectively.

Since every almost continuous function defined on I is a connectivity function [12], we obtain

COROLLARY. *Every $f \in \mathcal{D}^*(I, I)$ is a composition of two connectivity functions.*

III. It is well known that each real-valued function defined on an interval can be expressed as a sum of two almost continuous functions [6]. On the other hand, a general function may not be a product of Darboux functions [9] and therefore, it may not be a product of almost continuous functions. J. Ceder proved in [1], [3] that f is a product of two Darboux functions iff it has the following property:

(JC) f has a zero in each subinterval in which it changes sign.

In this section we shall prove that every function f with the property (JC) is a product of two almost continuous functions. From this theorem and the theorem of Ceder the following corollaries follow immediately:

(1) *f is a product of two almost continuous functions iff f has the property* (JC)

and (since every almost continuous function is a connectivity function)

(2) *f is a product of two connectivity functions iff it has the property* (JC).

We begin with several lemmata.

LEMMA 5. *If g and h are almost continuous functions on an open interval J then there exists a continuous function s such that $g+s$ and $h-s$ are almost continuous with cluster sets at the end points of J equal to $[-\infty, \infty]$.*

Proof. Let $J = (a, b)$. We can choose a sequence $(J_n)_n$ of pairwise disjoint, open subintervals of J such that:

(i) $J_n = (a_n, b_n)$ and $c_n = (a_n + b_n)/2$ for each $n \in \mathbf{N}$,

(ii) $J_{2n} \searrow a$ and $J_{2n+1} \nearrow b$,

(iii) if $\bar{J}_n \cap \bar{J}_m \neq \emptyset$ then $n = m$.

Let s be the function defined by

$$s(x) = \begin{cases} 0 & \text{for } x \notin \bigcup_n J_n, \\ |g(x)| + n & \text{for } x = c_n,\ r(n) = 0, 1, \\ -|g(x)| - n & \text{for } x = c_n,\ r(n) = 2, 3, \\ |h(x)| + n & \text{for } x = c_n,\ r(n) = 4, 5, \\ -|h(x)| - n & \text{for } x = c_n,\ r(n) = 6, 7, \\ \text{linear on } [a_n, c_n], [c_n, b_n], & n = 0, 1, \ldots \end{cases}$$

where $r(n)$ is the remainder of the division of n by 8.

Then s is continuous, and $\sup C^+(g+s, a) = \sup C^-(g+s, b) = \infty$ and $\inf C^+(g+s, a) = \inf C^-(g+s, b) = -\infty$. Since the sum of an almost continuous function and a continuous function is almost continuous [4], [10], $g+s$ is almost continuous and its cluster sets at a and b are intervals, i.e. $C^+(g+s, a) = C^-(g+s, b) = [-\infty, \infty]$. Similarly we can verify that $h-s$ is almost continuous and $C^+(h-s, a) = C^-(h-s, b) = [-\infty, \infty]$.

LEMMA 6. *Let f be a function defined on an open interval J. If $f > 0$ (or $f < 0$) on J, then there exist almost continuous functions g, h such that $f = g \cdot h$ on J, the cluster sets of g at the end points of J are $[0, \infty]$ if $f > 0$ and $[-\infty, 0]$ if $f < 0$, and the cluster sets of h at the end points of J are $[0, \infty]$.*

Proof. Assume that $f > 0$ on J. Then $\ln(f)$ is the sum of two almost continuous functions g_1 and h_1 on J. By Lemma 5 we can take g_1 and h_1 to have cluster sets $[-\infty, \infty]$ at the end points of J. Then $f = \exp(g_1) \cdot \exp(h_1)$ and the functions $g = \exp(g_1)$ and $h = \exp(h_1)$, being the compositions of the almost continuous functions g_1 and h_1 with the continuous function exp, are almost continuous [5]. Moreover, the cluster sets of g and h are $[0, \infty]$ at the end points of J.

When $f < 0$ carry out the above argument for $-f$.

LEMMA 7. *Let f be a real function defined on an open interval $J = (a, b)$ having the property* (JC) *and let $A = [f = 0]$ be a discrete set (i.e. $A^d \subset \{a, b\}$). Then there exist almost continuous functions g, h such that $f = g \cdot h$ and:*

(7.1) *if $a \notin \bar{A}$ and K is a component of $J \setminus \bar{A}$ with $a \in \bar{K}$, then $C^+(g, a) = C^+(h, a) = [0, \infty]$ if $f > 0$ on K and $C^+(g, a) = [-\infty, 0]$, $C^+(h, a) = [0, \infty]$ if $f < 0$ on K,*

(7.2) *if $a \in \bar{A}$ then there exists a sequence (c_n) in A such that $c_n \searrow a$, $C^+(g, c_{2n}) = C^+(h, c_{2n+1}) = [0, \infty]$ and $C^+(g, c_{2n+1}) = C^+(h, c_{2n}) = [-\infty, 0]$, or $C^+(g, c_{2n}) = C^+(h, c_{2n}) = [0, \infty]$ and $C^+(g, c_{2n+1}) = C^+(h, c_{2n+1}) = [-\infty, 0]$ for each $n \in \mathbf{N}$ (hence $C^+(g, a) = C^+(h, a) = [-\infty, \infty]$),*

(7.3) *the analogous left-hand conditions hold for b.*

Proof. There are four possible cases: $a, b \notin \bar{A}$; $a \notin \bar{A}$ and $b \in \bar{A}$; $a \in \bar{A}$ and $b \notin \bar{A}$; and $a, b \in \bar{A}$. We shall consider e.g. the case when $a \notin \bar{A}$ and $b \in \bar{A}$. The set A can be arranged in an increasing sequence (a_n) which converges to b. Put $a_0 = a$, $J_n = (a_{n-1}, a_n)$ and $f_n = f | J_n$ for $n \in \mathbf{N}$. Then there exists an increasing sequence (k_n) such that $k_1 > 1$ and either $f_{k_n} > 0$ for each $n \in \mathbf{N}$ or $f_{k_n} < 0$ for each n. We put $c_n = a_{k_n}$ for $n = 1, 2, \ldots$ By Lemma 6 for each $n \in \mathbf{N}$ there exist almost continuous functions v_n and w_n such that $|f_{k_n}| = v_n \cdot w_n$ and $C^-(v_n, c_n) = C^-(w_n, c_n) = (-1)^n \cdot [0, \infty]$ (i.e. $[-\infty, 0]$ if n is odd and $[0, \infty]$ if n is even) when $f_{k_n} > 0$ and $C^-(v_n, c_n) = C^-(w_n, c_n) = (-1)^n \cdot [-\infty, 0]$ when $f_{k_n} < 0$. Next we define $g_{k_n} = (\operatorname{sgn} f_{k_n}) \cdot v_n$ and $h_{k_n} = w_n$. For $m \notin \{k_n\colon n \in \mathbf{N}\}$ we choose almost continuous functions g_m, h_m which satisfy the condition of Lemma 6. Set

$$g(x) = \begin{cases} g_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}, \\ 0 & \text{for } x \in A; \end{cases} \qquad h(x) = \begin{cases} h_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}, \\ 0 & \text{for } x \in A. \end{cases}$$

Then $f = g \cdot h$. Lemma 3 yields the almost continuity of g and h. Moreover, it is obvious that g and h have the properties (7.1)–(7.3).

For every interval K and each function f on K let $\alpha(f, K)$ denote the following statement.

$\alpha(f, K)$ there exist four points $y_{K,0}, y_{K,1}, y_{K,2}, y_{K,3} \in K$ such that $y_{K,0} < y_{K,1} < y_{K,2} < y_{K,3}$ and $C^+(f, y_{K,0}) = C^-(f, y_{K,2}) = [0, \infty]$, $C^+(f, y_{K,1}) = C^-(f, y_{K,3}) = [-\infty, 0]$.

Observe that in Lemma 7, if the set A is infinite, then $\alpha(g, J)$ and $\alpha(h, J)$ hold.

LEMMA 8. *Assume that C is a non-empty, closed and nowhere dense subset of a closed interval $J = [a, b]$, $(J_n)_n$ is the sequence of all components of $J \setminus C$ and f is a real function defined on J. Let $\mathcal{K}$ denote the family of all components J_n for which the condition $\alpha(f, J_n)$ holds. Let f have the following properties*:

(8.1) *$f|J_n$ is almost continuous for each n with cluster sets at the end points of J_n equal to $[0, \infty]$ or $[-\infty, 0]$ or $[-\infty, \infty]$*,

(8.2) *if x_0 is the right (left) end point of some J_m, then $f(x_0) \in C^+(f, x_0)$ ($f(x_0) \in C^-(f, x_0)$)*,

(8.3) *if $x_0 \in C$ is the right (left) end point of no component of $J \setminus C$, then there exists a sequence (J_{n_k}) of components of $J \setminus C$ such that $J_{k_k} \nearrow x_0$ ($J_{k_n} \searrow x_0$) and all J_{k_n} belong to $\mathcal{K}$.*

Then f is almost continuous.

Proof. Let $G \subset J \times \mathbf{R}$ be an open set including f. For each $x \in C$ we choose two intervals U_x, V_x such that:

(i) $(x, f(x)) \in U_x \times V_x \subset G$,

(ii) the end points of U_x (other than those of J) do not belong to C,

(iii) if x_0 is the left (right) end point of some component K of $J \setminus C$ then $\sup(U_{x_0}) \in K$ ($\inf(U_{x_0}) \in K$),

(iv) if x_0 is the left (right) end point of no component of $J \setminus C$, then there exists a component $K \in \mathcal{K}$ for which $\inf(K) \in U_{x_0}$, $\sup(U_{x_0}) \in K$ and $y_{K,1} \in U_{x_0}$ (respectively: $\sup(K) \in U_{x_0}$, $\inf(U_{x_0}) \in K$ and $y_{K,2} \in U_{x_0}$).

From the covering $\{U_x: x \in C\}$ of the compact set C we choose a finite subcovering $\{U_{x_0}, \ldots, U_{x_n}\}$. Moreover, we can assume that:

(v) $a \in U_{x_0}$ and $b \in U_{x_n}$,

(vi) if $C \cap U_{x_i} \subset C \cap U_{x_j}$ then $i = j$,

(vii) if $i < j$ then $\sup(U_{x_i}) < \sup(U_{x_j})$.

Now we choose a sequence of pairwise disjoint open intervals $U_0, \ldots, U_n$ such that

(viii) $C \cap \bigcup_{i=0}^n U_i = C \cap \bigcup_{i=0}^n U_{x_i} = C$ and $U_i \subset U_{x_i}$ for $i = 0, \ldots, n$,

(ix) $\varphi(U_i, x_i)$ holds, i.e. there exist

$$c_i \in (\inf(U_i), \min(C \cap U_i)], \quad d_i \in [\max(C \cap U_i), \sup(U_i))$$

such that $C^+(f, d_i) \cap C^-(f, c_i) \supset [0, \infty]$ ($[-\infty, 0]$) if $f(x_i) > 0$ ($f(x_i) < 0$).

First we consider the sets U_{x_0} and U_{x_1}. There are two possible cases.

(a) x_0 is the left end point of a component J_{n_0} and x_1 is the right end point of a component J_{n_1}. Then by (vi) and (vii) it follows that $C \cap (x_0, x_1) = \emptyset$ and consequently $n_0 = n_1$. Set $z = \frac{1}{2}(x_0 + x_1)$.

(b) Either x_0 is the left end point of no component or x_1 is the right end point of no component. Then either the right end point of U_{x_0} or the left end point of U_{x_1} belongs to some component $K \in \mathcal{K}$. Let $z \in (y_{K,1}, y_{K,2})$.

Now we define the intervals $U_0 = U_{x_0} \cap (-\infty, z)$ and $W_1 = U_{x_1} \cap (z, \infty)$. Then $U_0 \subset U_{x_0}$, $W_1 \subset U_{x_1}$, $C \cap (U_0 \cup W_1) = C \cap (U_{x_0} \cap U_{x_1})$, $U_0 \cap W_1 = \emptyset$, $U_0 \cap U_{x_i} = \emptyset$ for $i = 2, \ldots, n$ and moreover, $\varphi(U_0, x_0)$ and $\varphi(W_1, x_1)$ hold. Assume that we have defined intervals $U_0, \ldots, U_k$ ($k < n$) and W_{k+1} such that $U_i \subset U_{x_i}$ for $i = 0, \ldots, k$, $W_{k+1} \subset U_{x_{k+1}}$, $C \cap (U_0 \cup \ldots \cup U_k \cup W_{k+1}) = C \cap (U_{x_0} \cup \ldots \cup U_{x_{k+1}})$, if $i < j$ then $\sup(U_i) \leqslant \inf(U_{i+1})$, $\sup(U_k) \leqslant \inf(W_{k+1})$, and $\varphi(U_i, x_i)$ for $i = 0, \ldots, k$ and $\varphi(W_{k+1}, x_{k+1})$ hold.

If $k+1 = n$ then $U_n = W_{k+1}$. Otherwise we consider W_{k+1} and $U_{x_{k+2}}$. In the same way as in (a) and (b) we define U_{k+1} and W_{k+2}.

Let $V_i = V_{x_i}$ for $i = 0, \ldots, n$. Then it follows from (ix) that $V_i \cap C^-(f, c_i) \neq \emptyset \neq V_i \cap C^+(f, d_i)$ and we can choose $s_i, t_i \in U_i$ such that $s_i < c_i$, $t_i > d_i$ and $f(s_i), f(t_i) \in V_i$. For each $i = 0, \ldots, n-1$, $[t_i, s_{i+1}]$ is included in some component of $J \setminus C$ and consequently, $f_i = f|[t_i, s_{i+1}]$ is almost continuous. Hence there exists a continuous function $g_i \subset G \cap [t_i, s_{i+1}] \times \mathbf{R}$ such that $g_i(t_i) \in V_i$ and $g_i(s_{i+1}) \in V_{i+1}$. We define a function g in the following way: $g(x) = g_i(x)$ for $x \in [t_i, s_{i+1}]$, $i = 0, \ldots, n-1$, and g is linear on each $[s_i, t_i]$, $i = 0, \ldots, n$. Then g is continuous and contained in G. Thus f is almost continuous.

LEMMA 9. *Let C be a subset of a closed interval J which includes no non-empty dense-in-itself subset. Let f be a function on J with the property* (JC) *for which $C = [f = 0]$. Then there exist almost continuous functions g, h such that $f = g \cdot h$ and*

(9.1) *if a is an end point of J then the cluster sets of g and h are $[0, \infty]$ or $[-\infty, 0]$ if $a \notin \overline{C \cap \operatorname{int}(J)}$, and $[-\infty, \infty]$ if $a \in \overline{C \cap \operatorname{int}(J)}$.*

Proof. Let A_0 be the union of all open subintervals K of J such that $K \cap C$ is discrete and infinite and let A_1 be the union of all open subintervals K of $J \setminus \bar{A}_0$ such that $K \cap C$ is finite (or empty). Then $A = A_0 \cup A_1$ is open and since C includes no non-empty dense-it-itself subset, A is dense in J. Consequently, $D = J \setminus A$ is closed, nowhere dense and included in $\bar{C}$. Let $(J_n)_n$ be the sequence of all components of A_0 or A_1. Then:

(i) if $x_0 \in D$ is the left end point of some J_n and the right end point of some J_m then either $x_0 \in \overline{J_n \cap C}$ or $x_0 \in \overline{J_m \cap C}$ (hence either $J_n \subset A_0$ or $J_m \subset A_0$),

(ii) if $x_0 \in D \setminus \{b\}$ is the left end point of no component of $J \setminus D = A$ then there exists a sequence (J_{k_n}) such that $J_{k_n} \subset A_0$ and $J_{k_n} \searrow x_0$,

(iii) the analogous right-hand condition holds for each x_0 in $D \setminus \{a\}$.

The condition (i) follows immediately from the fact that every isolated point of C belongs either to A_0 or to A_1. To verify (ii) let $x_0 \in D \setminus \{b\}$ be the left end point of no component of $J\setminus D$. Suppose that $(x_0, x_0+\delta)\cap A_0 = \emptyset$ for some $\delta > 0$. Then $E = (x_0, x_0+\delta)\cap D \neq \emptyset$ and, by (i), E has no isolated point. Consequently, E is dense-in-itself, a contradiction with the properties of C.

For every n let $f_n = f|J_n$ and let g_n, h_n be almost continuous functions for which $f_n = g_n \cdot h_n$ and the conditions (7.1)–(7.3) hold. Set

$$g(x) = \begin{cases} f(x) & \text{for } x\in D, \\ g_n(x) & \text{for } x\in J_n,\ n\in \mathbf{N}; \end{cases} \qquad h(x) = \begin{cases} 1 & \text{for } x\in D, \\ h_n(x) & \text{for } x\in J_n,\ n\in \mathbf{N}. \end{cases}$$

Observe that every component K of A_0 satisfies the conditions $\alpha(g, K)$ and $\alpha(h, K)$. Thus g and h satisfy all assumptions of Lemma 8 and therefore they are almost continuous. Moreover, it is easy to see that g and h satisfy (9.1).

Remark. If C satisfies the assumptions of Lemma 9 then it is nowhere dense and if C is infinite then A_0 is non-empty and consequently, the conditions $\alpha(g, J)$ and $\alpha(h, J)$ hold.

LEMMA 10. *Let f be a function defined on a closed interval J having the property* (JC). *Suppose that $C = [f = 0]$ is nowhere dense, the maximal dense-in-itself subset D of C is non-empty, and moreover*:

(10.1) *if K is a component of $J\setminus \bar{D}$ then $f|K$ is almost continuous,*

(10.2) *if (a, b) is a component of* $\operatorname{int}[f > 0]$ $(\operatorname{int}[f < 0])$ *then* $C^+(f, a) = C^-(f, b) = [0, \infty]$ $(= [-\infty, 0])$,

(10.3) *if $a\in \bar{C}$ is the left (right) end point of no component of $J\setminus \bar{C}$ then there exists a sequence (K_n) of components of $J\setminus \bar{C}$ such that $K_n \searrow a$ $(K_n \nearrow a)$ and* $\operatorname{sgn}(f|K_n) = (-1)^n$.

Then f is almost continuous.

Proof. Let G be an open neighbourhood of f. For every x from $\bar{D}$ we choose open intervals U_x, V_x such that:

(i) $(x, f(x))\in U_x \times V_x \subset G$,

(ii) the end points of U_x (other than those of J) do not belong to $\bar{C}$,

(iii) if x_0 is the left (right) end point of some component K of $J\setminus\bar{D}$, then $\sup(U_{x_0})\in K$ $(\inf(U_{x_0})\in K)$,

(iv) if $x_0 \in \bar{D}$ is the left (right) end point of some component K of $J\setminus\bar{C}$ then $\sup(U_{x_0})\in K$ $(\inf(U_{x_0})\in K)$,

(v) if $x_0\in\bar{D}$ is the left (right) end point of no component of $J\setminus\bar{C}$ then there exists a component K of $J\setminus\bar{C}$ such that $\sup(U_{x_0})\in K$ $(\inf(U_{x_0})\in K)$ and $f|K > 0$ if $f(x_0) > 0$, $f|K < 0$ if $f(x_0) < 0$.

A choice for $x\in\bar{D}$ of U_x satisfying (v) is possible by (10.3). From the covering $\{U_x\colon x\in\bar{D}\}$ of the compact set $\bar{D}$ we choose a finite subcovering $\{U_{x_0}, \ldots, U_{x_n}\}$. Additionally, we assume that:

(vi) the left end point of J belongs to U_{x_0}, and the right one to U_{x_n},

(vii) if $\bar{D}\cap U_{x_i} \subset \bar{D}\cap U_{x_j}$ then $i = j$,

(viii) if $i < j$ then $\sup(U_{x_i}) < \sup(U_{x_j})$.

Now we define (inductively) a sequence of open intervals $U_0, V_0, U_1, V_1, \ldots, U_m, V_m$ $(n \leqslant m \leqslant 3n)$ such that:

(ix) $\bar{D} \subset \bigcup_{i=0}^m U_i$ and $U_0, \ldots, U_m$ are pairwise disjoint,

(x) $V_i\cap C^+(f, d_i) \neq \emptyset \neq V_i\cap C^-(f, c_i)$ for $i = 0, \ldots, m$, where $c_i = \min(U_i\cap\bar{C})$ and $d_i = \max(U_i\cap\bar{C})$,

(xi) $U_i\times V_i \subset G$ for $i = 0, \ldots, m$.

First we consider U_{x_0} and U_{x_1}. There are two possible cases.

(a) If $f(x_0) = 0$ (or $f(x_1) = 0$) then we put $U_0 = U_{x_0}\cap(-\infty, z)$, $V_0 = V_{x_0}$, $W_1 = U_{x_1}\cap(z, \infty)$, where $z = \inf(U_{x_1})$ (or $z = \sup(U_{x_0})$).

(b) Now let $f(x_0)\cdot f(x_1) \neq 0$. Let $d = \max(U_{x_0}\cap\bar{D})$ and $c = \min(U_{x_1}\cap\bar{D})$. Then $d\neq c$. We shall consider the following subcases.

(b.0) $c > d$. Then (by (vii) and (viii)) $(d, c)\cap D$ is empty and we have the following possibilities.

(b.0.0) $(d, c)\cap C$ is infinite. Then, by (10.3), there exist $z_0, z_1 \in (c, d)\setminus\bar{C}$ such that $z_0 \leqslant z_1$ and $f(z_i) > 0$ $(f(z_i) < 0)$ if $f(x_i) > 0$ $(f(x_i) < 0)$ for $i = 0, 1$. Set $U_0 = U_{x_0}\cap(-\infty, z_0)$, $V_0 = V_{x_0}$, $W_1 = U_{x_1}\cap(z_1, \infty)$.

(b.0.1) $(d, c)\cap C$ is finite. Let K_c and K_d be the components of $(d, c)\setminus C$ which are contiguous to c and d, respectively.

(b.0.1.0) If $\operatorname{sgn}(f|K_d) = \operatorname{sgn}(f(x_0))$ and $\operatorname{sgn}(f|K_c) = \operatorname{sgn}(f(x_1))$ then we put $U_0 = U_{x_0}\cap(-\infty, z_0)$, $V_0 = V_{x_0}$ and $W_1 = U_{x_1}\cap(z_1, \infty)$, where $z_0\in K_d$, $z_1\in K_c$ and $z_0 \leqslant z_1$.

(b.0.1.1) If $\operatorname{sgn}(f|K_d) \neq \operatorname{sgn}(f(x_0))$ then let $z_0 \in D\cap U_d$ (notice that $z_0 < d$). By (10.3) we can choose $t_0, t_1 \in U_{z_0}\cap U_{x_0}\cap U_d$ such that $t_0 < t_1$, $(t_0, t_1)\cap D \neq \emptyset$ and t_0, t_1 belong to components K_0, K_1 of $J\setminus\bar{C}$ for which $\operatorname{sgn}(f|K_0) = \operatorname{sgn}(f(x_0))$ and $\operatorname{sgn}(f|K_1) = \operatorname{sgn}(f|K_d)$. In this case we define $U_0 = U_{x_0}\cap(-\infty, t_0)$, $V_0 = V_{x_0}$, $U_1 = (t_0, t_1)$, $V_1 = V_{z_0}$, $U_2 = (t_1, z)$, where $z\in K_d$, $V_2 = V_d$ and $W_1 = U_{x_1}\cap(z, \infty)$.

(b.0.1.2) The case $\operatorname{sgn}(f|K_c) \neq \operatorname{sgn}(f(x_1))$ is similar to (b.0.1.1).

(b.1) $c < d$. Then $(c, d)\cap D \neq \emptyset$. Let $z_0 \in D\cap(c, d)$. By (10.3) we can choose $t_0, t_1\in U_{z_0}\cap(c, d)$ such that $t_0 < t_1$, $(t_0, t_1)\cap D\neq\emptyset$ and t_0, t_1 belong to components K_0, K_1 of $J\setminus\bar{C}$ for which $f|K_i > 0$ $(f|K_i < 0)$ if $f(x_i) > 0$ $(f(x_i) < 0)$, $i = 0, 1$. In this case we define $U_0 = U_{x_0}\cap(-\infty, t_0)$, $V_0 = V_{x_0}$, $U_1 = (t_0, t_1)$, $V_1 = V_{z_0}$ and $W_1 = U_{x_1}\cap(t_1, \infty)$.

Assume that we have defined intervals $U_0, V_0, U_1, V_1, \ldots, U_{t(k)}, V_{t(k)}$ and W_{k+1} such that

$$D\cap(U_0\cup\ldots\cup U_{t(k)}\cup W_{k+1}) = D\cap(U_{x_0}\cup\ldots\cup U_{x_{k+1}}),$$

$$U_i\times V_i \subset G \quad \text{for } i\leqslant t(k), \qquad W_{k+1}\subset U_{k+1},$$

$$\sup(U_i)\leqslant\inf(U_{i+1}) \quad \text{for } i\leqslant t(k), \qquad \sup(U_{t(k)})\leqslant\inf(W_{k+1}),$$

and (x) holds for $U_0, U_1, \ldots, U_{t(k)}, W_{k+1}$.

If $k+1 = n$ then we put $m = t(k)+1$, $U_m = W_n$ and $V_m = V_{x_n}$. Otherwise we consider W_{k+1} and U_{k+2}, and in the same way as in (a), (b), we define $U_{t(k)+1}$, $V_{t(k)+1}, \ldots, U_{t(k+1)}, V_{t(k+1)}$ (where $0\leqslant t(k+1)-t(k)\leqslant 3$), and W_{k+2}.

In this way we obtain a sequence of intervals which satisfy (ix)–(xi). By (x) we can choose $s_i, v_i \in U_i$ such that $s_i < c_i$, $v_i > d_i$ and $f(s_i), f(v_i) \in V_i$. For each $i < m$, $[v_i, s_{i+1}]$ is included in some component of $J \setminus \bar{D}$ and consequently, by (10.1), $f_i = f|[v_i, s_{i+1}]$ is almost continuous. Thus there exists a continuous function $g_i \subset G \cap [v_i, s_{i+1}] \times \mathbf{R}$ such that $g_i(v_i) \in V_i$ and $g_i(s_{i+1}) \in V_{i+1}$. Let g be defined by $g(x) = g_i(x)$ for $x \in [v_i, s_{i+1}]$, $i = 0, \ldots, m-1$, and linear on each $[s_i, v_i]$ for $i = 0, \ldots, m$. Then g is continuous and included in G. Hence f is almost continuous.

Remark. Under the assumptions of Lemma 10 the condition $\alpha(f, J)$ holds.

LEMMA 11. *Let f be a function on $J = [a, b]$ with the property* (JC). *Suppose that $C = [f = 0]$ is nowhere dense, the maximal dense-in-itself subset D of C is non-empty and either*

$$\bar{D} \subset \overline{\{x \in J \setminus \bar{C}: f(x) > 0\}}, \tag{11.1}$$

or

$$\bar{D} \subset \overline{\{x \in J \setminus \bar{C}: f(x) < 0\}}. \tag{11.2}$$

Then there exist almost continuous functions g, h on J such that $f = g \cdot h$ and moreover, g and h satisfy (10.2) *and* (10.3). (*Hence the conditions $\alpha(g, J)$ and $\alpha(h, J)$ hold.*)

Proof. Let (J_n) be the sequence of all components of $J \setminus \bar{C}$. We choose (by induction) a sequence $(\mathcal{K}_n)$ of finite families of components of $J \setminus \bar{C}$ such that:

(i) if (11.1) holds then $\mathcal{K}_m \subset \{J_n: f|J_n > 0, n \in \mathbf{N}\} \setminus \bigcup_{i<m} \mathcal{K}_i$ for each $m \in \mathbf{N}$,

(ii) if (11.2) holds and (11.1) does not hold then $\mathcal{K}_m \subset \{J_n: f|J_n < 0, n \in \mathbf{N}\} \setminus \bigcup_{i<m} \mathcal{K}_i$ for each $m \in \mathbf{N}$,

(iii) $\bigcup \mathcal{K}_m \subset \{x \in J: \varrho(x, \bar{D}) < 1/m\}$ and $\varrho(x, \bigcup \mathcal{K}_m) < 2/m$ for each $x \in \bar{D}$ and $m \in \mathbf{N}$, where $\varrho(x, A)$ denotes the distance from x to A,

(iv) if (11.1) holds, x_0 is an end point of some component K of $J \setminus \bar{D}$, $x_0 \in \overline{\{x \in K \setminus \bar{C}: f(x) > 0\}}$ and $K_1 \subset K$ for some $K_1 \in \mathcal{K}_{m-1}$ then there exists $K_2 \in \mathcal{K}_m$ such that $K_2 \subset K$ and $\varrho(x_0, K_2) < \varrho(x_0, K_1)$,

(v) the similar condition holds if we assume (11.2).

Let $\mathcal{L}_0 = \bigcup_{i=1}^{\infty} \mathcal{K}_{2i}$ and $\mathcal{L}_1 = \bigcup_{i=0}^{\infty} \mathcal{K}_{2i+1}$. Observe that

(vi) for each $x_0 \in \bar{D}$ there exists a sequence (J_{k_n}) which converges to x_0 such that $J_{k_n} \in \mathcal{L}_0$ if n is even and $J_{k_n} \in \mathcal{L}_1$ if n is odd. Hence $x_0 \in \overline{\bigcup \mathcal{L}_0} \cap \overline{\bigcup \mathcal{L}_1}$.

Moreover,

(vii) if (J_{k_n}) is a sequence of intervals from $\mathcal{L}_0 \cup \mathcal{L}_1$ and (J_{k_n}) converges to x_0, then, by (iii), $x_0 \in \bar{D}$,

(viii) if K is a component of $J \setminus \bar{D}$, a is an end point of K and $a \in \overline{K \cap \bigcup (\mathcal{L}_0 \cup \mathcal{L}_1)}$, then, by (iv), $a \in \overline{K \cap \bigcup \mathcal{L}_i}$, $i = 0, 1$.

For each $J_n \in \mathcal{L}_0 \cup \mathcal{L}_1$ let $f_n = f|\bar{J}_n$. By Lemma 6 there exist almost continuous functions g_n, h_n such that $|f_n| = g_n \cdot h_n$ on $\bar{J}_n$ and all cluster sets of g_n and h_n at the end points of J_n are $[0, \infty]$. Moreover, if a_n, b_n are the end points of J_n then $g_n(a_n) = |f(a_n)|$, $g_n(b_n) = |f(b_n)|$, $h_n(a_n) = h_n(b_n) = 1$.

Let $(T_n)_n$ be the sequence of all components of $J \setminus (\overline{\bigcup \mathcal{L}_0} \cup \overline{\bigcup \mathcal{L}_1} \cup \bar{D})$. By the definition of D every T_n satisfies the assumptions of Lemma 9 and consequently, there exist almost continuous functions s_n, t_n such that $f|T_n = s_n \cdot t_n$ and s_n, t_n satisfy (10.2) and (10.3). Set

$$g(x) = \begin{cases} f(x) & \text{for } x \in (\bar{D} \cup \{a, b\}) \setminus \bigcup \{a_n, b_n: (a_n, b_n) \in \mathcal{L}_0 \cup \mathcal{L}_1\}, \\ \operatorname{sgn}(f(x)) \cdot (-1)^i g_n(x) & \text{for } x \in \bar{J}_n,\ J_n \in \mathcal{L}_i,\ i = 0, 1, \\ s_n(x) & \text{for } x \in T_n,\ n \in \mathbf{N}, \end{cases}$$

$$h(x) = \begin{cases} 1 & \text{for } x \in (\bar{D} \cup \{a, b\}) \setminus \bigcup \{a_n, b_n: (a_n, b_n) \in \mathcal{L}_0 \cup \mathcal{L}_1\}, \\ (-1)^i h_n(x) & \text{for } x \in \bar{J}_n,\ J_n \in \mathcal{L}_i,\ i = 0, 1, \\ t_n(x) & \text{for } x \in T_n,\ n \in \mathbf{N}. \end{cases}$$

Then $f = g \cdot h$ and g, h satisfy (10.2). Moreover, it is obvious that (10.3) holds if $a \notin \bar{D}$, or if $a \in \bar{D}$ and a is an end point of some component K of $J \setminus (\overline{\bigcup (\mathcal{L}_0 \cup \mathcal{L}_1)} \cup \bar{D})$. If $a \in \bar{D}$, a is an end point of some component K of $J \setminus \bar{D}$ and $a \in \overline{K \cap \bigcup (\mathcal{L}_0 \cup \mathcal{L}_1)}$, then (10.3) follows from (viii). Finally, if $a \in \bar{D}$ and $a_n \nearrow a$ ($a_n \searrow a$) for some sequence (a_n) in $\bar{D}$, then (10.3) follows from (vi).

Now we shall verify (10.1), e.g. for g. Let $K = (a, b)$ be some component of $J \setminus \bar{D}$. There are four possible cases: $a, b \notin M = \overline{K \cap \bigcup (\mathcal{L}_0 \cup \mathcal{L}_1)}$; $a \in M$ and $b \notin M$; $a \notin M$ and $b \in M$; and $a, b \in M$. Consider e.g. the case $a \notin M$ and $b \in M$. Then there exists a sequence $a_n \in C$ such that $a_0 = a$, $a_n \nearrow b$ and $(a_{n-1}, a_n) \in \mathcal{L}_0 \cup \mathcal{L}_1 \cup \{T_n: n \in \mathbf{N}\}$ for each $n \in \mathbf{N}$. Consequently, each $g|[a_{n-1}, a_n]$ is almost continuous and therefore, $g|K$ is almost continuous (see [4], Lemma 3.5). Since g and h satisfy all assumptions of Lemma 10, they are almost continuous. Moreover, the conditions $\alpha(g, J)$ and $\alpha(h, J)$ hold.

LEMMA 12. *Assume that C is a nowhere dense subset of the interval J and the maximal dense-in-itself subset D of C is non-empty. If f is a real function on J with the property* (JC) *and $C = [f = 0]$, then f is the product of two almost continuous functions g and h which satisfy the conditions $\alpha(g, J)$ and $\alpha(h, J)$.*

Proof. Let A be the union of all open subintervals K of J such that $K \cap D \neq \emptyset$ and $x \in \overline{\{z \in K \setminus \bar{C}: f(z) > 0\}}$ for each $x \in K \cap \bar{D}$, and let B be the union of all open subintervals K of $J \setminus \bar{A}$ such that $K \cap D \neq \emptyset$ and $x \in \overline{\{z \in K \setminus \bar{C}: f(z) < 0\}}$ for each $x \in K \cap \bar{D}$. List all components of A, B and $J \setminus (\bar{A} \cup \bar{B} \cup \bar{D})$ in a sequence $(J_n)_n$. Then:

(i) $E = J \setminus \bigcup_n J_n$ is a closed nowhere dense subset of $\bar{D}$,

(ii) if J_n is a component of $J \setminus (\bar{A} \cup \bar{B} \cup \bar{D})$, then $J_n \cap C$ has no non-empty dense-in-itself subset,

(iii) a point $x_0 \in E \setminus \{a\}$ is either the right end point of some component J_n, or there exists a sequence (J_{k_n}) such that $J_{k_n} \nearrow x_0$ and $J_{k_n} \subset A \cup B$ for each $n \in \mathbf{N}$,

(iv) the similar left-hand condition holds for points x_0 in $E \setminus \{b\}$.

The condition (i) follows from the obvious fact that the end points of each J_n belong to $\bar{D}$, and (ii) follows from the definition of D. To verify (iii), let $(x_0 - \delta, x_0) \cap A = \emptyset$ for some $x_0 \in E \setminus \{a\}$ and $\delta > 0$. Then either $(x_0 - \delta, x_0) \cap D$ is empty and x_0 is the right end point of some J_n, or there exists an interval $K \subset (x_0-\delta, x_0)$ such that $K \cap D \neq \emptyset$ and $f(x) < 0$ for each $x \in K \setminus \bar{C}$. Then $K \subset B$ and consequently, $(x_0-\delta, x_0)$ meets some component of B.

Let $f_n = f|J_n$ for each $n \in \mathbf{N}$. For $J_n \subset A \cup B$ let g_n, h_n be almost continuous functions such that $f_n = g_n \cdot h_n$ and $\alpha(g_n, J_n)$, $\alpha(h_n, J_n)$ hold. (Such functions exist by Lemma 11.) For $J_n \subset J \setminus (\bar{A} \cup \bar{B} \cup \bar{D})$ it follows by (ii) and Lemma 9 that f_n is the product of two almost continuous functions g_n, h_n which satisfy (9.1).

Now define

$$g(x) = \begin{cases} f(x) & \text{for } x \in E, \\ g_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}; \end{cases} \qquad h(x) = \begin{cases} 1 & \text{for } x \in E, \\ h_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}. \end{cases}$$

It is obvious that $f = g \cdot h$. Since $\alpha(g, J_n)$ and $\alpha(h, J_n)$ hold for every component J_n of $A \cup B$, Lemma 8 shows that g and h are almost continuous. Since $A \cup B$ is dense in J, $\alpha(g, J)$ and $\alpha(h, J)$ hold too.

LEMMA 13. *Let f be a function on an interval J for which the set $[f = 0]$ is of the first category and dense in J. Then f is a product of two almost continuous functions in $\mathscr{D}^*(J, \mathbf{R})$.*

Proof. Let $(K_\alpha)_{\alpha<\mathfrak{c}}$ be the sequence of all closed subsets of $J \times \mathbf{R}$ whose x-projection is a non-degenerate interval such that every set stands in this sequence exactly twice. We choose by induction a sequence $(x_\alpha, y_\alpha) \in K_\alpha$ such that:

(i) if $x_\alpha = x_\beta$ for some $\alpha, \beta < \mathfrak{c}$, then $y_\alpha = y_\beta = 0$ and $f(x_\alpha) = 0$,

(ii) $y_\alpha = 0$ iff $f(x_\alpha) = 0$.

Assume that we have chosen (x_β, y_β) for $\beta < \alpha$. Let $E_\alpha = \operatorname{dom}(K_\alpha) \setminus [f = 0]$. This set is residual in $\operatorname{dom}(K_\alpha)$ and therefore $\operatorname{card}(E_\alpha \cap L) = \mathfrak{c}$ for each subinterval L of $\operatorname{dom}(K_\alpha)$. Let $F_\alpha = E_\alpha \setminus \{x_\beta \colon \beta < \alpha\}$. This set is dense in $\operatorname{dom}(K_\alpha)$. We have two possibilities:

(a) If $(x, 0) \notin K_\alpha$ for some $x \in F_\alpha$, then we put $x_\alpha = x$ and $y_\alpha = y$ such that $(x, y) \in K_\alpha$.

(b) If $(x, 0) \in K_\alpha$ for each $x \in F_\alpha$, then (because K_α is closed) $\operatorname{dom}(K_\alpha) \times \{0\} \subset K_\alpha$. We choose $x \in \operatorname{dom}(K_\alpha) \cap [f = 0]$ and put $x_\alpha = x$, $y_\alpha = 0$.

It is obvious that (x_α, y_α) satisfies (i) and (ii). Now, if $x = x_\alpha$ for some $\alpha < \mathfrak{c}$ and $K_\alpha \neq K_\beta$ for each $\beta < \alpha$, then define $g(x) = y_\alpha$ and $h(x) = f(x)/g(x)$ if $y_\alpha \neq 0$, and $h(x) = 1$ if $y_\alpha = 0$. If $x = x_\alpha$ for some $\alpha < \mathfrak{c}$ and $K_\alpha = K_\beta$ for some $\beta < \alpha$, then $h(x) = y_\alpha$ and $g(x) = f(x)/h(x)$ if $y_\alpha \neq 0$, and $g(x) = 1$ if $y_\alpha = 0$. If $x \neq x_\alpha$ for each $\alpha < \mathfrak{c}$, then $g(x) = f(x)$, $h(x) = 1$.

It is easy to observe that $f = g \cdot h$ and $g, h \in \mathscr{D}^*(J, \mathbf{R})$. Since g and h meet each blocking set in $J \times \mathbf{R}$, they are almost continuous.

In the next lemmata we shall assume that a union of less than $\mathfrak{c}$ nowhere dense sets is a set of the first category (e.g. under MA).

LEMMA 14. (MA) *Let f be a function on an interval J for which $[f = 0]$ is of the second category at each point of J. Then f is a product of two almost continuous functions in $\mathscr{D}^*(J, \mathbf{R})$.*

Proof. Let $(K_\alpha)_{\alpha<\mathfrak{c}}$ be the sequence of closed subsets of $J \times \mathbf{R}$ defined in the proof of Lemma 13. We choose (inductively) a sequence $(x_\alpha, y_\alpha)_{\alpha<\mathfrak{c}}$ such that $(x_\alpha, y_\alpha) \in K_\alpha$ and $x_\alpha \in [f = 0]$ for $\alpha < \mathfrak{c}$. Assume that (x_β, y_β) are chosen for $\beta < \alpha$, where $\alpha < \mathfrak{c}$. By Martin's Axiom, $E_\alpha = \operatorname{dom}(K_\alpha) \cap [f = 0] \setminus \{x_\beta \colon \beta < \alpha\} \neq \emptyset$ and we choose $x_\alpha \in E_\alpha$ and y_α with $(x_\alpha, y_\alpha) \in K_\alpha$. Now assume first that $x = x_\alpha$ for some $\alpha < \mathfrak{c}$. If $K_\alpha \neq K_\beta$ for each $\beta < \alpha$, then define $g(x) = y_\alpha$ and $h(x) = 0$. If $K_\alpha = K_\beta$ for some $\beta < \alpha$, then $g(x) = 0$ and $h(x) = y_\alpha$. If $x \neq x_\alpha$ for each $\alpha < \mathfrak{c}$, then $g(x) = f(x)$ and $h(x) = 1$. Then $f = g \cdot h$, $g, h \in \mathscr{D}^*(J, \mathbf{R})$ and g, h are almost continuous.

LEMMA 15. (MA) *Let f be a function on an interval J with $[f = 0]$ dense in J. Then f is the product of two almost continuous functions g, h in $\mathscr{D}^*(J, \mathbf{R})$. (Hence $\alpha(g, J)$ and $\alpha(h, J)$ hold.)*

Proof. Let $A \subset J$ ($B \subset J$) be the set of all points at which $[f = 0]$ is of the first (second) category. Let $(J_n)_n$ be the sequence of all intervals $K \subset J$ such that K is a component of A ($= \operatorname{int}(A)$) or $\operatorname{int}(B)$. Then $C = J \setminus \bigcup_n J_n$ is closed and nowhere dense. Let $f_n = f|J_n$ for $n \in \mathbf{N}$. For each n such that $J_n \subset A$ ($J_n \subset B$) let g_n, h_n be almost continuous functions in $\mathscr{D}^*(J_n, \mathbf{R})$ and $f_n = g_n \cdot h_n$. Define

$$g(x) = \begin{cases} f(x) & \text{for } x \in C, \\ g_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}; \end{cases} \qquad h(x) = \begin{cases} 1 & \text{for } x \in C, \\ h_n(x) & \text{for } x \in J_n,\ n \in \mathbf{N}. \end{cases}$$

It is obvious that $f = g \cdot h$, $g, h \in \mathscr{D}^*(J, \mathbf{R})$ and $\alpha(g, J_n)$, $\alpha(h, J_n)$ hold for each $n \in \mathbf{N}$. By Lemma 8, g and h are almost continuous.

THEOREM 2. (MA) *A real function defined on the interval I is a product of two almost continuous functions iff it has a zero in each subinterval in which it changes sign.*

Proof. Let f be a function on I with the property (JC). Let $C = [f = 0]$. Let A_0 be the union of all subintervals of I in which C is dense, let A_1 be the union of all subintervals J of I for which $J \cap C$ is nowhere dense and includes a non-empty dense-in-itself subset, and let A_2 be the union of all subintervals K of I for which $K \cap \bar{C}$ is infinite and includes no non-empty dense-in-itself subset. List all components of A_0, A_1, A_2 and $I \setminus (\bigcup_{i<3} \bar{A}_i \cup \bar{C})$ in a sequence $(J_n)_n$. Then:

(i) $D = I \setminus \bigcup_n J_n$ is a closed nowhere dense subset of $\bar{C}$,

(ii) f has a constant sign on every $J_n \subset I \setminus (\bigcup_{i<3} \bar{A}_i \cup \bar{C})$,

(iii) each $x_0 \in D \setminus \{1\}$ is either the left end point of some J_n or there exists a sequence (J_{k_n}) such that $J_{k_n} \searrow x_0$ and $J_{k_n} \subset A_0 \cup A_1 \cup A_2$ for $n \in \mathbf{N}$,

(iv) the analogous right-hand condition holds for each x_0 in $D \setminus \{0\}$.

The first two conditions are obvious. We shall verify (iii). Fix x_0 in $D \setminus \{1\}$. Assume that $(x_0, x_0+\delta) \cap (A_0 \cup A_1 \cup A_2) = \emptyset$ for some $\delta > 0$. Since $(x_0, x_0+\delta) \cap A_0 = \emptyset$, $E = (x_0, x_0+\delta) \cap C$ is nowhere dense. Since $(x_0, x_0+\delta) \cap A_1 = \emptyset$, E contains no non-empty dense-in-itself subset. Since $(x_0, x_0+\delta) \cap A_2 = \emptyset$, E is finite. Therefore x_0 is the left end point of some component of $I \setminus (\bigcup_{i<3} \bar{A}_i \cup \bar{C})$.

For each $n \in \mathbf{N}$ let $f_n = f|J_n$. By Lemma 15 (respectively: Lemmata 12 and 9), if $J_n \subset A_0$ (respectively A_1 and A_2) then f_n is the product of two almost continuous functions g_n, h_n which satisfy the conditions $\alpha(g_n, J_n)$ and $\alpha(h_n, J_n)$.

If J_n is a component of $I\setminus(\bigcup_{i<3}\bar{A}_i\cup\bar{C})$ then by Lemma 6 there exist almost continuous functions g_n, h_n such that $f_n = g_n\cdot h_n$, the cluster sets of h_n at the end points of J_n are $[0, \infty]$ and the cluster sets of g_n at the end points of J_n are $\operatorname{sgn}(f_n)\cdot[0, \infty]$. Set

$$g(x) = \begin{cases} f(x) & \text{for } x\in D,\\ g_n(x) & \text{for } x\in J_n,\ n\in\mathbf{N}; \end{cases} \qquad h(x) = \begin{cases} 1 & \text{for } x\in D,\\ h_n(x) & \text{for } x\in J_n,\ n\in\mathbf{N}. \end{cases}$$

It is obvious that $f = g\cdot h$. Since g and h satisfy all assumptions of Lemma 8, they are almost continuous.

This completes the proof that every function with the property (JC) is a product of two almost continuous functions. The opposite implication follows immediately from the fact that a product of two Darboux functions has the property (JC).

Remark. Theorems 1 and 2 remain true for real functions defined on the whole real line.

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INSTITUTE OF MATHEMATICS
PEDAGOGICAL COLLEGE
Chodkiewicza 30, 85-064 Bydgoszcz, Poland

Received 4 June 1990